Improved direct and converse theorems in weighted Lorentz spaces

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Abstract

In the present work we prove the equivalence of the fractional modulus of smoothness to the realization functional and to the Peetre K-functional in weighted Lorentz spaces. Using this equivalence we obtain an improvement of the direct approximation theorem. Furthermore we prove the improved converse theorem in this space.

1 Introduction and main results

In approximation theory improvements of direct and inverse theorems have been investigated by several authors in different function spaces [1, 9, 12, 18, 20, 21]. In this paper we deal with the improved direct and inverse approximation theorems in the weighted Lorentz space $L^{pq}_{\omega}(\mathbb{T})$ with Muckenhoupt weights. To obtain the improved direct theorem we need the realization and characterization theorem in $L^{pq}_{\omega}(\mathbb{T})$. Therefore we will prove a realization result and an equivalence relation between the modulus of smoothness and the Peetre K-functional in $L^{pq}_{\omega}(\mathbb{T})$. Furthermore, the realization result has a lot of applications [6]. In particular, it is used to get Ul'yanov type inequalities [8]. First, we give some definitions and properties.

Let $\mathbb{T} := [-\pi, \pi]$ and $\omega : \mathbb{T} \to [0, \infty]$ be a weight function i.e., an almost everywhere positive measurable function. We define the decreasing rearrange-

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ment $f_\omega^*(t)$ [11] of $f : \mathbb{T} \to \mathbb{R}$ with respect to the Borel measure

$$\omega(e) = \int_e \omega(x)\mathrm{d}x,$$

by

$$f_\omega^*(t) = \inf\left\{\tau \geq 0 : \omega\left(x \in \mathbb{T} : |f(x)| > \tau\right) \leq t\right\}.$$

The weighted Lorentz space $L_\omega^{pq}(\mathbb{T})$ is defined [11] as

$$L_\omega^{pq}(\mathbb{T}) = \left\{f \in \mathbf{M}(\mathbb{T}) : \|f\|_{pq,\omega} = \Big(\int_{\mathbb{T}} (f^{**}(t))^q t^{\frac{q}{p}} \frac{\mathrm{d}t}{t}\Big)^{1/q} < \infty,\ 1 < p,q < \infty\right\},$$

where $\mathbf{M}(\mathbb{T})$ is the set of 2π periodic integrable functions on $\mathbb{T}$ and

$$f^{**}(t) = \frac{1}{t}\int_0^t f_\omega^*(u)\mathrm{d}u.$$

If $p = q$, $L_\omega^{pq}(\mathbb{T})$ turns into the weighted Lebesgue space $L_\omega^p(\mathbb{T})$ [11, p.20].
A weight function $\omega : \mathbb{T} \to [0,\infty]$ belongs to the Muckenhoupt class A_p [17], $1 < p < \infty$, if

$$\sup \frac{1}{|I|}\int_I \omega(x)\mathrm{d}x\Big(\frac{1}{|I|}\int_I \omega^{1-p'}(x)dx\Big)^{p-1} = C_{A_p} < \infty, \qquad p' := \frac{p}{p-1}$$

with a finite constant C_{A_p} independent of I, where the supremum is taken over all intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I. The constant C_{A_p} is called the Muckenhoupt constant of ω.
By the proof of [14, Prop. 3.3], we know that $L_w^{pq}(\mathbb{T}) \subset L^1(\mathbb{T})$. Let

$$S[f] := \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

be the Fourier series of a function $f \in L^1(\mathbb{T})$. Assume that

$$\int_{\mathbb{T}} f(x)\mathrm{d}x = 0. \tag{2}$$

For $\alpha \in \mathbb{R}^+$, we define the α-th fractional integral of f as [22, v.2, p.134]

$$I_\alpha(x,f) := \sum_{k\in\mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

with

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i\alpha \operatorname{sign} k}$$

as principal value.
We define the fractional derivative of a function $f \in L^1(\mathbb{T})$, satisfying (2), as

$$f^{(\alpha)}(x) := \frac{\mathrm{d}^{[\alpha]+1}}{\mathrm{d}x^{[\alpha]+1}} I_{1+\alpha-[\alpha]}(x,f),$$

whenever the right hand side exists.
For a function $f \in L_\omega^{pq}(\mathbb{T})$, $1 < p, q < \infty$, $w \in A_p$, Steklov's mean operator is defined as

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(u) \mathrm{d}u, \quad x \in \mathbb{T}.$$

Whenever $\omega \in A_p$, $1 < p, q < \infty$, the Hardy-Littlewood maximal function of $f \in L_\omega^{pq}(\mathbb{T})$ belongs to $L_\omega^{pq}(\mathbb{T})$ [5, Theorem 3]. Therefore the operator $\sigma_h f$ belongs to $L_\omega^{pq}(\mathbb{T})$. Using this fact and putting $x, t \in \mathbb{T}$, $r \in \mathbb{R}^+$, $\omega \in A_p$ and $f \in L_\omega^{pq}(\mathbb{T})$, $1 < p, q < \infty$, we define

$$\begin{aligned} \sigma_t^r f(x) &:= (I - \sigma_t)^r f(x) \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \frac{1}{(2t)^k} \int_{-t}^{t} \cdots \int_{-t}^{t} f(x + u_1 + \cdots + u_k) \mathrm{d}u_1 \ldots \mathrm{d}u_k, \end{aligned}$$

where $\binom{r}{k}$ are the binomial coefficients.
Since

$$\left| \binom{\alpha}{k} \right| \leq \frac{c}{k^{\alpha+1}}, \quad k \in \mathbb{N}$$

(see [19, p.14, (1.51)]), we have

$$\sum_{k=0}^{\infty} \left| \binom{\alpha}{k} \right| < \infty,$$

and therefore

$$\|\sigma_t^\alpha f\|_{pq,\omega} \leq c \|f\|_{pq,\omega} < \infty, \tag{3}$$

for $f \in L_\omega^{pq}(\mathbb{T})$, $1 < p, q < \infty$, $\omega \in A_p$.
For $1 < p, q < \infty$, $f \in L_\omega^{pq}(\mathbb{T})$ and $r \in \mathbb{R}^+$, we define the fractional modulus of smoothness of index r as

$$\Omega_r(f, \delta)_{pq,\omega} := \sup_{0 < h_i, t < \delta} \left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i})(I - \sigma_t)^{r-[r]} f \right\|_{pq,\omega}, \tag{4}$$

where $[r] := \max \{n \in \mathbb{N} : n \leq r\}$. Since the operator σ_t is bounded in $L_\omega^{pq}(\mathbb{T})$, $1 < p, q < \infty$, $\omega \in A_p$ we have by (3) that

$$\Omega_r(f, \delta)_{pq,\omega} \leq c \|f\|_{pq,\omega}$$

where the constant $c > 0$ only depends on r, p , q and C_{A_p}.

Remark 1.1. Let $r \in \mathbb{R}^+$, $1 < p, q < \infty$, $\omega \in A_p$ and $f \in L_\omega^{pq}(\mathbb{T})$. For $\delta > 0$, the modulus of smoothness $\Omega_r(f, \delta)_{pq,\omega}$ has the following properties.

(i) $\Omega_r(f, \delta)_{pq,\omega}$ is sub-additive in f, and a non-negative, non-decreasing function of $\delta \geq 0$.

(ii) $\lim\limits_{\delta \to 0} \Omega_r(f, \delta)_{pq,\omega} = 0$.

By $E_n(f)_{pq,\omega}$ we denote the best approximation of $f \in L^{pq}_{\omega}(\mathbb{T})$ by polynomials in $\mathcal{T}_n$, the set of trigonometric polynomials of degree $\leq n$:

$$E_n(f)_{pq,\omega} = \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_{pq,\omega}.$$

In this paper we will use the following notations:

$$\begin{array}{rcl} A(x) & \approx & B(x) \Leftrightarrow \exists c_1, c_2 > 0 : c_1 B(x) \leq A(x) \leq c_2 B(x) \\ A(x) & \preceq & B(x) \Leftrightarrow \exists c > 0 : A(x) \leq c B(x). \end{array}$$

Our main results are now the following.

Theorem 1.2. *If $r \in \mathbb{R}^+$, $1 < p, q < \infty$, $\omega \in A_p$ and $f \in L^{pq}_{\omega}(\mathbb{T})$, then there exists a constant $c > 0$ depending only on r, p, q and C_{A_p} such that*

$$E_n(f)_{pq,\omega} \leq c\, \Omega_r\Big(f, \frac{1}{n+1}\Big)_{pq,\omega} \tag{5}$$

holds for $n + 1 \in \mathbb{N}$.

The analogues of this direct approximation theorem were obtained in [10] for $r \in \mathbb{N}$, $f \in L^{p}_{\omega}(\mathbb{T}), 1 < p < \infty, \omega \in A_p$ and in [2] for $r \in \mathbb{N}$, $f \in L^{pq}_{\omega}(\mathbb{T})$, $1 < p, q < \infty$, $\omega \in A_p$ with the modulus of smoothness

$$W_r\Big(f, \frac{1}{n}\Big)_{L^{pq}_{\omega}} := \sup_{0 \leq h_i \leq 1/n} \Big\| \prod_{i=1}^{r} \left(I - \sigma_{h_i}\right) f \Big\|_{L^{pq}_{\omega}},$$

and in [1] for $r \in \mathbb{R}^+$, $f \in L^{p}_{\omega}(\mathbb{T}), 1 < p < \infty, \omega \in A_p$ with the fractional modulus of smoothness (4).
For $f \in L^{pq}_{\omega}(\mathbb{T})$, $t, r > 0$ and $1 < p, q < \infty$, the Peetre K-functional is defined as

$$K_r(f, t; L^{pq}_{\omega}, W^r_{pq,\omega}) := \inf_{g \in W^r_{pq,w}} \{\|f - g\|_{L^{pq}_{\omega}} + t^r \|g^{(r)}\|_{L^{pq}_{\omega}}\}.$$

Here $W^r_{pq,\omega} := \Big\{ g(x) \in L^{pq}_{\omega} \ : \ g^{(r)} \in L^{pq}_{\omega} \Big\}$.
We define the realization functional for $f \in L^{pq}_{\omega}(\mathbb{T})$ by

$$R_r(f, 1/n, L^{pq}_{\omega}) := \Big\{ \|f - t^*_n\|_{L^{pq}_{\omega}} + \frac{1}{n^r} \|(t^*_n)^{(r)}\|_{L^{pq}_{\omega}} \Big\},$$

for $r > 0$, $1 < p, q < \infty$, $n \in \mathbb{N}$. Here t^*_n denotes the best approximating trigonometric polynomial for f. The following theorem holds.

Theorem 1.3. *If $\mathbb{R}^+$, $f \in L^{pq}_{\omega}(\mathbb{T})$, $1 < p, q < \infty$ and $\omega \in A_p$, then the equivalence*

$$\Omega_r\Big(f, \frac{1}{n}\Big)_{pq,\omega} \approx R_{2r}\left(f, 1/n, L^{pq}_{w}\right) \tag{6}$$

holds for $n = 1, 2, 3, \ldots..$ Furthermore, we have, for $\delta \geq 0$,

$$\Omega_r(f, \delta)_{pq,\omega} \approx K_{2r}(f, \delta; L^{pq}_{w}, W^r_{pq,w}). \tag{7}$$

Here the equivalence constants only depend on r, p, q and C_{A_p}.

Corollary 1.4. *Let $r \in \mathbb{R}^+$, $f \in L_w^{pq}(\mathbb{T})$, $1 < p, q < \infty$, and $\omega \in A_p$. Then*

$$\Omega_r(f, \lambda\delta)_{pq,\omega} \preceq (1 + [\lambda])^{2r}\, \Omega_r(f, \delta)_{pq,\omega}, \quad \delta, \lambda > 0$$

and

$$\Omega_r(f, \delta)_{pq,\omega}\, \delta^{-2r} \preceq \Omega_r(f, \delta_1)_{pq,\omega}\, \delta_1^{-2r}, \quad 0 < \delta_1 \le \delta.$$

An improvement of (5) is given by the following theorem.

Theorem 1.5. *If $r \in \mathbb{R}^+$, $f \in L_\omega^{pq}(\mathbb{T})$, $1 < p, q < \infty$ and $\omega \in A_p$, then there exists a constant $c > 0$ depending on r, p, q and C_{A_p} such that for $n = 1, 2, 3, \ldots$*

$$\Big(\prod_{j=1}^{n} E_j(f)_{pq,\omega}\Big)^{1/n} \le c\, \Omega_r\Big(f, \frac{1}{n}\Big)_{pq,\omega}. \tag{8}$$

Remark 1.6. The inequality (8) is never worse than the classical Jackson inequality. Since $E_n(f)_{pq,\omega} \to 0$ as $n \to \infty$ we obtain that

$$E_n(f)_{pq,\omega} \le \Big(\prod_{j=1}^{n} E_j(f)_{pq,\omega}\Big)^{1/n} \le c\, \Omega_r\Big(f, \frac{1}{n}\Big)_{pq,\omega}.$$

On the other hand, in some cases the inequality (8) gives better results than the classical Jackson inequality. For example, if $E_n(f)_{pq,\omega} = 2^{-n}$, then the classical Jackson inequality implies $\Omega_r\left(f, \frac{1}{n}\right)_{pq,\omega} \ge c2^{-n}$ but inequality (8) yields $\Omega_r\left(f, \frac{1}{n}\right)_{pq,\omega} \ge c2^{-n/2}$.

An analogue of Theorem 1.5 for the space L^∞ was proved in [18]. In [2], the weak converse of (5)

$$\Omega_r\Big(f, \frac{1}{n}\Big)_{L_\omega^{pq}} \le \frac{c}{n^{2r}} \sum_{\nu=0}^{n} (\nu + 1)^{2r-1} E_\nu(f)_{L_\omega^{pq}}, \tag{9}$$

for $r \in \mathbb{N}$, $f \in L_\omega^{pq}(\mathbb{T})$, $\omega \in A_p$ and $1 < p, q < \infty$ was obtained.

Theorem 1.7. *Let $1 < p < \infty$ and $1 < q \le 2$ or $p > 2$ and $q \ge 2$, $\omega \in A_p$, $f \in L_\omega^{pq}(\mathbb{T})$. If $n \in \mathbb{N}$, $r \in \mathbb{R}^+$ and $\gamma := \min\{2, q\}$, then there is a constant $c > 0$ only depending on r, q, p and C_{A_p} such that*

$$\Omega_r\Big(f, \frac{1}{n}\Big)_{pq,\omega} \le \frac{c}{n^{2r}} \Big(\sum_{\nu=1}^{n} \nu^{2\gamma r-1} E_{\nu-1}^{\gamma}(f)_{pq,\omega}\Big)^{1/\gamma}. \tag{10}$$

The analogues of this improved converse theorem were proven in [15] for $r \in \mathbb{N}$, $f \in L_\omega^p(\mathbb{T})$, $1 < p < \infty$, $\omega \in A_p$ with the modulus of smoothness $W_r\left(f, \frac{1}{n}\right)_{L_\omega^p}$; in [1] for $r \in \mathbb{R}^+$, $f \in L_\omega^p(\mathbb{T})$, $1 < p < \infty$, $\omega \in A_p$ with the fractional modulus of smoothness (4); in [14] for $r \in \mathbb{N}$, $f \in L_\omega^{pq}(\mathbb{T})$, $1 < p < \infty$ and $1 < q \le 2$ or $p > 2$ and $q \ge 2$, $\omega \in A_p$ with $W_r\left(f, \frac{1}{n}\right)_{L_\omega^{pq}}$; in [20] for $r \in \mathbb{R}^+$, $f \in L_\omega^{pq}(\mathbb{T})$, $1 < p < \infty$

and $1 < q \leq 2$ or $p > 2$ and $q \geq 2$, $\omega \in A_p$ with a modulus of smoothness defined by Ky [16].
The inequality (10) is better than (9). Indeed, using the fact that x^γ is convex for $\gamma = \min\{2, p\}$ we obtain that

$$\left(\nu\nu^{2r-1}E_\nu(f)_{pq,\omega}\right)^\gamma - \left((\nu-1)\,\nu^{2r-1}E_\nu(f)_{pq,\omega}\right)^\gamma$$
$$\leq \left(\sum_{\mu=1}^{\nu}\mu^{2r-1}E_\mu(f)_{pq,\omega}\right)^\gamma - \left(\sum_{\mu=1}^{\nu-1}\mu^{2r-1}E_\mu(f)_{pq,\omega}\right)^\gamma.$$

Taking the summation over ν, we obtain that

$$\sum_{\nu=1}^{n}\left\{\left(\nu\nu^{2r-1}E_\nu(f)_{pq,\omega}\right)^\gamma - \left((\nu-1)\,\nu^{2r-1}E_\nu(f)_{pq,\omega}\right)^\gamma\right\}$$
$$\leq \sum_{\nu=1}^{n}\left\{\left(\sum_{\mu=1}^{\nu}\mu^{2r-1}E_\mu(f)_{pq,\omega}\right))^\gamma - \left(\sum_{\mu=1}^{\nu-1}\mu^{2r-1}E_\mu(f)_{pq,\omega}\right)^\gamma\right\},$$

hence we have the inequality

$$\left\{\sum_{\nu=1}^{n}\nu^{2\gamma r-1}E_{\nu-1}^{\gamma}(f)_{pq,\omega}\right\}^{1/\gamma} \leq 2\sum_{\nu=1}^{n}\nu^{2r-1}E_{\nu-1}(f)_{pq,\omega}.$$

We give the Marcinkiewicz multiplier and Littlewood-Paley theorems in $L_\omega^{pq}(\mathbb{T})$ which are used in the proofs of previous Theorems.

Theorem 1.8. *Let $\lambda_0, \lambda_1, \cdots$ be a sequence of real numbers such that*

$$|\lambda_l| \leq M \text{ and } \sum_{\nu=2^{l-1}}^{2^l-1}|\lambda_\nu - \lambda_{\nu+1}| \leq M,$$

for all $\nu, l \in \mathbb{N}$. If $1 < p, q < \infty$, $\omega \in A_p$ and $f \in L_\omega^{pq}(\mathbb{T})$ with Fourier series $\sum_{\nu=0}^{\infty}(a_\nu(f)\cos\nu x + b_\nu(f)\sin\nu x)$, then there exists $h \in L_\omega^{pq}(\mathbb{T})$ such that the series $\sum_{\nu=0}^{\infty}\lambda_\nu(a_\nu(f)\cos\nu x + b_\nu(f)\sin\nu x)$ is the Fourier series of h and

$$\|h\|_{pq,\omega} \leq C\,\|f\|_{pq,\omega}, \tag{11}$$

where C does not depend on f.

Theorem 1.9. *Let $\nu \in \mathbb{N}$, $1 < p, q < \infty$, $\omega \in A_p$ and $f \in L_\omega^{pq}(\mathbb{T})$ with Fourier series $\sum_{\nu=0}^{\infty}(a_\nu(f)\cos\nu x + b_\nu(f)\sin\nu x)$, then there exist constants c_1 and c_2 independent of f such that*

$$c_1\left\|\left(\sum_{\mu=\nu}^{\infty}|\Delta_\mu|^2\right)^{1/2}\right\|_{pq,\omega} \leq \|f\|_{pq,\omega} \leq c_2\left\|\left(\sum_{\mu=\nu}^{\infty}|\Delta_\mu|^2\right)^{1/2}\right\|_{pq,\omega}, \tag{12}$$

where

$$\Delta_\mu := \Delta_\mu(x, f) := \sum_{\nu=2^{\mu-1}}^{2^\mu-1}(a_\nu(f)\cos\nu x + b_\nu(f)\sin\nu x).$$

2 Proof of the main results

From [5, 14] we recall four important properties of the spaces $L^{pq}_{\omega}(\mathbb{T})$.

Lemma A ([5] or [13, prop. 5.1.2]) *For* $1 < p, q < \infty$, there exists a $c > 0$ *such that for every* $f \in L^{pq}_{\omega}(\mathbb{T})$

$$c^{-1}\|f\|_{pq,\omega} \leq \sup\Big|\int_{\mathbb{T}} f(x)g(x)w(x)\mathrm{d}x\Big| \leq c\,\|f\|_{pq,\omega},$$

where the supremum is taken over all functions g *for which* $\|g\|_{p'q',\omega} \leq 1$.

Lemma B [14]. *Let* $1 < p < \infty$ *and* $1 < q \leq 2$. *Then for an arbitrary system of functions* $\{\varphi_j(x)\}_{j=1}^{m}$, $\varphi_j \in L^{pq}_{\omega}(\mathbb{T})$ *we have*

$$\Big\|\Big(\sum_{j=1}^{m}\varphi_j{}^2\Big)^{1/2}\Big\|_{pq,\omega} \leq c\Big(\sum_{j=1}^{m}\|\varphi_j\|^{q}_{pq,\omega}\Big)^{1/q}$$

with a constant c independent of φ_j *and m.*

Lemma C [14]. *Let* $2 < p < \infty$ *and* $q \geq 2$. *For an arbitrary system* $\{\varphi_j(x)\}_{j=1}^{m}$, $\varphi_j \in L^{pq}_{\omega}(\mathbb{T})$, *we have*

$$\Big\|\Big(\sum_{j=1}^{m}\varphi_j{}^2\Big)^{1/2}\Big\|_{pq,\omega} \leq c\Big(\sum_{j=1}^{m}\|\varphi_j\|^{2}_{pq,\omega}\Big)^{1/2}$$

with a constant c independent of φ_j *and m.*

Lemma D [14]. *Let* $1 < p, q < \infty$, $f \in L^{pq}_{\omega}(\mathbb{T})$ *and* $w \in A_p$. *If* $B_{k,\mu}(x) = a_k(f)\cos\left(k + \mu\frac{\pi}{2}\right)x + b_k(f)\sin\left(k + \mu\frac{\pi}{2}\right)x$, *where* a_k, b_k *are the Fourier coefficients of* f, *then*

$$\Big\|\sum_{k=2^i+1}^{2^{i+1}} k^{\mu}B_{k,\mu}\Big\|_{pq,\omega} \leq c2^{i\mu}E_{2^i}(f)_{pq,\omega},$$

where the constant c is independent of f and i.

Proof of Theorem 1.8. We define a linear operator

$$Tf(x) := \sum_{\nu=0}^{\infty}\lambda_{\nu}\left(a_{\nu}(f)\cos\nu x + b_{\nu}(f)\sin\nu x\right)$$

for $f \in L^{pq}_{\omega}(\mathbb{T})$ which is bounded (in particular is of weak type (p, p)) in $L^{p}_{\omega}(\mathbb{T})$ for every $p > 1$ by [4, Th. 4.4]. Therefore the hypothesis of the interpolation theorem for Lorentz spaces [3, Th. 4.13] is fulfilled. Applying this theorem we get the desired result (11). ∎

Proof of Theorem 1.9. Let us define a quasilinear operator

$$Tf(x) := \Big(\sum_{\mu=\nu}^{\infty}\left|\Delta_{\mu}(x, f)\right|^2\Big)^{1/2}.$$

This operator is bounded in $L^p_\omega(\mathbb{T})$ for every $p > 1$ by [4, Th. 4.5]. Therefore the left hand side of the required result (12) is derived by means of the interpolation theorem for Lorentz spaces [3, Th. 4.13].

Using Hölder's inequality for $f \in L^{pq}_\omega(\mathbb{T}) \cap L^2_\omega(\mathbb{T})$, $g \in L^{p'q'}_\omega(\mathbb{T}) \cap L^2_\omega(\mathbb{T})$ and the left hand side of (12) we obtain

$$\begin{aligned}
\int_{\mathbb{T}} |f(x)g(x)|\,\omega(x)\mathrm{d}x &= \int_{\mathbb{T}} \left|\sum_{\mu=1}^{\infty}\Delta_\mu(x,f)\Delta_\mu(x,g)\right| \omega(x)\mathrm{d}x \\
&\le \int_{\mathbb{T}} \sum_{\mu=1}^{\infty} |\Delta_\mu(x,f)\Delta_\mu(x,g)|\omega(x)\mathrm{d}x \\
&\le \int_{\mathbb{T}} \Big[\sum_{\mu=1}^{\infty} |\Delta_\mu(x,f)|^2\Big]^{1/2} \Big[\sum_{\mu=1}^{\infty} |\Delta_\mu(x,g)|^2\Big]^{1/2} \omega(x)\mathrm{d}x \\
&\le \left\| \Big[\sum_{\mu=1}^{\infty} \Big|\Delta_\mu(x,f)\Big|^2\Big]^{1/2} \right\|_{pq,\omega} \left\| \Big[\sum_{\mu=1}^{\infty} \Big|\Delta_\mu(x,g)\Big|^2\Big]^{1/2} \right\|_{p'q',\omega} \\
&\le C \left\| \Big[\sum_{\mu=1}^{\infty} |\Delta_\mu(x,f)|^2\Big]^{1/2} \right\|_{pq,\omega} \|g\|_{p'q',\omega}.
\end{aligned}$$

where $p' = p/(p-1)$, $q' = q/(q-1)$. Taking the supremum in the last inequality over all functions $g \in L^{p'q'}_\omega(\mathbb{T})$ satisfying $\|g\|_{p'q',\omega} \le 1$, we find, applying Lemma A that

$$\|f\|_{pq,\omega} \le C \left\| \Big(\sum_{\mu=1}^{\infty} \Big|\Delta_\mu\Big|^2\Big)^{1/2} \right\|_{pq,\omega}.$$

The density of $L^{pq}_\omega(\mathbb{T}) \cap L^2_\omega(\mathbb{T})$ in $L^{pq}_\omega(\mathbb{T})$ yields the last inequality for any $f \in L^{pq}_\omega(\mathbb{T})$. ■

Lemma 2.1. *If* $0 < \alpha \le \beta$, $\omega \in A_p$, $1 < p, q < \infty$ *and* $f \in L^{pq}_\omega(\mathbb{T})$ *then*

$$\Omega_\beta(f,\cdot)_{pq,\omega} \le c\,\Omega_\alpha(f,\cdot)_{pq,\omega}. \tag{13}$$

Proof. The proof of Lemma 2.1 is similar to the proof of [1, Lemma 1]. ■

Lemma 2.2. *Let* $r \in \mathbb{R}^+$, $1 < p, q < \infty$, $\omega \in A_p$ *and* $T_n \in \mathcal{T}_n$ *for* $n = 1, 2, \cdots$. *Then*

$$\Omega_r\Big(T_n, \frac{1}{n}\Big)_{pq,\omega} \preceq \frac{1}{n^{2r}} \left\| T_n^{(2r)} \right\|_{pq,\omega}$$

holds with some constant only depending on r, p, q *and* C_{A_p}.

Proof. For all $x \ge 0$, we have that

$$\left(1 - \frac{\sin x}{x}\right)_* \le x^2,$$

where

$$\left(1 - \frac{\sin x}{x}\right)_* := \begin{cases} 1 - \frac{\sin x}{x} & \text{if } x \ge 0; \\ 0 & \text{if } x = 0. \end{cases}$$

For $0 < t$ and $h_i \leq \frac{1}{n}$, we have that

$$\begin{aligned}
&\left\|\prod_{i=1}^{[r]} \left(I-\sigma_{h_i}\right) \left(I-\sigma_t\right)^{r-[r]} T_n\right\|_{pq,\omega} \\
&= \left\|\sum_{\nu=0}^{n} \left(1-\frac{\sin \nu h_1}{\nu h_1}\right)_* \cdots \left(1-\frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right)_* \left(1-\frac{\sin \nu t}{\nu t}\right)_*^{r-[r]} \right. \\
&\qquad \left. A_\nu(T_n,x)\right\|_{pq,\omega} \\
&= \left\|\sum_{\nu=1}^{n} \frac{\left(1-\frac{\sin \nu h_1}{\nu h_1}\right)(\nu h_1)^2}{(\nu h_1)^2} \cdots \frac{\left(1-\frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right)\left(\nu h_{[r]}\right)^2}{\left(\nu h_{[r]}\right)^2} \left(\frac{1-\frac{\sin \nu t}{\nu t}}{(\nu t)^2}\right)^{r-[r]} \right. \\
&\qquad \left. (\nu t)^{2(r-[r])} A_\nu(T_n,x)\right\|_{pq,\omega} \\
&\leq n^{-2r}\left\|\sum_{\nu=1}^{n} \frac{\left(1-\frac{\sin \nu h_1}{\nu h_1}\right)}{(\nu h_1)^2}\nu^2 \cdots \frac{\left(1-\frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right)}{\left(\nu h_{[r]}\right)^2}\nu^2 \left(\frac{1-\frac{\sin \nu t}{\nu t}}{(\nu t)^2}\right)^{r-[r]} \right. \\
&\qquad \left. \nu^{2(r-[r])} A_\nu(T_n,x)\right\|_{pq,\omega} \\
&\leq n^{-2r}\left\|\sum_{\nu=1}^{n} \nu^{2r} \frac{\left(1-\frac{\sin \nu h_1}{\nu h_1}\right)}{(\nu h_1)^2} \cdots \frac{\left(1-\frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right)}{\left(\nu h_{[r]}\right)^2} \left(\frac{1-\frac{\sin \nu t}{\nu t}}{(\nu t)^2}\right)^{r-[r]} A_\nu(T_n,x)\right\|_{pq,\omega}.
\end{aligned}$$

Applying Theorem 1.8 we obtain that

$$\left\|\prod_{i=1}^{[r]} \left(I-\sigma_{h_i}\right) \left(I-\sigma_t\right)^{r-[r]} T_n\right\|_{pq,\omega} \preceq n^{-2r}\left\|\sum_{\nu=1}^{n} \nu^{2r} A_\nu(T_n,x)\right\|_{pq,\omega}.$$

For $\nu = 1, 2, 3, \ldots$ we have

$$A_\nu(T_n,x) = A_\nu(T_n, x+\frac{r\pi}{\nu})\cos r\pi + A_\nu(\tilde{T}_n, x+\frac{r\pi}{\nu})\sin r\pi,$$

where $\tilde{T}_n$ is the Fourier conjugate of T_n. Therefore

$$\begin{aligned}
&\left\|\prod_{i=1}^{[r]}(I-\sigma_{h_i})(I-\sigma_t)^{r-[r]}T_n\right\|_{pq,\omega} \\
&\preceq n^{-2r}\left\|\sum_{\nu=1}^{n} \nu^{2r}(A_\nu(T_n, x+\frac{r\pi}{\nu})\cos r\pi + A_\nu(\tilde{T}_n, x+\frac{r\pi}{\nu})\sin r\pi)\right\|_{pq,\omega} \\
&\preceq n^{-2r}\Big(\left\|\sum_{\nu=1}^{n} \nu^{2r} A_\nu(T_n, x+\frac{r\pi}{\nu})\right\|_{pq,\omega} + \left\|\sum_{\nu=1}^{n} \nu^{2r} A_\nu(\tilde{T}_n, x+\frac{r\pi}{\nu})\Big)\right\|_{pq,\omega}\Big).
\end{aligned}$$

Since

$$A_\nu(T_n^{(2r)}, x) = \nu^{2r} A_\nu(T_n, x+\frac{r\pi}{\nu}),$$

for $\nu = 1, 2, 3, \cdots$, we find

$$\begin{aligned}
&\Omega_r\left(T_n, \frac{1}{n}\right)_{pq,\omega} \\
&\preceq n^{-2r}\left(\left\|\sum_{\nu=1}^{n} \nu^{2r} A_\nu(T_n, x + \frac{r\pi}{\nu})\right\|_{pq,\omega} + \left\|\sum_{\nu=1}^{n} \nu^{2r} A_\nu(\tilde{T}_n, x + \frac{r\pi}{\nu})\right\|_{pq,\omega}\right) \\
&\preceq n^{-2r}\left(\left\|T_n^{\left(2r\right)}\right\|_{pq,\omega} + \left\|\tilde{T}_n^{\left(2r\right)}\right\|_{pq,\omega}\right) \preceq n^{-2r}\left\|T_n^{\left(2r\right)}\right\|_{pq,\omega}.
\end{aligned}$$

■

Lemma 2.3. *Let $r \in \mathbb{R}_+$, $1 < p, q < \infty$, $\omega \in A_p$ and $T_n \in \mathcal{T}_n$. For $n = 1, 2, \cdots$, we have that*

$$\frac{1}{n^{2r}}\|T_n^{(2r)}\|_{pq,\omega} \preceq \Omega_r\left(T_n, \frac{1}{n}\right)_{pq,\omega}$$

with some constant depending only on r, p, q and C_{A_p}.

Proof.

$$\begin{aligned}
n^{-2r}\left\|T_n^{(2r)}\right\|_{pq,\omega} &= n^{-2r}\left\|\sum_{\nu=1}^{n} \nu^{2r} A_\nu(T_n, x + \frac{r\pi}{\nu})\right\|_{pq,\omega} \\
&= n^{-2r}\left\|\sum_{\nu=1}^{n} \nu^{2r}\left(A_\nu(T_n, x)\cos r\pi + A_\nu(\tilde{T}_n, x)\sin r\pi\right)\right\|_{pq,\omega} \\
&\leq n^{-2r}\left\|\sum_{\nu=1}^{n} \nu^{2r} A_\nu(T_n, x)\cos r\pi\right\|_{pq,\omega} \\
&\quad + n^{-2r}\left\|\sum_{\nu=1}^{n} \nu^{2r} A_\nu(\tilde{T}_n, x)\sin r\pi\right\|_{pq,\omega} \\
&= \left\|\sum_{\nu=1}^{n} \cos r\pi \left(\frac{\left(\frac{\nu}{n}\right)^2}{1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}}\right)^r \left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r A_\nu(T_n, x)\right\|_{pq,\omega} \\
&\quad + \left\|\sum_{\nu=1}^{n} \sin r\pi \left(\frac{\left(\frac{\nu}{n}\right)^2}{1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}}\right)^r \left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r A_\nu(\tilde{T}_n, x)\right\|_{pq,\omega}.
\end{aligned}$$

Applying Theorem 1.8 and the linearity of the conjugate operator we get

$$\begin{aligned}
n^{-2r}\left\|T_n^{(2r)}\right\|_{pq,\omega} &\preceq \left\|\sum_{\nu=1}^{n}\left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r A_\nu(T_n, x)\right\|_{pq,\omega} \\
&\quad + \left\|\sum_{\nu=1}^{n}\left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r A_\nu(\tilde{T}_n, x)\right\|_{pq,\omega} \\
&= \left\|\sum_{\nu=1}^{n}\left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r A_\nu(T_n, x)\right\|_{pq,\omega} \\
&\quad + \left\|\left(\sum_{\nu=1}^{n}\left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r A_\nu(T_n, x)\right)^{\sim}\right\|_{pq,\omega}.
\end{aligned}$$

From the boundedness of the conjugate operator [14] we have

$$\begin{aligned} & n^{-2r}\left\|T_n^{(2r)}\right\|_{pq,\omega} \\ \preceq & \left\|\sum_{\nu=1}^{n}\left(1-\frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^{r}A_\nu(T_n,x)\right\|_{pq,\omega}+\left\|\sum_{\nu=1}^{n}\left(1-\frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^{r}A_\nu(T_n,x)\right\|_{pq,\omega} \\ \preceq & \left\|\left(I-\sigma_{\frac{1}{n}}\right)^{r}T_n\right\|_{pq,\omega}\preceq \sup_{0<h_i,u<1/n}\left\|\prod_{i=1}^{[r]}\left(I-\sigma_{h_i}\right)\left(I-\sigma_u\right)^{r-[r]}T_n\right\|_{pq,\omega} \\ \preceq & \Omega_r\left(T_n,\frac{1}{n}\right)_{pq,\omega}. \end{aligned}$$ ■

Proof of Theorem 1.2. From Lemma 2.1 and [2, Th. 1.1] we have

$$E_n(f)_{pq,\omega}\leq c\,\Omega_{[r]+1}\left(f,\frac{1}{n+1}\right)_{pq,\omega}\leq C\Omega_r\left(f,\frac{1}{n+1}\right)_{pq,\omega}$$

for $n+1\in\mathbb{N}$ and the assertion (5) follows. ■

Lemma 2.4. *Let* $1<p,q<\infty$, $\omega\in A_p$, $f\in L_\omega^{pq}(\mathbb{T})$ *and* $\gamma>0$ *. Then for any* $0<t<2$,

$$\Omega_\gamma\left(f,t\right)_{pq,\omega}\preceq t^{\gamma}\left\|f^{(\gamma)}\right\|_{pq,\omega}.$$

Proof. There is some $n=1,2,3,...$ such that $(1/n)<t\leq(2/n)$. From Lemma 2.2 we get

$$\Omega_\gamma\left(f,t\right)_{pq,\omega}\leq\Omega_\gamma\left(f-T_n,t\right)_{pq,\omega}+\Omega_\gamma\left(T_n,t\right)_{pq,\omega}\preceq E_n(f)_{pq,\omega}+t^{2\gamma}\left\|T_n^{(2\gamma)}\right\|_{pq,\omega}.$$

On the other hand applying [20, (3.9) and Th. 1.3] and Theorem 1.2 we have

$$E_n(f)_{pq,\omega}\preceq\frac{1}{n^{2\gamma}}E_n(f^{(2\gamma)})_{pq,\omega}\preceq\frac{1}{n^{2\gamma}}\Omega_\gamma\left(f^{(2\gamma)},\frac{1}{n}\right)_{pq,\omega}\preceq t^{2\gamma}\left\|f^{(2\gamma)}\right\|_{pq,\omega}.$$

Using Theorem 1.2 and [20, Th. 1.3] the proof is completed. ■

Proof of Theorem 1.3. We have to show that (6) holds. Let T_n be the near best approximating trigonometric polynomial to f. From Theorem 1.2

$$\|f-T_n\|_{pq,\omega}\preceq E_n(f)_{pq,\omega}\leq c\,\Omega_r\left(f,\frac{1}{n+1}\right)_{pq,\omega}.$$

Applying Lemma 2.3, we find that

$$\begin{aligned} \frac{1}{n^{2r}}\left\|T_n^{\left(2r\right)}\right\|_{pq,\omega} & \preceq\Omega_r\left(T_n,\frac{1}{n}\right)_{pq,\omega}\preceq\Omega_r\left(T_n-f,\frac{1}{n}\right)_{pq,\omega}+\Omega_r\left(f,\frac{1}{n}\right)_{pq,\omega} \\ & \preceq\left\|f-T_n\right\|_{pq,\omega}+\Omega_r\left(f,\frac{1}{n}\right)_{pq,\omega}\preceq\Omega_r\left(f,\frac{1}{n}\right)_{pq,\omega} \end{aligned}$$

and

$$\|f - T_n\|_{pq,\omega} + \frac{1}{n^{2r}}\left\|T_n^{(2r)}\right\|_{pq,\omega} \preceq \Omega_r\left(f, \frac{1}{n}\right)_{pq,\omega}.$$

On the other hand using Lemma 2.2

$$\begin{aligned}\Omega_r\left(f, \frac{1}{n}\right)_{pq,\omega} &\leq \Omega_r\left(f - T_n, \frac{1}{n}\right)_{pq,\omega} + \Omega_r\left(T_n, \frac{1}{n}\right)_{pq,\omega} \\ &\preceq \left\|f - T_n\right\|_{pq,\omega} + \frac{1}{n^{2r}}\left\|T_n^{(2r)}\right\|_{pq,\omega} = R_{2r}\left(f, \frac{1}{n}, L_\omega^{pq}\right).\end{aligned}$$

This completes the proof of (6). Using Lemma 2.4, properties of the modulus of smoothness and of the $K-$functional (7) are proven.

Proof of Theorem 1.5. By Corollary 1.4 we have for $v \leq n$

$$\Omega_r\left(f, 1/v\right)_{pq,\omega} \leq \left(1 + n/v\right)^{2r} \Omega_r\left(f, 1/n\right)_{pq,\omega}$$

and

$$\prod_{v=1}^{n} \Omega_r\left(f, 1/v\right)_{pq,\omega} \leq \prod_{v=1}^{n} \left(1 + n/v\right)^{2r} \left(\Omega_r\left(f, 1/n\right)_{pq,\omega}\right)^n.$$

For every n we have

$$\prod_{v=1}^{n} \left(1 + n/v\right)^{2r} \leq \left(\frac{2n}{\sqrt[n]{n!}}\right)^{2r}.$$

Using Stirling's formula

$$n! \asymp \sqrt{2\pi n} n^n e^{-n} e^{\theta(n)} \text{ with } \left|\theta\left(n\right)\right| \leq 1/\left(12n\right)$$

we get

$$\prod_{v=1}^{n} \left(1 + n/v\right)^{2r} \leq 2^{2r} e^{4r}.$$

Thus

$$\left(\prod_{v=1}^{n} \Omega_r\left(f, 1/v\right)_{pq,\omega}\right)^{1/n} \leq c\, \Omega_r\left(f, 1/n\right)_{pq,\omega}.$$

From (5) and the property $E_n(f)_{pq,\omega} \to 0$ as $n \to \infty$ we find

$$\left(\prod_{v=1}^{n} E_v(f)_{pq,\omega}\right)^{1/n} \leq \left(\prod_{v=1}^{n} \Omega_r\left(f, 1/v\right)_{pq,\omega}\right)^{1/n} \leq c\, \Omega_r\left(f, 1/n\right)_{pq,\omega}.$$

Proof of Theorem 1.7. For $1 < p, q < \infty$, $\omega \in A_p$, let $f \in L_\omega^{pq}\left(\mathbb{T}\right)$ be such that $\int_0^{2\pi} f(x)\mathrm{d}x = 0$. We assume that f has Fourier series (1). We choose $m \in \mathbb{N}$ such that $2^m \leq n < 2^{m+1}$. Let us denote $S_n\left(x\right) := S_n\left(x, f\right) := \sum_{k=0}^{n} A_k(f, x)$, for $x \in \mathbb{T}$, where $A_k(f, x) = a_k(f)\cos kx + b_k(f)\sin kx$. By [14, Prop. 3.4], we have that

$$\|f - S_n\|_{pq,\omega} \leq cE_n(f)_{pq,\omega}. \tag{14}$$

It is well-known that $\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} f := \prod_{i=1}^{[r]} \left(I - \sigma_{h_i}\right) \left(I - \sigma_t\right)^{r-[r]} f$ has Fourier series

$$\begin{aligned} &\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} f\left(\cdot\right) \\ &\sim \sum_{\nu=0}^{\infty} \left(1 - \frac{\sin \nu t}{\nu t}\right)_*^{r-[r]} \left(1 - \frac{\sin \nu h_1}{\nu h_1}\right)_* \ldots \left(1 - \frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right)_* A_\nu(f, x). \end{aligned}$$

Moreover

$$\begin{aligned} &\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} f\left(\cdot\right) \\ &= \sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} \left(f\left(\cdot\right) - S_{2^{m-1}}\left(\cdot, f\right)\right) + \sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} S_{2^{m-1}}\left(\cdot, f\right). \end{aligned}$$

From (14) and $E_n(f)_{p,\omega} \to 0$ we have

$$\begin{aligned} &\left\| \sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} \left(f\left(\cdot\right) - S_{2^{m-1}}\left(\cdot, f\right)\right) \right\|_{pq,\omega} \\ &\leq c \left\| f\left(\cdot\right) - S_{2^{m-1}}\left(\cdot, f\right) \right\|_{pq,\omega} \leq c E_{2^{m-1}}(f)_{pq,\omega} \\ &\leq \frac{c}{n^{2r}} \left\{ \sum_{\nu=1}^{n} \nu^{2\gamma r-1} E^\gamma_{\nu-1}(f)_{p,\omega} \right\}^{1/\gamma}. \end{aligned}$$

On the other hand, it follows from (12) that

$$\left\| \sigma^r_{t,h_1,h_2,\ldots,h_{[r]}} S_{2^{m-1}}\left(\cdot, f\right) \right\|_{pq,\omega} \leq c \left\| \left\{ \sum_{\mu=1}^{m} \left|\delta_\mu\right|^2 \right\}^{1/2} \right\|_{pq,\omega}$$

where

$$\delta_\mu := \sum_{\nu=2^{\mu-1}}^{2^\mu-1} \left(1 - \frac{\sin \nu t}{\nu t}\right)^{r-[r]} \left(1 - \frac{\sin \nu h_1}{\nu h_1}\right) \ldots \left(1 - \frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right) A_\nu(f, x).$$

By Lemmas B and C

$$\left\| \left\{ \sum_{\mu=1}^{m} \left|\delta_\mu\right|^2 \right\}^{1/2} \right\|_{pq,\omega} \leq \left\{ \sum_{\mu=1}^{m} \left\|\delta_\mu\right\|^\gamma_{pq,\omega} \right\}^{1/\gamma}.$$

By Abel's transformation we obtain

$$\begin{aligned} &\|\delta_\mu\|_{pq,\omega} \\ &\leq \sum_{\nu=2^{\mu-1}}^{2^\mu-2} \Bigg| \left(1 - \frac{\sin \nu t}{\nu t}\right)^{r-[r]} \left(1 - \frac{\sin \nu h_1}{\nu h_1}\right) \ldots \left(1 - \frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right) \\ &\quad - \left(1 - \frac{\sin(\nu+1)t}{(\nu+1)t}\right)^{r-[r]} \left(1 - \frac{\sin(\nu+1)h_1}{(\nu+1)h_1}\right) \\ &\quad \cdots \left(1 - \frac{\sin\left(\nu+1\right)h_{[r]}}{(\nu+1)h_{[r]}}\right) \Bigg| \left\| \sum_{l=2^{\mu-1}}^{\nu} A_l(f, x) \right\|_{pq,\omega} \end{aligned}$$

$$+\left|\left(1-\frac{\sin(2^{\mu}-1)t}{(2^{\mu}-1)t}\right)^{r-[r]}\left(1-\frac{\sin(2^{\mu}-1)h_1}{(2^{\mu}-1)h_1}\right)\right.$$
$$\left.\cdots\left(1-\frac{\sin(2^{\mu}-1)h_{[r]}}{(2^{\mu}-1)h_{[r]}}\right)\right|\left\|\sum_{l=2^{\mu-1}}^{2^{\mu}-1}A_l(f,x)\right\|_{pq,\omega}$$

and by Lemma D

$$\left\|\sum_{l=2^{\mu-1}}^{\nu}A_l(f,x)\right\|_{pq,\omega}\le cE_{2^{\mu-1}-1}(f)_{p,\omega}$$

and

$$\left\|\sum_{l=2^{\mu-1}}^{2^{\mu}-1}A_l(f,x)\right\|_{pq,\omega}\le CE_{2^{\mu-1}-1}(f)_{pq,\omega}.$$

Since $x^r\left(1-\frac{\sin x}{x}\right)^r$ is non decreasing for positive x we have

$$\left\|\delta_\mu\right\|_{pq,\omega}\le c2^{2\mu r}t^{2(r-[r])}h_1^2h_2^2\ldots h_{[r]}^2E_{2^{\mu-1}-1}(f)_{pq,\omega}$$

and hence

$$\begin{aligned}
&\left\|\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}S_{2^{m-1}}\left(\cdot,f\right)\right\|_{pq,\omega}\\
\le\ & ct^{2\left(r-[r]\right)}h_1^2h_2^2\ldots h_{[r]}^2\left\{\sum_{\mu=1}^{m}2^{\mu r\gamma}E^{\gamma}_{2^{\mu-1}-1}(f)_{pq,\omega}\right\}^{1/\gamma}\\
\le\ & ct^{2\left(r-[r]\right)}h_1^2h_2^2\ldots h_{[r]}^2\left\{2^{\gamma r}E_0^{\gamma}(f)_M\right\}^{1/\alpha}\\
+\ & ct^{2\left(r-[r]\right)}h_1^2h_2^2\ldots h_{[r]}^2\left\{\sum_{\mu=2}^{m}\sum_{\nu=2^{\mu-2}}^{2^{\mu-1}-1}\nu^{2\gamma r-1}E^{\gamma}_{\nu-1}(f)_{pq,\omega}\right\}^{1/\gamma}\\
\le\ & ct^{2\left(r-[r]\right)}h_1^2h_2^2\ldots h_{[r]}^2\left\{\sum_{\nu=1}^{2^{m-1}-1}\nu^{2\gamma r-1}E^{\gamma}_{\nu-1}(f)_{pq,\omega}\right\}^{1/\gamma}.
\end{aligned}$$

Therefore we find

$$\Omega_r\left(f,\frac{1}{n}\right)_{pq,\omega}\le\frac{c}{n^{2r}}\left\{\sum_{\nu=1}^{n}\nu^{2\gamma r-1}E^{\gamma}_{\nu-1}\left(f\right)_{pq,\omega}\right\}^{1/\gamma}$$

finishing the proof of Theorem 1.7. ■

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