Upper-Lower and Left-Right Semi-Fredholmness

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Abstract

Upper-lower and left-right approaches coincide for semi-Fredholm operators on Hilbert spaces but, in general, they are distinct for operators on a Banach space. The purpose of this paper is to investigate the difference between the upper-lower and left-right approaches for semi-Fredholm operators on a Banach space and, in particular, to verify when these approaches coincide. The program is based on the classes $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ of all operators on a Banach space $\mathcal{X}$ with complemented range and complemented kernel. It is shown that the intersection $\Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]$ is algebraically and topologically large, and also that if $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ are either open, or closed, or if they coincide, then there is no difference between the upper-lower and left-right approaches for semi-Fredholm operators on a Banach space.

Introduction

Fredholm operators comprise a crucial class of Banach space operators which play a central role in operator theory, being rather relevant in many areas of analysis, both from theoretical and applied points of view. For a survey on Fredholm theory in Hilbert space the reader is referred to [10]. Semi-Fredholm operators are defined either as the union of the class of all upper semi-Fredholm and the class of all lower semi-Fredholm operators on the one hand or, on the other hand, as the union of the class of all left semi-Fredholm and the class of all

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right semi-Fredholm operators. These two ways of handling semi-Fredholm operators are referred to as the upper-lower and left-right approaches. On a Hilbert space these approaches are coincident, and the reason for such a coincidence is that in a Hilbert space every (closed) subspace is complemented. Such a complementation property may fail in an arbitrary Banach space, and this is the reason why the upper-lower and left-right approaches may not coincide for semi-Fredholm operators on a Banach space.

The purpose of this paper is to draw a parallel between upper-lower and left-right approaches for semi-Fredholm operators on a Banach space. The difference between them is investigated, and it is shown when the upper-lower and left-right approaches coincide. In particular, it isshown when these approaches coincide as far as the existence of pseudoholes of the essential (Fredholm) spectrum is concerned. The main results appear in Theorems 4.1 and 5.1, and an application is discussed in Section 6 and summarized in Corollary 6.1.

Let $\mathcal{B}[\mathcal{X}]$ be the Banach algebra of all bounded linear operators on a Banach space $\mathcal{X}$, and consider the class $\Gamma_R[\mathcal{X}]$ of all operators with complemented range and the class $\Gamma_N[\mathcal{X}]$ of all operators with complemented kernel. Since in a Hilbert space every subspace is complemented, it follows that $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}] = \mathcal{B}[\mathcal{X}]$ if $\mathcal{X}$ is a Hilbert space, and this is sufficient (but not necessary) to ensure that there is no difference between the upper-lower and left-right approaches. However, such an identity does not hold in general. The program for investigating when the upper-lower and left-right approaches coincide starts by showing that the intersection $\Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]$ is algebraically and topologically large, and then it is shown that if $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ are either open, or closed, or if they coincide, then there is no difference between the upper-lower and left-right approaches.

The paper is organized as follows. Section 1 contains notation, terminology, and only the basic results that will be frequently required in the sequel. Let T be a bounded linear operator acting on a Banach space $\mathcal{X}$. Section 2 shows in Theorem 2.1 that the classes of all operators with complemented range $\Gamma_R[\mathcal{X}]$ and complemented kernel $\Gamma_N[\mathcal{X}]$ are algebraically and topologically large. The relationship between the classes of upper or lower and left or right semi-Fredholm operators is discussed in Section 3. Section 4 paves the way to compare upper and lower semi-Fredholm spectra, $\sigma_{e_+}(T)$ and $\sigma_{e_-}(T)$, with left and right essential spectra, $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$, which differ by the sets $\zeta_R(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Gamma_R[\mathcal{X}]\}$ and $\zeta_N(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Gamma_N[\mathcal{X}]\}$, as carried out in Section 5. Theorem 4.1 says that if $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ are closed, so that $\zeta_R(T)$ and $\zeta_N(T)$ are open, then the left and right essential spectra coincide with the upper and lower semi-Fredholm spectra. Theorem 5.1 says that if either $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ are open (are they always open?) so that $\zeta_R(T)$ and $\zeta_N(T)$ are closed, or if $\zeta_R(T)$ and $\zeta_N(T)$ coincide, then the essential spectrum has no pseudoholes if and only if the left and right essential spectra coincide. An application in Section 6 shows when the previous results ensure that biquasitriangularity in a Banach space boils down to the same thing as its Hilbert space counterpart.

1 Preliminaries

Let $\mathcal{X}$ be a linear space, let $I\colon \mathcal{X} \to \mathcal{X}$ be the identity transformation, and let $\mathcal{M}$ be a linear manifold of $\mathcal{X}$. For any linear transformation $L\colon \mathcal{X} \to \mathcal{X}$ let $\mathcal{N}(L) = L^{-1}(\{0\})$ be the kernel of L and $\mathcal{R}(L) = L(\mathcal{X})$ the range of L, which are linear manifolds of $\mathcal{X}$. An algebraic complement of $\mathcal{M}$ is any linear manifold $\mathcal{N}$ of $\mathcal{X}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$. Every linear manifold $\mathcal{M}$ has an algebraic complement; every algebraic complement of $\mathcal{M}$ has the same dimension, referred to as the codimension of $\mathcal{M}$. If $E\colon \mathcal{X} \to \mathcal{X}$ is a projection (an idempotent linear transformation), then $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are complementary linear manifolds; conversely, if $\mathcal{M}$ and $\mathcal{N}$ are complementary linear manifolds, then there is a unique projection $E\colon \mathcal{X} \to \mathcal{X}$ with $\mathcal{R}(E) = \mathcal{M} = \mathcal{N}(I-E)$ and $\mathcal{N}(E) = \mathcal{N} = \mathcal{R}(I-E)$, where $I - E\colon \mathcal{X} \to \mathcal{X}$ is the complementary projection of E. Let $\mathcal{X}/\mathcal{M}$ be the quotient space of $\mathcal{X}$ modulo $\mathcal{M}$: the linear space of all cosets $[x] = x + \mathcal{M}$ of x modulo $\mathcal{M}$. Every algebraic complement of $\mathcal{M}$ is isomorphic to the quotient space $\mathcal{X}/\mathcal{M}$, and $\operatorname{codim} \mathcal{M} = \dim \mathcal{X}/\mathcal{M}$: codimension of a linear manifold is the dimension of any algebraic complement of it, which is constant and coincides with the dimension of the quotient space (see, e.g., [11, Sections 2.8 and 2.9]).

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are (complex) normed spaces. A subspace of a normed space is a *closed* linear manifold; the closure $\mathcal{M}^-$ of a linear manifold $\mathcal{M}$ is a subspace. Let $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ denote the normed space of all linear bounded (i.e., continuous) transformations of $\mathcal{X}$ into $\mathcal{Y}$. For any $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$, its kernel $\mathcal{N}(T)$ is a subspace (i.e., a closed linear manifold) of $\mathcal{X}$, and its range $\mathcal{R}(T)$ is a linear manifold of $\mathcal{Y}$. By an operator we mean a bounded linear transformation of $\mathcal{X}$ into itself. Set $\mathcal{B}[\mathcal{X}] = \mathcal{B}[\mathcal{X},\mathcal{X}]$, the normed algebra of all operators on $\mathcal{X}$, which is a Banach algebra if $\mathcal{X}$ is a Banach space. Take the map $\|\cdot\|\colon \mathcal{X}/\mathcal{M} \to \mathbb{R}$ such that $\|[x]\| = \inf_{u\in\mathcal{M}} \|x+u\| = d(x,\mathcal{M})$, the distance of x to $\mathcal{M}$. This is a seminorm that becomes the usual norm on $\mathcal{X}/\mathcal{M}$ if $\mathcal{M}$ is closed. If $\mathcal{M}$ is a subspace, then equip $\mathcal{X}/\mathcal{M}$ with its usual norm.

A subspace $\mathcal{M}$ of a normed space $\mathcal{X}$ is *complemented* if it has a subspace as an algebraic complement; that is, a *closed* linear manifold $\mathcal{M}$ of a normed space $\mathcal{X}$ is complemented if there is a *closed* linear manifold $\mathcal{N}$ of $\mathcal{X}$ such that $\mathcal{M}$ and $\mathcal{N}$ are algebraic complements. In this case, $\mathcal{M}$ and $\mathcal{N}$ are *complementary subspaces*.

Remark 1.1. If $\mathcal{X}$ is a normed space and $E\colon \mathcal{X} \to \mathcal{X}$ is a continuous projection, then $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are complementary subspaces of $\mathcal{X}$. Conversely, if $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces of a Banach space $\mathcal{X}$, then the (unique) projection $E\colon \mathcal{X} \to \mathcal{X}$ with $\mathcal{R}(E) = \mathcal{M}$ and $\mathcal{N}(E) = \mathcal{N}$ is continuous (in fact, a projection with closed range and closed kernel on a Banach space is continuous). See, e.g., [11, Problem 4.35]. So, if $\mathcal{X}$ is a Banach space, then the assertions below are equivalent.

(a) A subspace $\mathcal{M}$ of $\mathcal{X}$ is complemented.

(b) There exists a projection $E \in \mathcal{B}[\mathcal{X}]$ with $\mathcal{R}(E) = \mathcal{M}$.

(c) There exists a projection $I - E \in \mathcal{B}[\mathcal{X}]$ with $\mathcal{N}(I - E) = \mathcal{M}$.

Therefore, in a finite-dimensional normed space every subspace is complemented (in a finite-dimensional normed space every linear manifold is a subspace).

Proposition 1.1. *Finite-dimensional subspaces of a Banach space are complemented. If a subspace of a Banach space has finite codimension, then it is complemented.*

Proof. See, e.g., [16, Theorem A.1.25(i,ii)]. ∎

Remark 1.2. The above result does not hold if the normed space $\mathcal{X}$ is not Banach. For instance, let $C[0,1]$ be the linear space of all real-valued continuous function on the interval $[0,1]$ equipped with the norm $\|\cdot\|_1$. Take a discontinuous function, say, $v(t)=0$ for $t\in[0,\frac{1}{2})$ and $v(t)=1$ for $t\in[\frac{1}{2},1]$, and consider the normed space $\mathcal{X}=C[0,1]+\operatorname{span}\{v\}$, again equipped the norm $\|\cdot\|_1$, which is not Banach (reason: $C[0,1]$ is dense in $L^1[0,1]$ and so is $\mathcal{X}\subsetneqq L^1[0,1]$). $C[0,1]$ is not closed in $\mathcal{X}$ (there are sequences of continuous functions converging to v in the norm $\|\cdot\|_1$).

Proposition 1.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of a normed space $\mathcal{X}$. If $\dim\mathcal{N}<\infty$, then $\mathcal{M}+\mathcal{N}$ is closed.*

Proof. See, e.g., [5, Proposition III.4.3]. ∎

Consider the converse to Proposition 1.2: is it true that if $\mathcal{M}+\mathcal{N}$ is closed and $\dim\mathcal{N}<\infty$, then $\mathcal{M}$ is closed? If $\mathcal{M}$ is the range of a bounded linear transformation between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, then the answer is yes.

Proposition 1.3. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, take $T\in\mathcal{B}[\mathcal{X},\mathcal{Y}]$, and let $\mathcal{N}$ be a finite-dimensional subspace of $\mathcal{Y}$. If $\mathcal{R}(T)+\mathcal{N}$ is closed, then so is $\mathcal{R}(T)$.*

Proof. See, e.g., [16, Lemma 16.2]. ∎

Corollary 1.1. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. If $T\in\mathcal{B}[\mathcal{X},\mathcal{Y}]$ is such that $\operatorname{codim}\mathcal{R}(T)<\infty$, then $\mathcal{R}(T)$ is closed.*

Proof. Apply Proposition 1.3. ∎

Remark 1.3. This does not ensure that $\mathcal{R}(T)$ is closed if $\operatorname{codim}\mathcal{R}(T)^-$ is finite. There are $T\in\mathcal{B}[\mathcal{X},\mathcal{Y}]$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\operatorname{codim}\mathcal{R}(T)^-=0$ and $\mathcal{R}(T)$ is not closed. Example: if $T=\operatorname{diag}\{\frac{1}{k}\}$ on $\mathcal{X}=\ell^2$, then 0 lies in the continuous spectrum $\sigma_C(T)$ of T, and hence $\mathcal{R}(T)$ is not closed but is dense in ℓ^2 so that the unique algebraic complement of $\mathcal{R}(T)^-=\ell^2$ is $\{0\}$, and so $\operatorname{codim}\mathcal{R}(T)^-=0$.

2 Range-Kernel Complemented

Recall that if a Banach space $\mathcal{X}$ is complemented (i.e., if every subspace of $\mathcal{X}$ is complemented), then it is isomorphic (i.e., topologically isomorphic) to a Hilbert space [15]. Thus complemented Banach spaces are identified with Hilbert spaces — only Hilbert spaces (up to an isomorphism) are complemented.

Definition 2.1. Given a Banach space $\mathcal{X}$, consider the following subsets of $\mathcal{B}[\mathcal{X}]$.

$$\Gamma_R[\mathcal{X}] = \{T \in \mathcal{B}[\mathcal{X}] \colon \mathcal{R}(T)^- \text{ is complemented}\},$$

$$\Gamma_N[\mathcal{X}] = \{T \in \mathcal{B}[\mathcal{X}] \colon \mathcal{N}(T) \text{ is complemented}\},$$

$$\Gamma[\mathcal{X}] = \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] = \{T \in \mathcal{B}[\mathcal{X}] \colon \mathcal{R}(T)^- \text{ and } \mathcal{N}(T) \text{ are complemented}\}.$$

(Note: The class $\Gamma_R[\mathcal{X}]$ has been denoted by $\tilde{\zeta}(\mathcal{X})$ in [4].)

We say that a Banach space $\mathcal{X}$ is *range complemented* if $\Gamma_R[\mathcal{X}] = \mathcal{B}[\mathcal{X}]$, *kernel complemented* if $\Gamma_N[\mathcal{X}] = \mathcal{B}[\mathcal{X}]$, and *range-kernel complemented* if $\Gamma[\mathcal{X}] = \mathcal{B}[\mathcal{X}]$ (i.e., if $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}] = \mathcal{B}[\mathcal{X}]$). Hilbert spaces are complemented, and consequently they are range-kernel complemented. Actually, if a Banach space $\mathcal{X}$ is complemented (i.e., if $\mathcal{X}$ is essentially a Hilbert space), then it is trivially range-kernel complemented. Is the converse true? However, for an arbitrary Banach space $\mathcal{X}$, the set $\Gamma[\mathcal{X}] = \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]$ is algebraically and topologically large in the sense that it includes nonempty open groups. Indeed, the group $\mathcal{G}[\mathcal{X}]$ of all invertible operators from $\mathcal{B}[\mathcal{X}]$ (i.e., of all operators from $\mathcal{B}[\mathcal{X}]$ with a bounded inverse) is open in $\mathcal{B}[\mathcal{X}]$ and is included in $\Gamma[\mathcal{X}]$. In fact, as we shall see later in Section 3, $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ include open regularities, namely the lower and upper semi-Fredholm operators $\Phi_-[\mathcal{X}]$ and $\Phi_+[\mathcal{X}]$, which include $\mathcal{G}[\mathcal{X}]$.

Theorem 2.1. *Consider operators on a Banach space $\mathcal{X}$.*

(a) *Every finite-rank operator lies in $\Gamma_R[\mathcal{X}]$, and every operator with finite-dimensional kernel lies in $\Gamma_N[\mathcal{X}]$.*

(b) *An operator whose range has a finite codimension lies in $\Gamma_R[\mathcal{X}]$; an operator whose kernel has a finite codimension lies in $\Gamma_N[\mathcal{X}]$.*

(c) *Every invertible operator has closed and complemented range and kernel, so that $\mathcal{G}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}] = \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]$, and $\mathcal{G}[\mathcal{X}]$ is open in $\mathcal{B}[\mathcal{X}]$.*

(d) *If $T \in \Gamma_R[\mathcal{X}]$ is bounded below, then $\mathcal{R}(T+K)$ is closed and $T+K \in \Gamma[\mathcal{X}] = \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]$ for every compact $K \in \mathcal{B}[\mathcal{X}]$.*

(e) *If $T \in \mathcal{G}[\mathcal{X}]$, then $\mathcal{R}(T+K)$ is closed and $T+K \in \Gamma[\mathcal{X}] = \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]$ for every compact $K \in \mathcal{B}[\mathcal{X}]$.*

(f) *If T is a compact operator on a reflexive Banach space $\mathcal{X}$ with a Schauder basis, then $T \in \Gamma_R[\mathcal{X}]$ and its adjoint $T^* \in \Gamma_R[\mathcal{X}^*]$.*

Proof. (a,b) Clear by Proposition 1.1 and Corollary 1.1.

(c) If $T \in \mathcal{G}[\mathcal{X}]$, then $\mathcal{R}(T) = \mathcal{X}$ and $\mathcal{N}(T) = \{0\}$. Moreover, $\mathcal{G}[X]$ is open in the uniform topology of $\mathcal{B}[\mathcal{X}]$ (see, e.g., [11, Problem 4.48]).

(d) As for the range issue (i.e., $T+K \in \Gamma_R[\mathcal{X}]$), see [9, Theorem 2]. As for the kernel issue (i.e., $T+K \in \Gamma_N[\mathcal{X}]$), this is straightforward by the forthcoming Section 3 (which is independent of Theorem 2.1), since bounded below are injective with closed range (see e.g., [11, Corollary 4.24]) and so lie in $\mathcal{F}_\ell[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}]$ (Proposition 3.1), and $\mathcal{F}_\ell[\mathcal{X}]$ is invariant under compact perturbation by its very definition (Definition 3.2), since the compact operators comprise an ideal of $\mathcal{B}[\mathcal{X}]$.

(e) Particular case of (d). (Also see [9, Proposition 1].)

(f) If T is finite-rank, then the result follows from item (a). Thus suppose $T \in \mathcal{B}[\mathcal{X}]$ is not finite-rank but compact, and $\mathcal{X}$ has a Schauder basis, so that there exists a sequence $\{T_n\}$ of finite-rank operators $T_n \in \mathcal{B}[\mathcal{X}]$ such that $\{T_n\}$ converges uniformly (thus strongly) to T (see, e.g., [11, Problem 4.58]) and $\mathcal{R}(T_n) \subseteq \mathcal{R}(T)^-$. Since each T_n is finite-rank, it follows that $\mathcal{R}(T_n) = \mathcal{R}(T_n)^-$ and, from item (a), $T_n \in \Gamma_R[\mathcal{X}]$, which means that there exist continuous projections $E_n \in \mathcal{B}[\mathcal{X}]$ and $I - E_n \in \mathcal{B}[\mathcal{X}]$ such that $\mathcal{R}(E_n) = \mathcal{N}(I - E_n) = \mathcal{R}(T_n)$ (cf. Remark 1.1) and $\mathcal{R}(I - E_n) = \mathcal{N}(E_n)$ with $\{\mathcal{R}(E_n)\}$ is increasing. Since $\{\mathcal{R}(E_n)\}$ is monotone, $\lim_n \mathcal{R}(E_n)$ exists in the sense that $\lim_n \mathcal{R}(E_n) = \bigcap_{n\geq 1}\bigvee_{k\geq n} \mathcal{R}(E_k) = \big(\bigcup_{n\geq 1}\bigcap_{k\geq n} \mathcal{R}(E_k)\big)^-$ (cf. [4, Definition 1]), so that $\lim_n \mathcal{R}(I - E_n)$ also exists. Set $\mathcal{R} = \lim_n \mathcal{R}(E_n)$ so that $\mathcal{R} \subseteq \mathcal{R}(T)^-$ (since $\mathcal{R}(T_n) \subseteq \mathcal{R}(T)^-$). Moreover, since $\mathcal{X}$ has a Schauder basis and T is compact, it follows that the sequence $\{E_n\}$ converges strongly (see, e.g., [11, Hint to Problem 4.58], and hence $\{E_n\}$ is bounded. Thus, since $\mathcal{X}$ is reflexive, [4, Theorem 2] ensures that $T \in \Gamma_R[\mathcal{X}]$. Since reflexivity and compactness are $*$-preserved [5, Theorems V.4.2, VI.3.4], the range of the compact T^* on $\mathcal{X}^*$ is complemented. ■

3 Fredholm Operators

Definition 3.1. (See, e.g., [16, Definition 16.1]). Let $\mathcal{X}$ be a Banach space.

$$\Phi_+[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}] \colon\ \mathcal{R}(T) \text{ is closed and } \dim \mathcal{N}(T) < \infty\big\}$$

is the class of *upper semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$,

$$\Phi_-[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}] \colon\ \operatorname{codim} \mathcal{R}(T) < \infty\big\}$$

is the class of *lower semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$, and

$$\Phi[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}]$$

is the class of *Fredholm* operators from $\mathcal{B}[\mathcal{X}]$. Since $\operatorname{codim} \mathcal{R}(T) = \dim \mathcal{X}/\mathcal{R}(T)$, and since $\operatorname{codim} \mathcal{R}(T) < \infty$ implies that $\mathcal{R}(T)$ is closed (Corollary 1.1), the class of lower semi-Fredholm operators can be written as

$$\Phi_-[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}] \colon\ \mathcal{R}(T) \text{ is closed and } \dim \mathcal{X}/\mathcal{R}(T) < \infty\big\},$$

and so the class of Fredholm operators can be written as

$$\begin{aligned}\Phi[\mathcal{X}] &= \big\{T \in \mathcal{B}[\mathcal{X}] \colon\ \dim \mathcal{N}(T) < \infty \text{ and } \operatorname{codim} \mathcal{R}(T) < \infty\big\}\\ &= \big\{T \in \mathcal{B}[\mathcal{X}] \colon\ \mathcal{R}(T) \text{ is closed},\ \dim \mathcal{N}(T) < \infty,\ \dim \mathcal{X}/\mathcal{R}(T) < \infty\big\}.\end{aligned}$$

The classes of operators $\Phi_+[\mathcal{X}]$ and $\Phi_-[\mathcal{X}]$ are open in $\mathcal{B}[\mathcal{X}]$ (see, e.g., [16, Proposition 16.11]), and the sets $\Phi_+[\mathcal{X}]\backslash\Phi_-[\mathcal{X}]$ and $\Phi_-[\mathcal{X}]\backslash\Phi_+[\mathcal{X}]$ are closed in $\mathcal{B}[\mathcal{X}]$ (see, e.g., [16, Proof of Corollary 18.2]).

Definition 3.2. (See, e.g., [12, Section 5.1]). Let $\mathcal{X}$ be a Banach space.

$$\begin{aligned}\mathcal{F}_\ell[\mathcal{X}] &= \{T \in \mathcal{B}[\mathcal{X}] : T \text{ is left essentially invertible}\}\\ &= \{T \in \mathcal{B}[\mathcal{X}] : AT = I + K \text{ for some } A \in \mathcal{B}[\mathcal{X}] \text{ and some compact } K \in \mathcal{B}[\mathcal{X}]\}\end{aligned}$$

is the class of *left semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$,

$$\begin{aligned}\mathcal{F}_r[\mathcal{X}] &= \{T \in \mathcal{B}[\mathcal{X}] : T \text{ is right essentially invertible}\}\\ &= \{T \in \mathcal{B}[\mathcal{X}] : TA = I + K \text{ for some } A \in \mathcal{B}[\mathcal{X}] \text{ and some compact } K \in \mathcal{B}[\mathcal{X}]\}\end{aligned}$$

is the class of *right semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$, and

$$\mathcal{F}[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}] \cap \mathcal{F}_r[\mathcal{X}] = \{T \in \mathcal{B}[\mathcal{X}] : T \text{ is essentially invertible}\},$$

the class of *Fredholm* operators from $\mathcal{B}[\mathcal{X}]$. The definitions of $\mathcal{F}_\ell[\mathcal{X}]$ and $\mathcal{F}_r[\mathcal{X}]$ can be equivalently stated if "compact" is replaced with "finite-rank" (see e.g., [3, Remark 3.3.3] or [16, Theorems 16.14 and 16.15]).

The classes $\mathcal{F}_\ell[\mathcal{X}]$ and $\mathcal{F}_r[\mathcal{X}]$ are open in $\mathcal{B}[\mathcal{X}]$, since they are the inverse images under the natural map $\pi : \mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}]/\mathcal{B}_\infty[\mathcal{X}]$ (which is continuous) of the left and right invertible elements, respectively, in the Calkin algebra $\mathcal{B}[\mathcal{X}]/\mathcal{B}_\infty[\mathcal{X}]$ of $\mathcal{B}[\mathcal{X}]$ modulo the ideal $\mathcal{B}_\infty[\mathcal{X}]$ of compact operators (see e.g., [5, Proposition XI.2.6]).

Left and right and upper and lower semi-Fredholm operators are linked by range and kernel complementation: $T \in \mathcal{F}_\ell[\mathcal{X}]$ if and only if $T \in \Phi_+[\mathcal{X}]$ and $\mathcal{R}(T)$ is complemented, and $T \in \mathcal{F}_r[\mathcal{X}]$ if and only if $T \in \Phi_-[\mathcal{X}]$ and $\mathcal{N}(T)$ is complemented.

Proposition 3.1. *Left-right and upper-lower semi-Fredholm are related as follows.*

$$\mathcal{F}_\ell[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}] = \{T \in \Phi_+[\mathcal{X}] :\ \mathcal{R}(T) \textit{ is a complemented subspace}\},$$

$$\mathcal{F}_r[\mathcal{X}] = \Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] = \{T \in \Phi_-[\mathcal{X}] :\ \mathcal{N}(T) \textit{ is a complemented subspace}\}.$$

Proof. [16, Theorems 16.14, 16.15] (since $\mathcal{R}(T)^- = \mathcal{R}(T)$ if $T \in \Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]$). ■

Question 3.1. Are the sets $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ open in $\mathcal{B}[\mathcal{X}]$?

In view of Proposition 3.1, operators in $\mathcal{F}_\ell[\mathcal{X}]$ and $\mathcal{F}_r[\mathcal{X}]$ are also referred to as *Atkinson operators* [7, Theorem 2.3] (left and right respectively).

Lemma 3.1. *The classes of operators in Definitions* 2.1, 3.1, *and* 3.2 *satisfy the following relations.*

(a) $\Phi_+[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}]$ *and* $\Phi_-[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}]$, *and so*

$\Phi[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$ *and* $\Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}] \cup \Gamma_N[\mathcal{X}]$;

(b) $\mathcal{F}_\ell[\mathcal{X}]\backslash\mathcal{F}_r[\mathcal{X}] = (\Phi_+[\mathcal{X}]\backslash\Phi_-[\mathcal{X}]) \cap \Gamma_R[\mathcal{X}] = (\Phi_+[\mathcal{X}]\backslash\Phi_-[\mathcal{X}]) \cap \Gamma[\mathcal{X}]$, *and*

$\mathcal{F}_r[\mathcal{X}]\backslash\mathcal{F}_\ell[\mathcal{X}] = (\Phi_-[\mathcal{X}]\backslash\Phi_+[\mathcal{X}]) \cap \Gamma_N[\mathcal{X}] = (\Phi_-[\mathcal{X}]\backslash\Phi_+[\mathcal{X}]) \cap \Gamma[\mathcal{X}]$;

(c) $\mathcal{F}_\ell[\mathcal{X}] \cap \mathcal{F}_r[\mathcal{X}] = \mathcal{F}[\mathcal{X}] = \Phi[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}]$;

(d) $\mathcal{F}_\ell[\mathcal{X}] \cup \mathcal{F}_r[\mathcal{X}] = (\Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]) \cap \Gamma[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$;

for every Banach space $\mathcal{X}$. *Now take a Banach space* $\mathcal{X}$.

(e) *If* $\Phi_+[\mathcal{X}] = \Phi_-[\mathcal{X}]$, *then* $\Phi_+[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}]$ *and* $\Phi_-[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$,

so that $\mathcal{F}_\ell[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$.

(f) *If* $\mathcal{F}_\ell[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$, *then* $\Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}] = \Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] =$

$\Phi_+[\mathcal{X}] \cap \Gamma[\mathcal{X}] = \Phi_-[\mathcal{X}] \cap \Gamma[\mathcal{X}] =$

$\Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}] = (\Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]) \cap \Gamma[\mathcal{X}]$;

and also $(\Phi_+[\mathcal{X}]\backslash\Phi_-[\mathcal{X}]) \cap \Gamma_R[\mathcal{X}] = (\Phi_-[\mathcal{X}]\backslash\Phi_+[\mathcal{X}]) \cap \Gamma_N[\mathcal{X}] = \varnothing$.

(g) *If* $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}]$, *then* $\Phi_+[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}]$ *and* $\Phi_-[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$.

Moreover,

(h) $\Phi_+[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}]$ *and* $\Phi_-[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$ *if and only if* $\Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$.

Proof. The proof is elementary, mostly based on standard set-theoretical relations.

(a) If $T \in \Phi_+[\mathcal{X}]$, then the subspace $\mathcal{N}(T)$ is finite-dimensional, and hence it is complemented by Proposition 1.1. Thus $\Phi_+[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}]$. On the other hand, if $T \in \Phi_-[\mathcal{X}]$, then $\mathcal{R}(T)$ is closed and codim $\mathcal{R}(T) < \infty$, so that the subspace $\mathcal{R}(T)$ is trivially complemented (codim $\mathcal{R}(T) < \infty$ implies that every algebraic complement of $\mathcal{R}(T)$ is finite-dimensional, thus closed). Hence $\Phi_-[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}]$.

(b) By Proposition 3.1 we get $\mathcal{F}_\ell\mathcal{X}]\backslash\mathcal{F}_r[\mathcal{X}] = (\Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}])\backslash(\Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}])$ and $\mathcal{F}_r\mathcal{X}]\backslash\mathcal{F}_\ell[\mathcal{X}] = (\Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}])\backslash(\Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}])$. By item (a) it follows that $\Phi_+[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}]$ and $\Phi_-[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}]$. So $\mathcal{F}_\ell[\mathcal{X}]\backslash\mathcal{F}_r[\mathcal{X}] = (\Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}])\backslash \Phi_-[\mathcal{X}] = (\Phi_+[\mathcal{X}]\backslash\Phi_-[\mathcal{X}]) \cap \Gamma_R[\mathcal{X}]$. Similarly, $\mathcal{F}_r[\mathcal{X}]\backslash\mathcal{F}_\ell[\mathcal{X}] = (\Phi_-[\mathcal{X}]\backslash\Phi_+[\mathcal{X}]) \cap \Gamma_N[\mathcal{X}]$.

(c) According to Definition 3.1 (where $\mathcal{R}(T)$ must be closed for operators T in $\Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]$) and item (a), if $T \in \Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}]$, then $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are complemented subspaces, which implies by Proposition 3.1 that $\Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}] \subseteq \mathcal{F}_\ell[\mathcal{X}] \cap \mathcal{F}_r[\mathcal{X}] \subseteq \Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}]$. Therefore, $\mathcal{F}_\ell[\mathcal{X}] \cap \mathcal{F}_r[\mathcal{X}] = \mathcal{F}[\mathcal{X}] = \Phi[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}]$ by Definitions 3.1 and 3.2.

(d) By Proposition 3.1, $\mathcal{F}_\ell[\mathcal{X}] \cup \mathcal{F}_r[\mathcal{X}] = (\Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}]) \cup (\Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]) = (\Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]) \cap (\Phi_+[\mathcal{X}] \cup \Gamma_N[\mathcal{X}]) \cap (\Gamma_R[\mathcal{X}] \cup \Phi_-[\mathcal{X}]) \cap (\Gamma_R[\mathcal{X}] \cup \Gamma_N[\mathcal{X}])$. But

$(\Phi_+[\mathcal{X}] \cup \Gamma_N[\mathcal{X}]) \cap (\Gamma_R[\mathcal{X}] \cup \Phi_-[\mathcal{X}]) \cap (\Gamma_R[\mathcal{X}] \cup \Gamma_N[\mathcal{X}]) = \Gamma[\mathcal{X}]$ according to item (a) and Definition 2.1. The inclusion is trivial.

(e) If $\Phi_+[\mathcal{X}] = \Phi_-[\mathcal{X}]$ then, by item(a), $\Phi_+[\mathcal{X}] = \Phi_-[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]$, which implies by Proposition 3.1 that $\Phi_+[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}]$ and $\Phi_-[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$, and so (since $\Phi_+[\mathcal{X}] = \Phi_-[\mathcal{X}]$) $\mathcal{F}_\ell[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$.

(f) If $\mathcal{F}_\ell[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$, then (by Definition 3.2) $\mathcal{F}[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}] \cap \mathcal{F}_r[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}] \cup \mathcal{F}_r[\mathcal{X}]$, and so (by Proposition 3.1 and items (c) and (d)) $\Phi[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}] = \Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] = (\Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]) \cap \Gamma[\mathcal{X}]$. But $\Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Gamma[\mathcal{X}]$ and $\Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] = \Phi_-[\mathcal{X}] \cap \Gamma[\mathcal{X}]$ by (a). Moreover, in this case, $\mathcal{F}_\ell[\mathcal{X}] \backslash \mathcal{F}_r[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}] \backslash \mathcal{F}_\ell[\mathcal{X}] = \varnothing$, thus apply item (b).

(g) If $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}]$, then $\Phi_+[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}] = \Gamma_R[\mathcal{X}]$ and $\Phi_-[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}]$ by item (a), which implies that $\mathcal{F}_\ell[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}] = \Phi_+[\mathcal{X}]$ and $\mathcal{F}_r[\mathcal{X}] = \Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] = \Phi_-[\mathcal{X}]$ by Proposition 3.1.

(h) If $\Phi_+[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}]$, then $\Phi_+[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}]$ (Proposition 3.1). Hence $\Phi_+[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}]$ and so $\Phi_+[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] = \Gamma[\mathcal{X}]$ (by item (a)). Similarly, If $\Phi_-[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$, then $\Phi_-[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$. Conversely, if $\Phi_+[\mathcal{X}] \cup \Phi_+[\mathcal{X}] \subseteq \Gamma[\mathcal{X}] = \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]$, then $\mathcal{F}_\ell[\mathcal{X}] = \Phi_+[\mathcal{X}]$ and $\mathcal{F}_r[\mathcal{X}] = \Phi_-[\mathcal{X}]$ (Proposition 3.1). ■

Remark 3.1. If $\Phi_+[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}]$ and $\Phi_-[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}]$ (i.e., if $\Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$ — in particular, if $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}]$, more particularly, if $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}] = \mathcal{B}[\mathcal{X}]$; even more particularly, if $\mathcal{X}$ is complemented), then (cf. Lemma 3.1(a,g,h))

$$\Phi_+[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}] \quad \text{and} \quad \Phi_-[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}].$$

Thus, in these cases, the classes $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ play no role in Proposition 3.1.

4 Essential Spectra

Definition 4.1. Corresponding to the classes $\Phi_+[\mathcal{X}]$, $\Phi_-[\mathcal{X}]$, and $\Phi[\mathcal{X}]$ there are the following spectra. Take $T \in \mathcal{B}[\mathcal{X}]$.

$$\begin{aligned}\sigma_{e_+}(T) = \sigma_{\pi e}(T) &= \big\{\lambda \in \mathbb{C} \colon \lambda I - T \notin \Phi_+[\mathcal{X}]\big\} \\ &= \big\{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \text{ is not closed or } \dim \mathcal{N}(\lambda I - T) = \infty\big\}\end{aligned}$$

is the *upper semi-Fredholm spectrum* or the *essential approximate point spectrum*,

$$\begin{aligned}\sigma_{e-}(T) = \sigma_{\delta e}(T) &= \big\{\lambda \in \mathbb{C} \colon\ \lambda I - T \notin \Phi_-[\mathcal{X}]\big\} \\ &= \big\{\lambda \in \mathbb{C} \colon\ \operatorname{codim} \mathcal{R}(\lambda I - T) = \infty\big\} \\ &= \big\{\lambda \in \mathbb{C} \colon\ \mathcal{R}(\lambda I - T) \text{ is not closed or } \dim \mathcal{X}/\mathcal{R}(\lambda I - T) = \infty\big\}\end{aligned}$$

is the *lower semi-Fredholm spectrum* or the *essential surjective spectrum*, and

$$\begin{aligned}\sigma_e(T) &= \sigma_{e_+}(T) \cup \sigma_{e_-}(T) \\ &= \big\{\lambda \in \mathbb{C} \colon \lambda I - T \notin (\Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}])\big\} = \big\{\lambda \in \mathbb{C} \colon \lambda I - T \notin \Phi[\mathcal{X}]\big\}\end{aligned}$$

is the *Fredholm spectrum* or the *essential spectrum* (cf. Corollary 1.1).

The sets $\sigma_{e_+}(T)$, $\sigma_{e_-}(T)$, and $\sigma_e(T)$ are compact subsets of the spectrum $\sigma(T)$ (reason: the sets $\Phi_+[\mathcal{X}]$, $\Phi_-[\mathcal{X}]$, and $\Phi[\mathcal{X}]$ are open regularities in the Banach algebra $\mathcal{B}[\mathcal{X}]$ [16, Proposition 6.2 and Theorems 16.7 and 16.11]).

Definition 4.2. Corresponding to the classes $\mathcal{F}_\ell[\mathcal{X}]$, $\mathcal{F}_r[\mathcal{X}]$, and $\mathcal{F}[\mathcal{X}]$ there are the following spectra. Take $T \in \mathcal{B}[\mathcal{X}]$.

$$\begin{aligned}\sigma_{\ell e}(T) &= \big\{\lambda \in \mathbb{C}\colon\, \lambda I - T \notin \mathcal{F}_\ell[\mathcal{X}]\big\}\\ &= \big\{\lambda \in \mathbb{C}\colon\, \lambda I - T \text{ is not left essentially invertible}\big\}\end{aligned}$$

is the *left essential spectrum*,

$$\begin{aligned}\sigma_{re}(T) &= \big\{\lambda \in \mathbb{C}\colon\, \lambda I - T \notin \mathcal{F}_r[\mathcal{X}]\big\}\\ &= \big\{\lambda \in \mathbb{C}\colon\, \lambda I - T \text{ is not right essentially invertible}\big\}\end{aligned}$$

is the *right essential spectrum*, and

$$\begin{aligned}\sigma_e(T) &= \sigma_{\ell e}(T) \cup \sigma_{re}(T)\\ &= \big\{\lambda \in \mathbb{C}\colon \lambda I - T \notin (\mathcal{F}_\ell[\mathcal{X}] \cap \mathcal{F}_r[\mathcal{X}])\big\} = \big\{\lambda \in \mathbb{C}\colon \lambda I - T \notin \mathcal{F}[\mathcal{X}]\big\}\end{aligned}$$

is the *essential spectrum*.

The sets $\sigma_{\ell e}(T)$, $\sigma_{re}(T)$, and $\sigma_e(T)$ are compact subsets of the spectrum $\sigma(T)$ (reason: the sets $\mathcal{F}_\ell[\mathcal{X}]$ and $\mathcal{F}_r[\mathcal{X}]$ are open regularities in the Banach algebra $\mathcal{B}[\mathcal{X}]$). Moreover, $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ are the left and right spectra of the natural image $\pi(T)$ of T in the Calkin algebra $\mathcal{B}[\mathcal{X}]/\mathcal{B}_\infty[\mathcal{X}]$. See [16, p.52,53,160,172].

Definition 4.3. Corresponding to the classes $\Gamma_R[\mathcal{X}]$, $\Gamma_N[\mathcal{X}]$, and $\Gamma[\mathcal{X}]$ there are the following subsets of $\mathbb{C}$ (actually, subsets of $\sigma(T)$). Take $T \in \mathcal{B}[\mathcal{X}]$.

$$\begin{aligned}\zeta_R(T) &= \big\{\lambda \in \mathbb{C}\colon\, \lambda I - T \notin \Gamma_R[\mathcal{X}]\big\}\\ &= \big\{\lambda \in \mathbb{C}\colon\, \mathcal{R}(\lambda I - T)^- \text{ is not complemented}\big\},\end{aligned}$$

$$\begin{aligned}\zeta_N(T) &= \big\{\lambda \in \mathbb{C}\colon\, \lambda I - T \notin \Gamma_N[\mathcal{X}]\big\}\\ &= \big\{\lambda \in \mathbb{C}\colon\, \mathcal{N}(\lambda I - T) \text{ is not complemented}\big\},\end{aligned}$$

$$\begin{aligned}\zeta(T) &= \zeta_R(T) \cup \zeta_N(T)\\ &= \big\{\lambda \in \mathbb{C}\colon\, \lambda I - T \notin \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}]\big\} = \big\{\lambda \in \mathbb{C}\colon\, \lambda I - T \notin \Gamma[\mathcal{X}]\big\}\\ &= \big\{\lambda \in \mathbb{C}\colon\, \mathcal{R}(\lambda I - T)^- \text{ or } \mathcal{N}(\lambda I - T) \text{ is not complemented}\big\}.\end{aligned}$$

Proposition 4.1. *Left-right and upper-lower essential spectra are related as follows.*

$$\begin{aligned}\sigma_{\ell e}(T) &= \sigma_{e_+}(T) \cup \zeta_R(T)\\ &= \big\{\lambda \in \mathbb{C}\colon\, \mathcal{R}(\lambda I - T) \text{ is not closed or}, \dim \mathcal{N}(\lambda I - T) = \infty,\\ &\qquad \text{or } \mathcal{R}(\lambda I - T) \text{ is not a complemented subspace}\big\}.\end{aligned}$$

$$\begin{aligned}\sigma_{re}(T) &= \sigma_{e_-}(T) \cup \zeta_N(T)\\ &= \big\{\lambda \in \mathbb{C}\colon\, \mathcal{R}(\lambda I - T) \text{ is not closed, or} \dim \mathcal{X}/\mathcal{R}(\lambda I - T) = \infty;\\ &\qquad \text{or } \mathcal{N}(\lambda I - T) \text{ is not a complemented subspace}\big\}.\end{aligned}$$

Proof. Straightforward by Definition 4.2, Lemma 3.1(a), Definitions 4.3 and 3.2. ∎

Lemma 4.1. *The sets in Definitions* 4.1, 4.2, *and* 4.3 *satisfy the relations.*

(a) $\zeta_N(T) \subseteq \sigma_{e_+}(T)$ *and* $\zeta_R(T) \subseteq \sigma_{e_-}(T)$, *and so*
$\zeta(T) \subseteq \sigma_e(T)$ *and* $\zeta_R(T) \cap \zeta_N(T) \subseteq \sigma_{e_+}(T) \cap \sigma_{e_-}(T)$;

(b) $\sigma_{\ell e}(T)\backslash\sigma_{re}(T) = (\sigma_{e_+}(T)\backslash\sigma_{e_-}(T))\backslash\zeta_N(T) = (\sigma_{e_+}(T)\backslash\sigma_{e_-}(T))\backslash\zeta(T)$, *and*
$\sigma_{re}(T)\backslash\sigma_{\ell e}(T) = (\sigma_{e_-}(T)\backslash\sigma_{e_+}(T))\backslash\zeta_R(T) = (\sigma_{e_-}(T)\backslash\sigma_{e_+}(T))\backslash\zeta(T)$;

(c) $\sigma_e(T) = \sigma_{\ell e}(T) \cup \sigma_{re}(T) = \sigma_{e_+}(T) \cup \sigma_{e_-}(T)$;

(d) $\zeta(T) \subseteq (\sigma_{e_+}(T) \cap \sigma_{e_-}(T)) \cup \zeta(T) = \sigma_{\ell e}(T) \cap \sigma_{re}(T)$;

for every T on any Banach space $\mathcal{X}$. Now take T on a Banach space $\mathcal{X}$.

(e) *If* $\sigma_{e_+}(T) = \sigma_{e_-}(T)$, *then* $\sigma_{e_+}(T) = \sigma_{\ell e}(T)$ *and* $\sigma_{e_-}(T) = \sigma_{re}(T)$,
so that $\sigma_{\ell e}(T) = \sigma_{re}(T)$.

(f) *If* $\sigma_{\ell e}(T) = \sigma_{re}(T)$, *then* $\sigma_{e_+}(T) \cup \zeta_R(T) = \sigma_{e_-}(T) \cup \zeta_N(T) =$
$\sigma_{e_+}(T) \cup \zeta(T) = \sigma_{e_-}(T) \cup \zeta(T) =$
$\sigma_{e_+}(T) \cup \sigma_{e_-}(T) = \big(\sigma_{e_+}(T) \cap \sigma_{e_-}(T)\big) \cup \zeta(T)$;
and also $\sigma_{e_+}(T)\backslash\sigma_{e_-}(T)\backslash\zeta_N(T) = \sigma_{e_-}(T)\backslash\sigma_{e_+}(T)\backslash\zeta_R(T)$.

(g) *If* $\zeta_R(T) = \zeta_N(T)$, *then* $\sigma_{e_+}(T) = \sigma_{\ell e}(T)$ *and* $\sigma_{e_-}(T) = \sigma_{re}(T)$.
Moreover,

(h) $\sigma_{e_+}(T) = \sigma_{\ell e}(T)$ *and* $\sigma_{e_-}(T) = \sigma_{re}(T)$ *if and only if* $\zeta(T) \subseteq \sigma_{e_+}(T) \cap \sigma_{e_-}(T)$.

Proof. This is the dual of Lemma 3.1, from item (a) to item (h), respectively. ∎

Theorem 4.1. *Take $T \in \mathcal{B}[\mathcal{X}]$ on a Banach space $\mathcal{X}$.*

(a) *If $\Gamma_R[\mathcal{X}]$ is open (closed) in $\mathcal{B}[\mathcal{X}]$, then $\zeta_R(T)$ is closed (open) in $\mathbb{C}$. If $\Gamma_N[\mathcal{X}]$ is open (closed) in $\mathcal{B}[\mathcal{X}]$, then $\zeta_N(T)$ is closed (open) in $\mathbb{C}$.*

(b) *If $\zeta_R(T)$ is open in $\mathbb{C}$, then $\sigma_{e_+}(T) = \sigma_{\ell e}(T)$. If $\zeta_N(T)$ is open in $\mathbb{C}$, then $\sigma_{e_-}(T) = \sigma_{re}(T)$.*

Proof. Let T be an operator on a Banach space $\mathcal{X}$.

(a) Readily verified by the complementary character between $\Gamma_R[\mathcal{X}]$ and $\zeta_R(T)$, and between $\Gamma_N[\mathcal{X}]$ and $\zeta_N(T)$.

(b) By Proposition 4.1, $\sigma_{\ell e}(T) = \sigma_{e_+}(T) \cup \zeta_R(T) = \sigma_{e_+}(T) \cup (\zeta_R(T)\backslash\sigma_{e_+}(T))$, where $\{\sigma_{e_+}(T), \zeta_R(T)\backslash\sigma_{e_+}(T)\}$ is a partition of $\sigma_{\ell e}(T)$. Suppose $\sigma_{\ell e}(T) \neq \sigma_{e_+}(T)$, which means that $\zeta_R(T) \not\subseteq \sigma_{e_+}(T)$; equivalently, $\zeta_R(T)\backslash\sigma_{e_+}(T) \neq \varnothing$. Since $\sigma_{\ell e}(T)$

and $\sigma_{e_+}(T)$ are closed subsets of $\mathbb{C}$, it follows that $\zeta_R(T)\backslash\sigma_{e_+}(T)$ is not open. If $\zeta_R(T)$ is open, then $\zeta_R(T)\backslash\sigma_{e_+}(T) = \zeta_R(T) \cap (\mathbb{C}\backslash\sigma_{e_+}(T))$ is open (because $\sigma_{e_+}(T)$ is closed), which is a contradiction. Thus $\zeta_R(T)$ is not open. Similarly, since $\sigma_{re}(T) = \sigma_{e_-}(T) \cup \zeta_N(T)$ (Proposition 4.1), and since $\sigma_{re}(T)$ and $\sigma_{e_-}(T)$ are closed sets in $\mathbb{C}$, the same argument ensures that if $\sigma_{re}(T) \neq \sigma_{e_-}(T)$, then $\zeta_N(T)$ is not open. ■

Remark 4.1. Consider Lemma 4.1(a,g,h). If $\zeta_R(T) \subseteq \sigma_{e_+}(T)$ and $\zeta_N(T) \subseteq \sigma_{e_-}(T)$ (i.e., if $\zeta(T) \subseteq \sigma_{e_+}(T) \cap \sigma_{e_+}(T)$ — in particular, if $\zeta_R(T) = \zeta_N(T)$; more particularly, if $\zeta_R(T) = \zeta_N(T) = \varnothing$; even more particularly, if $\mathcal{X}$ is complemented), then

$$\sigma_{e_+}(T) = \sigma_{\ell e}(T) \quad \text{and} \quad \sigma_{e_-}(T) = \sigma_{re}(T).$$

Thus, in these cases, the sets $\zeta_R(T)$ and $\zeta_N(T)$ play no role in Proposition 4.1.

Remark 4.2. If $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ are closed, then $\zeta_R(T)$ and $\zeta_N(T)$ are open. So

$$\sigma_{e_+}(T) = \sigma_{\ell e}(T) \quad \text{and} \quad \sigma_{e_-}(T) = \sigma_{re}(T)$$

(Theorem 4.1). There may be sequences of operators T_n in $\Gamma_R[\ell^p]$ converging uniformly to an operator T not in $\Gamma_R[\ell^p]$ for $1 < p \neq 2$ [4, Example 1]) and so $\Gamma_R[\mathcal{X}]$ may not be closed. On the other hand, $\Gamma_R[\mathcal{X}]$ and $\Gamma_N[\mathcal{X}]$ include nonempty open regularities (e.g., $\Phi_+[\mathcal{X}]$ and $\Phi_-[\mathcal{X}]$, which include the open group $\mathcal{G}[\mathcal{X}]$).

5 Holes and Pseudo Holes of the Essential Spectrum

Let $\mathbb{Z}$ denote the set of all integers, and let $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ be the set of all extended integers. If $T \in \Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]$, then $\operatorname{ind}(T) = \dim \mathcal{N}(T) - \operatorname{codim} \mathcal{R}(T)$ is an element of $\overline{\mathbb{Z}}$. This is the index of $T \in \Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]$. If $T \in \Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]$ and $\operatorname{ind}(T)$ is finite, then $T \in \Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}] = \Phi[\mathcal{X}] = \mathcal{F}[\mathcal{X}]$. Set

$$\begin{aligned}\sigma_w(T) &= \big\{\lambda \in \mathbb{C} \colon\ \lambda I - T \text{ is not a Fredholm operator of index zero}\big\}\\ &= \big\{\lambda \in \mathbb{C} \colon\ \lambda \in \sigma_e(T) \text{ or } \lambda I - T \in \Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}] \text{ and } \operatorname{ind}(\lambda I - T) \neq 0\big\},\end{aligned}$$

the Weyl spectrum of $T \in \mathcal{B}[\mathcal{X}]$. Recall that $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma(T)$. Let $\sigma_0(T)$ denote the complement of $\sigma_w(T)$ in $\sigma(T)$,

$$\begin{aligned}\sigma_0(T) &= \sigma(T)\backslash\sigma_w(T)\\ &= \big\{\lambda \in \sigma(T) \colon\ \lambda \notin \sigma_e(T) \text{ and } \operatorname{ind}(\lambda I - T) = 0\big\}\\ &= \big\{\lambda \in \sigma(T) \colon\ \lambda I - T \text{ is a Fredholm operator of index zero}\big\}.\end{aligned}$$

Moreover, for each nonzero extended integer $k \in \overline{\mathbb{Z}}\backslash\{0\}$ set

$$\sigma_k(T) = \big\{\lambda \in \mathbb{C} \colon\ \lambda I - T \in \Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}] \text{ and } \operatorname{ind}(\lambda I - T) = k \neq 0\big\},$$

which are open subsets of $\sigma(T)$. For every nonzero (finite) integer $k \in \mathbb{Z}\backslash\{0\}$ the sets $\sigma_k(T)$ are the holes of $\sigma_{e_+}(T) \cap \sigma_{e_-}(T)$, and therefore they are the holes of the essential spectrum $\sigma_e(T)$, since

$$\sigma_{+\infty}(T) = \sigma_{e_+}(T)\backslash\sigma_{e_-}(T) \quad \text{and} \quad \sigma_{-\infty}(T) = \sigma_{e_-}(T)\backslash\sigma_{e_+}(T)$$

are the holes of $\sigma_{e_-}(T)$ and $\sigma_{e_+}(T)$ in $\sigma_e(T)$, respectively. The sets $\sigma_{+\infty}(T)$ and $\sigma_{-\infty}(T)$ are the pseudoholes of $\sigma_e(T)$ — which are not holes of $\sigma_e(T)$ (see e.g., [17, p.3], [10, p.162], and [12, p.147]). Summing up, $\{\sigma_k(T)\}_{k\in\mathbb{Z}\setminus\{0\}}$ is the collection of all holes of the essential spectrum $\sigma_e(T)$, and $\{\sigma_{-\infty}(T), \sigma_{+\infty}(T)\}$ is the collection of pseudoholes of $\sigma_e(T)$, all of them open subsets of the spectrum $\sigma(T)$.

Remark 5.1. At this point a word on notation is in order. Here we have defined $\sigma_{e_+}(T)$ and $\sigma_{e_-}(T)$ coincidently with what has been defined as $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ in [1], [2], [5], [10], [12], [13], [14], and [17] — the same happening with respect to $\Phi_+[\mathcal{X}]$, $\Phi_-[\mathcal{X}]$ and $\mathcal{F}_\ell[\mathcal{X}]$, $\mathcal{F}_r[\mathcal{X}]$ — since there (up to [14]) they work on a Hilbert space, which is complemented, where these sets indeed coincide (i.e., in that case, $\Gamma[\mathcal{X}] = \mathcal{B}[\mathcal{X}]$, which implies that $\Phi_+[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}]$ and $\Phi_-[\mathcal{X}] = \mathcal{F}_r[\mathcal{X}]$ — thus $\zeta(T) = \varnothing$, which in turn implies that $\sigma_{e_+}(T) = \sigma_{\ell e}(T)$ and $\sigma_{e_-}(T) = \sigma_{re}(T)$).

Keeping Remark 5.1 in mind, recall that

$$\sigma(T) = \sigma_e(T) \cup \bigcup\nolimits_{k\in\mathbb{Z}} \sigma_k(T),$$

$$\sigma_e(T) = \big(\sigma_{e_+}(T) \cap \sigma_{e_-}(T)\big) \cup \sigma_{+\infty}(T) \cup \sigma_{-\infty}(T),$$

(this partition of $\sigma(T)$ is the Spectral Picture) so that (Schechter Theorem)

$$\sigma_w(T) = \sigma(T)\backslash\sigma_0(T) = \sigma_e(T) \cup \bigcup\nolimits_{k\in\mathbb{Z}\setminus\{0\}} \sigma_k(T) :$$

the Weyl spectrum is the union of the essential spectrum and all its holes [5, Proposition XI.6.10 and Theorem XI.6.12] or [12, Corollary 5.18 and Theorem 5.24]. Thus,

$\sigma_e(T)$ *has no holes if and only if* $\sigma_e(T) = \sigma_w(T)$.

Theorem 5.1. *The essential spectrum has no pseudoholes if and only if the upper and lower semi-Fredholm spectra coincide, which implies that the right and left essential spectra coincide; that is,*

(i) $\quad \sigma_{+\infty}(T) = \sigma_{-\infty}(T) = \varnothing \iff \sigma_{e_+}(T) = \sigma_{e_-}(T) \implies \sigma_{\ell e}(T) = \sigma_{re}(T).$

Conversely,

(ii) $\sigma_{\ell e}(T) = \sigma_{re}(T) \implies \sigma_{+\infty}(T) = \zeta_N(T)\backslash\sigma_{e_-}(T)$ *and* $\sigma_{-\infty}(T) = \zeta_R(T)\backslash\sigma_{e_+}(T)$.

Special cases. Suppose $\sigma_{\ell e}(T) = \sigma_{re}(T)$.

(ii′) *If* $\zeta_R(T)$ *and* $\zeta_N(T)$ *are closed, then* $\sigma_{+\infty}(T) = \sigma_{-\infty}(T) = \varnothing$.

(ii″) *If* $\zeta_R(T) \cup \zeta_N(T) \subseteq \sigma_{e_+}(T) \cap \sigma_{e_-}(T)$ *(in particular, if* $\zeta_R(T) = \zeta_N(T)$*),*
then $\sigma_{+\infty}(T) = \sigma_{-\infty}(T) = \varnothing$.

(ii‴) *If* $\zeta_R(T) \cap \sigma_{e_+}(T) = \varnothing$ *and* $\zeta_N(T) \cap \sigma_{e_-}(T) = \varnothing$,
then $\sigma_{+\infty}(T) = \zeta_N(T)$ *and* $\sigma_{-\infty}(T) = \zeta_R(T)$.

Proof. Let $\sigma_{+\infty}(T) = \sigma_{e_+}(T)\backslash\sigma_{e_-}(T)$ and $\sigma_{-\infty}(T) = \sigma_{e_-}(T)\backslash\sigma_{e_+}(T)$ be the pseudoholes of $\sigma_e(T)$, and consider the following assertions.

(a) $\sigma_{+\infty}(T) = \varnothing$ and $\sigma_{-\infty}(T) = \varnothing$ (i.e., $\sigma_e(T)$ has no pseudoholes).

(b) $\sigma_{e_+}(T) = \sigma_{e_-}(T)$.

(c) $\sigma_{\ell e}(T) = \sigma_{re}(T)$.

(d) $\sigma_{+\infty}(T) = \zeta_N(T)\backslash\sigma_{e_-}(T)$ and $\sigma_{-\infty}(T) = \zeta_R(T)\backslash\sigma_{e_+}(T)$.

(i) The very definition of $\sigma_{+\infty}(T)$ and $\sigma_{-\infty}(T)$ ensures that (a) and (b) are equivalent, and from Lemma 4.1(e) it follows that (b) implies (c). Thus (i) holds.

(ii) If $\sigma_{\ell e}(T) = \sigma_{re}(T)$, then $\sigma_{\ell e}(T)\backslash\sigma_{re}(T) = \sigma_{re}(T)\backslash\sigma_{\ell e}(T) = \varnothing$. By Lemma 4.1(b) we get $\sigma_{e_+}(T)\backslash\sigma_{e_-}(T) \subseteq \zeta_N(T)$ and $\sigma_{e_-}(T)\backslash\sigma_{e_+}(T) \subseteq \zeta_R(T)$. By Lemma 4.1(a), $\sigma_{e_+}(T)\backslash\sigma_{e_-}(T) \subseteq \zeta_N(T)\backslash\sigma_{e_-}(T) \subseteq \sigma_{e_+}(T)\backslash\sigma_{e_-}(T)$ and $\sigma_{e_-}(T)\backslash\sigma_{e_+}(T) \subseteq \zeta_R(T)\backslash\sigma_{e_+}(T) \subseteq \sigma_{e_-}(T)\backslash\sigma_{e_+}(T)$. Thus (c) implies (d), and so (ii) holds.

(ii′) Suppose $\sigma_{\ell e}(T) = \sigma_{re}(T)$ so that $\sigma_{+\infty}(T) = \zeta_N(T)\backslash\sigma_{e_-}(T)$ by (ii). Suppose $\zeta_N(T)$ is closed in $\mathbb{C}$. Recall that $\sigma_{+\infty}(T)$ is bounded. Assume that $\sigma_{+\infty}(T) \neq \varnothing$. If $\zeta_N(T) \cap \sigma_{e_-}(T) \neq \varnothing$, then $\sigma_{+\infty}(T) = \zeta_N(T)\backslash\sigma_{e_-}(T) \subseteq \zeta_N(T)$ is not open (because $\zeta_N(T)$ and $\sigma_{e_-}(T)$ are closed), which is a contradiction (since $\sigma_{+\infty}(T)$ is open). If $\zeta_N(T) \cap \sigma_{e_-}(T) = \varnothing$, then $\sigma_{+\infty}(T) = \zeta_N(T)$, which is another contradiction (since $\sigma_{+\infty}(T)$ is open and bounded and $\zeta_N(T)$ is closed). Thus $\sigma_{+\infty}(T) = \varnothing$. Outcome: $\sigma_{\ell e}(T) = \sigma_{re}(T)$ and $\zeta_N(T)$ closed imply $\sigma_{+\infty}(T) = \varnothing$. Similarly (same argument), $\sigma_{\ell e}(T) = \sigma_{re}(T)$ and $\zeta_R(T)$ closed imply $\sigma_{-\infty}(T) = \varnothing$.

(ii″) If $\zeta_R(T) = \zeta_N(T)$, then $\zeta_R(T) \cup \zeta_N(T) \subseteq \sigma_{e_+}(T) \cap \sigma_{e_-}(T)$ according to Lemma 4.1(a). If $\sigma_{\ell e}(T) = \sigma_{re}(T)$ and $\zeta_R(T) \cup \zeta_N(T) \subseteq \sigma_{e_+}(T) \cap \sigma_{e_-}(T)$, then $\sigma_{+\infty}(T) = \zeta_N(T)\backslash\sigma_{e_-}(T) = \varnothing$ and $\sigma_{-\infty}(T) = \zeta_R(T)\backslash\sigma_{e_+}(T) = \varnothing$ from (ii).

(ii‴) On the other hand, if $\zeta_R(T) \cap \sigma_{e_+}(T) = \zeta_N(T) \cap \sigma_{e_-}(T) = \varnothing$ (in addition to $\sigma_{\ell e}(T) = \sigma_{re}(T)$), we get from (ii) that $\sigma_{+\infty}(T) = \zeta_N(T)$ and $\sigma_{-\infty}(T) = \zeta_R(T)$. ∎

Remark 5.2. The identity $\sigma_{e_+}(T) = \sigma_{e_-}(T)$ implies the identity $\sigma_{\ell e}(T) = \sigma_{re}(T)$ by Theorem 5.1(i). The reverse implication holds if $\zeta_R(T)$ and $\zeta_N(T)$ are open, or closed, or $\zeta_R(T) \cup \zeta_N(T) \subseteq \sigma_{e_+}(T) \cap \sigma_{e_-}(T)$ by Theorems 4.1(b) and 5.1(ii′,ii″), but the reverse implication might fail if it is possible that $\zeta_R(T)$ and $\zeta_N(T)$ are nonclosed, or nonopen, or $\zeta_R(T) \cup \zeta_N(T) \not\subseteq \sigma_{e_+}(T) \cap \sigma_{e_-}(T)$. For instance, take a compact nonempty subset $\mathbb{D}$ of $\mathbb{C}$ (e.g., the closed unit disk) and two proper subsets D_R and D_N of $\mathbb{D}$, not both closed nor both open, such that $(\mathbb{D}\backslash D_R)^- \neq (\mathbb{D}\backslash D_N)^-$. If it is possible that $\zeta_R(T) = D_R$ and $\zeta_N(T) = D_N$, and if $\sigma_{e_+}(T) = (\mathbb{D}\backslash D_R)^-$ and $\sigma_{e_-}(T) = (\mathbb{D}\backslash D_N)^-$, then $\sigma_{\ell e}(T) = \sigma_{re}(T) = \mathbb{D}$ and $\sigma_{e_+}(T) \neq \sigma_{e_-}(T)$.

6 An Application

An operator T on a (complex) infinite-dimensional *separable* Hilbert space is quasitriangular if there exists a sequence $\{E_n\}$ of finite-rank projections such that (i) $\{E_n\}$ converges strongly to the identity operator I and (ii) $\{(I - E_n)TE_n\}$ converges uniformly to the null operator [8, Section 2]. If both T and its adjoint T^* are quasitriangular, then T is biquasitriangular. Biquasitriangular operators were equivalently described in [1, Theorem 5.4], [2, Theorem 2.1] (also see [17, p.37]) as follows. An operator T is biquasitriangular if and only if

$$(1) \qquad \sigma_{\ell e}(T) = \sigma_{re}(T) = \sigma_e(T) = \sigma_w(T),$$

which in a Hilbert space setting means that $\sigma_e(T)$ has no holes and no pseudoholes [13, Section 4]. This immediately extends the definition of biquasitriangularity from separable to general Hilbert spaces. Note that, since the above result was worked out in a Hilbert space setting (where $\sigma_{e_+}(T) = \sigma_{\ell e}(T)$ and $\sigma_{e_-}(T) = \sigma_{re}(T)$ — see Remark 4.1), it actually means that a Hilbert-space operator T is biquasitriangular if and only if

$$(1') \qquad \sigma_{e_+}(T) = \sigma_{e_-}(T) = \sigma_e(T) = \sigma_w(T),$$

and this can be naturally extended to an arbitrary (complex) Banach space. However, as we saw in the preceding section (Spectral Picture and Schechter Theorem) the identities in (1′) are equivalent to saying that T has no holes or pseudoholes. Therefore, an operator T is biquasitriangular if and only if

$$(1'') \qquad \sigma_e(T) \text{ has no holes and no pseudoholes,}$$

where (1′) and (1″) are equivalent definitions of biquasitriangularity for operators T acting on any Banach space [14, Lemma 3.1(e)] and [6]. (Recall that here we have defined $\sigma_{e_+}(T)$ and $\sigma_{e_-}(T)$ coincidently with what has been defined as $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ in [1], [2], [5], [10], [12], [13], [14], and [17] — see Remark 5.1.)

Assertions (1′) and (1″) are always equivalent (i.e., (1′) and (1″) are equivalent in an arbitrary Banach space) by the very definition of holes and pseudoholes in Section 5. Assertions (1) and (1′) are equivalent in a Hilbert space. The next corollary summarizes when assertions (1) and (1′) are equivalent in a Banach space.

Corollary 6.1. *Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator acting on a Banach space $\mathcal{X}$. If either*

(a) *$\zeta_R(T)$ and $\zeta_N(T)$ are closed, or*

(b) *$\zeta_R(T)$ and $\zeta_N(T)$ are open, or*

(c) *$\zeta_R(T) \subseteq \sigma_{e_+}(T)$ and $\zeta_N(T) \subseteq \sigma_{e_-}(T)$ (in particular, if $\zeta_R(T) = \zeta_N(T)$),*

then (1) *and* (1′) *are equivalent; and each of* (a), (b), (c) *holds whenever each of*

(a′) *$\Gamma_R[\mathcal{X}]$ and $\Gamma_R[\mathcal{X}]$ are open,*

(b′) *$\Gamma_R[\mathcal{X}]$ and $\Gamma_R[\mathcal{X}]$ are closed,* (c′) $\Phi_-[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}]$ *and* $\Phi_+[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}]$ (*in particular, if* $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}]$)

holds, respectively.

Proof. Apply Theorems 4.1 and 5.1 (see Question 3.1). ∎

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