Boundaries and peak points in topological algebras

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Abstract

We examine the relation of the set of peak points with several boundaries defined in the spectrum of a given topological algebra relative to a subspace of it. In this respect, we show that the peak points are contained in the Choquet boundary and, under suitable conditions, are dense in the Šilov boundary. Furthermore, the set of points at issue coincide with the Bishop boundary if, and only if, it constitutes a weakly boundary set. On the other hand, in appropriate topological algebras, the Bishop, Choquet and strong boundaries coincide with the peak points, so that they are dense in the Šilov boundary. Finally there are topological algebras for which all the above boundaries and points remain invariant, under restriction of the Gel'fand transform algebras to subsets of the spectra of the topological algebras involved, containing the Šilov boundary.

1 Introduction

In this paper we employ the notion of a *peak subset* of the spectrum of a topological algebra, extending the relevant situation, as appeared, for instance, in *E.*

*Some of the main results extend relevant ones presented in the author's Doctoral Thesis under the supervision of Professor A. Mallios. I express my sincere thanks to the referee for careful reading of the manuscript and pertinent remarks.

Received by the editors in August 2015.

Communicated by E. Colebunders.

2010 *Mathematics Subject Classification :* 46H05, 46H99, 46J20.

Key words and phrases : Representing measure, Choquet point, Choquet boundary, Šilov boundary, peak set (resp. point), peaking function, weakly peak set (resp. point), strong boundary point, strong boundary, weakly spectral map, Weierstrass algebra, weakly boundary set, Bishop boundary, geometric hull, equicontinuous set, semi-simple algebra, Q-space, bornological space, perfectly normal space.

Bishop [4] and *H. G. Dales* [9] within the context of *"function algebras"* on a compact metrizable space. In this framework, one proves that the intersections of peak sets are sets of the same type, while the peak points are dense in the Šilov boundary; in effect, their set constitutes the *unique minimal boundary* called, *Bishop boundary* (ibid.).

Precisely, we generalize the above for an arbitrary topological algebra E, with non-empty spectrum, defining the notion of a peak subset of its spectrum $\mathfrak{M}(E)$, with respect to a subspace H of E. We thus prove that the family of peak sets of E, relative to H, is closed under finite intersections; in particular, when the *"Gel'fand transform"* space $H^\wedge$ is a σ-complete subspace of $E^\wedge$, then, it is closed for countable intersections, as well (Lemmas 4.1, 4.2). The latter result is extended to arbitrary intersections, provided *peak sets* and *weakly peak sets relative to H* coincide (Corollaries 4.3, 4.5). The same sets, when reduced to singletons, are called *peak and weakly peak points relative to H*, respectively. This coincidence is still attained, within appropriate condition for spectrum (Corollary 4.4).

Investigating the relation of peak points with the various boundaries in the spectrum of a topological algebra E, relative to a subspace H of it, we remark that the aforementioned points are, in general, contained in the *relative Choquet boundary of E*, while, under suitable conditions, are dense in the *relative Šilov boundary of E* (Lemma 5.1, Theorems 3.1, 3.5 and Corollary 4.9 along with Remark 4.10).

On the other hand, when the set of peak points is a *weakly boundary set of E relative to H*, then, it is precisely the *relative Bishop boundary of E* (Theorem 6.3, Corollary 6.4). Of course, the relative Choquet, peak and *strong boundary points* that coincide, under appropriate conditions (Theorem 5.5, Corollary 5.7), constitute, in suitable topological algebras, relative weakly boundary sets. So, the latter sets are identified with the relative Bishop boundary and, consequently, are dense in the relative Šilov boundary (Corollary 6.7).

Finally, we prove that in certain topological algebras the various boundaries and points, as above, remain invariant when we restrict the Gel'fand transform algebra of the given algebra to subsets of its spectrum that contain the Šilov boundary (Theorems 7.4, 7.6, 7.9).

2 Preliminaries

In all that follows, by a *topological algebra E* we mean a topological $\mathbb{C}$-vector space, which is also an algebra with a separately continuous multiplication, having a non-empty *spectrum* or *Gel'fand space* $\mathfrak{M}(E)$, endowed with the *Gel'fand topology*. The *Gel'fand map* is given by

$$\begin{aligned} \mathcal{G}_E : E \longrightarrow \mathcal{C}(\mathfrak{M}(E)) : x \longmapsto \mathcal{G}_E(x) \equiv \hat{x} : \mathfrak{M}(E) \longrightarrow \mathbb{C} \\ : f \longmapsto \hat{x}(f) := f(x). \end{aligned} \tag{2.1}$$

The image of $\mathcal{G}_E$, denoted by $E^\wedge$, is called the *Gel'fand transform algebra* of E and is topologized as a *locally m-convex algebra* by the inclusion $E^\wedge \subseteq \mathcal{C}_c(\mathfrak{M}(E))$, where the algebra $\mathcal{C}(\mathfrak{M}(E))$ carries the topology "c" of compact convergence in $\mathfrak{M}(E)$ [26, p. 19, Example 3.1]. If $\mathfrak{B}$ stands for the *Borel σ-algebra* generated by the closed subsets of $\mathfrak{M}(E)$ and $\mathcal{M}_c(\mathfrak{M}(E))$ denotes the *vector space of regular complex Borel* *measures on* $\mathfrak{M}(E)$ *with compact supports*, then, one gets (ibid. p. 474, Lemma 2.1)

$$\mathcal{M}_c(\mathfrak{M}(E)) = (\mathcal{C}_c(\mathfrak{M}(E)))', \tag{2.2}$$

within an isomorphism *of* $\mathbb{C}$*-vector spaces* defined by

$$l_\mu(h) = \int_{\mathfrak{M}(E)} h d\mu, \tag{2.3}$$

for every $h \in \mathcal{C}_c(\mathfrak{M}(E))$. Furthermore, $\mathcal{M}_c^+(\mathfrak{M}(E))$ is the subspace of $\mathcal{M}_c(\mathfrak{M}(E))$ of all positive measures on $\mathfrak{M}(E)$, in the sense that

$$\mu(h) \geq 0, \ \ for \ \ every \ \ 0 \leq h \in \mathcal{C}_c(\mathfrak{M}(E)). \tag{2.4}$$

Now, given a subspace H of E and an $f \in \mathfrak{M}(E)$, we call *representing measure of* f *with respect to* H, a measure $\mu \equiv \mu_f \in \mathcal{M}_c^+(\mathfrak{M}(E))$, such that

$$\mu_f(\hat{x}) \equiv \int_{\mathfrak{M}(E)} \hat{x} d\mu_f = \delta_f(\hat{x}) := \hat{x}(f) \equiv f(x), \ \ x \in H, \tag{2.5}$$

or, equivalently,

$$\mu_f(Re\hat{x}) = \delta_f(Re\hat{x}) := Re\hat{x}(f), \ \ x \in H. \tag{2.6}$$

The set of representing measures of f with respect to H is denoted by $r_c^+(f)_H \equiv r_H(f)$, and when it is reduced to the singleton $\{\delta_f\}$, the latter being, of course, such a measure, then, f is said to be a *Choquet point of* E *relative to* H, or a *relative Choquet point of* E. The set of the latter points is defined as the *Choquet boundary of* E *relative to* H, or *relative Choquet boundary of* E, denoted by $Ch_H(E)$ (cf. [14] or [16]). If *E has an identity contained in H*, one proves that

$$\begin{aligned} \overline{co}(\tau_H(\mathfrak{M}(E))) = \{\phi \in (H^\wedge)' : \phi(\hat{x}) = \mu(\hat{x}), \ x \in H, \\ for\ some\ \mu \in S^+_{\mathfrak{M}(E)}(1)\} \end{aligned} \tag{2.7}$$

and

$$Ext(\overline{co}(\tau_H(\mathfrak{M}(E)))) \subseteq \tau_H(\mathfrak{M}(E)); \tag{2.8}$$

here $\overline{co}$ and Ext stand for *closed convex hull* and *extreme points* respectively, while $S^+_{\mathfrak{M}(E)}(1)$ denotes the measures of $\mathcal{M}_c^+(\mathfrak{M}(E))$ with total variation norm 1,

$$\begin{aligned} &\tau_H : \mathfrak{M}(E) \longrightarrow (H^\wedge)'_s : H^\wedge \longrightarrow \mathbb{C} \\ &: \hat{x} \mapsto \tau_H(f)(\hat{x}) := \hat{x}(f) \equiv f(x), \end{aligned} \tag{2.9}$$

being a continuous map. The latter becomes a *homeomorphism into when* H *separates the points of* $\mathfrak{M}(E)$, yielding that

$$Ch_H(E) = \tau_H^{-1}(Ext(\overline{co}(\tau_H(\mathfrak{M}(E))))) \tag{2.10}$$

(cf. [14] or [16]). Yet, the presence of the identity in H makes the representing measures *probability measures*, in the sense that, for every $f \in \mathfrak{M}(E)$, one has

$$\mu_f(\mathfrak{M}(E)) = \mu_f(1_{E^\wedge}) = 1, \ \ \mu_f \in r_H(f). \tag{2.11}$$

Furthermore, the *Šilov boundary of E relative to H*, or the *relative Šilov boundary of E*, denoted by $\partial_H(E)$, is *the least boundary set of E relative to H*, or *the least relative boundary set of E*; that is, the smallest closed subset of $\mathfrak{M}(E)$, on which every $\hat{x}$, $x \in H$, attains its maximum absolute value [26, p. 189, Definitions 2.1 and 2.2] or [14, p. 6, Definition 1.4, (1.16)]. Its elements,

(2.12) *the relative Šilov points, are characterized by the fact that for every open neighbourhood V of f in $\mathfrak{M}(E)$, there exists $x \in H$, such that $M_{\hat{x}} \subseteq V$, with*

$$M_{\hat{x}} = \{f \in \mathfrak{M}(E) : |\hat{x}(f)| = \sup_{h \in \mathfrak{M}(E)} |\hat{x}(h)| \equiv p_{\mathfrak{M}(E)}(\hat{x}) \equiv \alpha\},$$

being a closed subset of $\mathfrak{M}(E)$, by the continuity of $\hat{x}$, $x \in H$ [26, p. 190, (2.4) and Lemma 2.1] or [14, p. 8, (1.20) and p. 9, Proposition 1.4]. Moreover, the relation $M_{\hat{x}} = |\hat{x}|^{-1}(\{\alpha\})$ implies that $M_{\hat{x}}$ is a G_δ-set. *The existence of $\partial_H(E)$ is accomplished in a unital topological algebra with $\mathfrak{M}(E)$ compact and H a subspace of E containing the constants and separating the points of $\mathfrak{M}(E)$* [3, p. 484, Theorem] or [14, p. 10, Theorem 1.1].

On the other hand, a subset $B \subseteq \mathfrak{M}(E)$ is called a *peak set of E relative to H*, or *relative peak set of E*, if there exists $x \in H$, such that

$$\hat{x} = \alpha|_B \quad and \quad |\hat{x}| < \alpha|_{B^c}, \tag{2.13}$$

where obviously $\alpha \in \mathbb{R}_+^*$ and $B^c \equiv \mathfrak{M}(E) \setminus B$. Besides an $f \in \mathfrak{M}(E)$ is said to be a *peak point of E relative to H*, or a *relative point of E*, if the singleton $\{f\}$ is a relative peak set of E; thus, if there exists $x \in H$, with

$$\hat{x}(f) = \alpha \quad and \quad |\hat{x}| < \alpha|_{\{f\}^c}. \tag{2.14}$$

The function $\hat{x}$ in (2.13) and (2.14) is called *peaking function*, while the sets of relative peak points and sets of E are denoted by $P_H(E)$ and $\mathcal{P}_H(E)$, respectively (cf. [14]).

Remark 2.1. In (2.13) and (2.14), we can have 1 instead of α, by considering $y = \frac{1}{\alpha}x \in H$. Based on (2.13) and the continuity of $\hat{x}$, $x \in H$, one has, of course, that *every relative peak set B of E is closed*, since $B = \hat{x}^{-1}(\{\alpha\})$, while the defined element $x \in H$ satisfies

$$\|\hat{x}\|_\infty \equiv p_{\mathfrak{M}(E)}(\hat{x}) = \sup_{h \in \mathfrak{M}(E)} |\hat{x}(h)| = \alpha.$$

Of course, the empty set is not a relative peak set. Now, a set (or point), when an intersection of relative peak sets of E, is defined as a *weakly peak set* (or *point*) *of E relative to H*, or a *relative weakly peak set* (or *point*) *of E*. The set of the respective objects is denoted by $\mathcal{P}_H^w(E))$ (or $P_H^w(E)$). By the preceding relation, one concludes that

(2.15) *every relative peak set (or point) of E is of the form $M_{\hat{x}}$, for some $x \in H$, while the converse is not in general true* (see [25, p. 160]).

In particular, if the relative Šilov boundary $\partial_H(E)$ of E exists, then, due to the characterization of its elements (cf. (2.12)), one gets

$$B \cap \partial_H(E) \neq \emptyset, \;\; B \in \mathcal{P}_H(E), \tag{2.16}$$

and consequently,

$$P_H(E) \subseteq \partial_H(E). \tag{2.17}$$

A *strong boundary point of E relative to H*, or a *relative strong point of E*, is an $f \in \mathfrak{M}(E)$, for every open neighbourhood U of which, there exists $x \in H$, such that

$$\hat{x}(f) = \alpha = \|\hat{x}\|_\infty \;\; and \;\; |\hat{x}| < \alpha|_{U^c}, \tag{2.18}$$

where, of course, $\alpha \in \mathbb{R}^*_+$. The set of the latter points is called the *strong boundary of E relative to H*, or the *relative strong boundary of E*, and denoted by $s_H(E)$. By taking $y = \frac{1}{\alpha}x \in H$ in (2.18), we can have 1 instead of α. Yet, in conjunction with (2.12), one concludes that

(2.19) *for every $f \in s_H(E)$, there exists $x \in H$, with $f \in M_{\hat{x}} \subseteq U$*, so that $s_H(E) \subseteq \partial_H(E)$.

A continuous map $\phi : \mathfrak{M}(E) \to X$, where X is a complex manifold, is called a *weakly spectral map relative to H*, if for every holomorphic function h on X, one has [26, p. 295]

$$h \circ \phi = \hat{x}, \;\; for \;\; some \;\; x \in H. \tag{2.20}$$

The latter is actually applied for an *infinite dimensional complex manifold*; as a matter of fact, we consider a given (complex) normed space as a complex manifold, as above (cf. [28, p. 16, Definition 1] or [26, p. 315, Definition 10.2]). Furthermore, if $(x_i)_{i\in I}$ is a set of *topological generators of E*, so that one has $E = \overline{\mathbb{C}_0[(x_i)_{i\in I}]}$, the following *canonical map* is defined

$$\phi : \mathfrak{M}(E) \longrightarrow \mathbb{C}^I : f \longmapsto \phi(f) \equiv (\hat{x}_i(f))_{i\in I}, \tag{2.21}$$

being a *continuous bijection onto its image* [26, p. 292]. Finally, we say that a topological algebra E is *Weierstrass relative to a subspace H* of it, if $\mathfrak{M}(E)$ is a *Weierstrass space with respect to $H^\wedge$*, in the sense that every $|\hat{x}|$, $x \in H$, realizes its supremum at a point of $\mathfrak{M}(E)$. Based on (2.12), we obtain the following characterization

(2.22) *a topological algebra E is Weierstrass relative to a subspace H of it, if, and only if, $M_{\hat{x}} \neq \emptyset$, for every $x \in H$.*

Yet E is called *H-bounded*, whenever $\hat{x}$ is bounded, for every $x \in H$. Evidently, according to (2.22), *a Weierstrass algebra E relative to H is H-bounded*. Besides, we recall that *E is a σ-complete (or else sequentially complete) topological algebra*, if this is the case for the respective topological vector space E (i.e., every Cauchy sequence in E converges).

Given a subset H of a topological vector space E, a closed real hyperplane M is called a *supporting hyperplane of H*, if $H \cap M \neq \emptyset$ and H lies in one of the two closed half-spaces defined by M, while the elements of $H \cap M$ are called *points of support of H*. Moreover, an $x \in H \subseteq E$ is said to be an *exposed point of H*, if x is the only point of support for a supporting hyperplane M of H; that is $H \cap M = \{x\}$. The set of exposed points of H is denoted by $Exp(H)$. In case H is a locally compact convex subset of a normed space E containing no line, then, the exposed points of H are dense in its extreme points $Ext(H)$, that is (cf. [22, p. 91, Theorem (2.3)])

$$\overline{Exp(H)} = Ext(H). \tag{2.23}$$

In this context, an $x \in H$ is said to be an *extreme point of* H, if for every $y_1, y_2 \in H$, such that $x = \lambda y_1 + (1-\lambda) y_2$, with $0 < \lambda < 1$, one has $y_1 = x = y_2$.

3 Relative peak and Šilov points

We have already mentioned that every relative peak point is a relative Šilov point (cf. (2.17)). In this section we prove that, in a suitable topological algebra, *the relative Šilov points adhere to relative peak points*. First, we note that every (relative) peak set does not necessarily contain a (relative) peak point, even though this is the case for a Banach function algebra on a metrizable space [9, p. 126, §3]. However, under appropriate conditions, every neighbourhood of a (relative) peak set contains a (relative) peak point, as the next result proves. In this concern, we say that *the exponential map acts on the Gel'fand transform algebra $E^\wedge$ of a given topological algebra E*, if for every $x \in E$, there is $y \in E$, such that $exp(\hat{x}) \equiv e^{\hat{x}} = \hat{y}$.

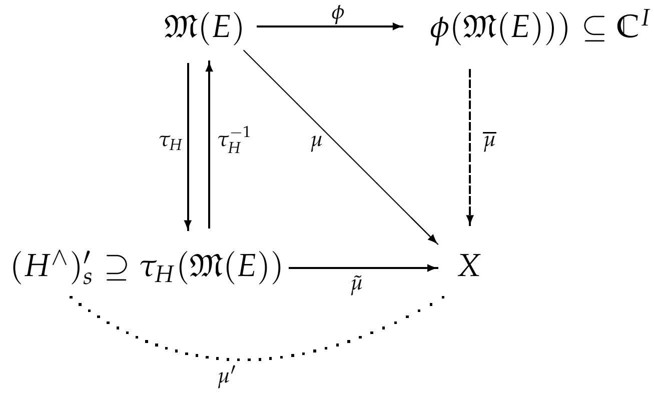
Theorem 3.1. *Let E be a unital topological algebra, H a subspace of E containing the constants and separating the points of $\mathfrak{M}(E)$, and $Ch_H(E)$ the relative Choquet boundary of E, such that*

$$B \cap Ch_H(E) \neq \emptyset, \quad for \quad every \quad B \in \mathcal{P}_H(E), \tag{3.1}$$

while the exponential map acts on $H^\wedge \subseteq E^\wedge$. Moreover, let $(x_i)_{i\in I}$ be a system of generators of E, $\phi : \mathfrak{M}(E) \to \mathbb{C}^I$ the canonical continuous injection, X a normed subspace of $\mathbb{C}^I$, and $\mu : \mathfrak{M}(E) \to X$ a homeomorphism into, being also a weakly spectral map relative to H, such that the closed convex hull of $\mu(\mathfrak{M}(E))$ is locally compact in X, with no lines in it. Furthermore, assume that μ is the restriction of a continuous 1-1 linear map $\mu' : (H^\wedge)'_s \to X$ on $\mathfrak{M}(E) \equiv \phi(\mathfrak{M}(E)) \subseteq \mathbb{C}^I$. Then, every neighbourhood of a relative peak set of E contains a relative peak point of E.

Proof. If $B \in \mathcal{P}_H(E)$, by (3.1) there exists $f \in B \cap Ch_H(E)$, with (cf. (2.8), (2.10)) $\tau_H(f) \in Ext(\overline{co}(\tau_H(\mathfrak{M}(E))))$. By the following commutative diagram,

(3.2)



one gets

$$\mu \approx \tilde{\mu} = \mu'|_{\tau_H(\mathfrak{M}(E))=\mathfrak{M}(E)=\phi(\mathfrak{M}(E))}, \tag{3.3}$$

hence $\mu(f) = \mu'(\tau_H(f)) \in Ext(\overline{co}(\mu(\mathfrak{M}(E)))) \subseteq X$ and also $\mu(f) \in \mu(B)$. Taking a neighbourhood U of B in $\mathfrak{M}(E)$, then, $\mu(U)$ is a neighbourhood of $\mu(B)$ in $\mu(\mathfrak{M}(E))$, so, for some neighbourhood V of $\mu(B)$ in X, one has

$$\mu(U) = V \cap \mu(\mathfrak{M}(E)). \tag{3.4}$$

Thus, $V \cap \overline{co}(\mu(\mathfrak{M}(E)))$ is a neighbourhood of $\mu(f)$ in $\overline{co}(\mu(\mathfrak{M}(E)))$, and, in view of (2.23), contains an exposed point of $\overline{co}(\mu(\mathfrak{M}(E)))$, say z. By (2.8), (3.3) and (3.4), $z \in V \cap \mu(\mathfrak{M}(E)) = \mu(U)$, hence $f_1 \equiv \mu^{-1}(z) \in U$, which is a relative peak point of E : Indeed, if M is a closed supporting hyperplane of $\overline{co}(\mu(\mathfrak{M}(E)))$, with $M \cap \overline{co}(\mu(\mathfrak{M}(E))) = \{z\}$, assume that $M = \{x \in X_{\mathbb{R}} : h(x) = \lambda\}$ and $\mu(\mathfrak{M}(E))\backslash\{z\} \subseteq \{x \in X_{\mathbb{R}} : h(x) < \lambda\}$, where $X_{\mathbb{R}}$ is the underlying real vector space of X, $h : X_{\mathbb{R}} \to \mathbb{R}$ is a continuous linear form on $X_{\mathbb{R}}$ and λ is a constant. Considering the respective complex-valued continuous linear form on X,

$$h_1 : X \longrightarrow \mathbb{C} : x \longmapsto h_1(x) := h(x) - ih(ix),$$

by the hypothesis for μ (cf. also (2.20)), there exists $y \in H$, such that $h_1 \circ \mu = \hat{y}$. By defining $\hat{x} = exp(\hat{y} - h_1(z)) \in H^{\wedge}$, we have $\hat{x}(f_1) = 1$ and $|\hat{x}(g)| < 1$, for every $g \neq f_1$, proving that f_1 is a relative peak point of E in U. ■

In particular, if the topological algebra E is finitely generated, say n-generated, then, in the previous theorem, we can take $\mathbb{C}^n$ instead of the normed space X, and the map ϕ in place of μ. So, by applying a relevant argument with that one in the proof of Theorem 3.1, we get at the next. (In this respect, we consider finite dimensional complex manifolds).

Corollary 3.2. *Let E be a unital n-generated topological algebra, H a subspace of E containing the constants and separating the points of $\mathfrak{M}(E)$, and $Ch_H(E)$ the relative Choquet boundary of E, such that*

$$B \cap Ch_H(E) \neq \emptyset, \quad B \in \mathcal{P}_H(E), \tag{3.5}$$

while the exponential map acts on $H^{\wedge} \subseteq E^{\wedge}$. Moreover, let $\phi : \mathfrak{M}(E) \to \mathbb{C}^n$ be a homeomorphism (into), which is also a weakly spectral map relative to H, and derives from the restriction of a continuous 1-1 linear map $\phi' : (H^{\wedge})'_s \to \mathbb{C}^n$ on $\mathfrak{M}(E) \equiv \phi(\mathfrak{M}(E))$, such that the closed convex hull of $\phi(\mathfrak{M}(E)$ in $\mathbb{C}^n$ contains no lines. Then, every neighbourhood of a relative peak set of E contains a relative peak point of E.

We show now that although sets of the form $M_{\hat{x}}$, $x \in H$, need not be relative peak sets, they do contain relative peak sets (cf. [7, p. 105], [32, p. 139, Corollary (3.3.11)]).

Lemma 3.3. *Let E be a unital topological algebra and H a subspace of E containing the constants. Then, for every $x \in H$, with $M_{\hat{x}} \neq \emptyset$, there exists a relative peak set B of E, such that $B \subseteq M_{\hat{x}}$.*

Proof. If $x \in H$, with $M_{\hat{x}} \neq \emptyset$, let $\alpha = \sup_{h \in \mathfrak{M}(E)} |\hat{x}(h)| \equiv \|\hat{x}\|_{\infty}$. Setting

$$y = \frac{1}{2}(e^{it}x + \alpha) \in H, \tag{3.6}$$

we can take $t = iln(f(x)\alpha^{-1})$, so that $f(y) = \alpha$, with $f \in \mathfrak{M}(E)$. In particular, for a fixed $f_0 \in M_{\hat{x}}$ (and the corresponding t, hence y), we consider the non-empty set

$$B = \{f \in M_{\hat{x}} : \hat{y}(f) = \alpha\} \subseteq M_{\hat{x}}. \tag{3.7}$$

(Concerning B, we remark that by taking $f_0 \in M_{\hat{x}}$, as above, for every $f \in \mathfrak{M}(E)$, with $f(y) = \alpha$, we get $f \in M_{\hat{x}}$). Then, B is a relative peak set of E, with peaking function

$$\hat{z} = \frac{1}{2}(\alpha^{-1}\hat{y} + 1_{E^\wedge}) \in H^\wedge.$$

In fact, $\hat{z} = 1|_B$, while $\|\hat{y}\|_\infty = \alpha$ (cf. (3.6), (3.7))). Now, if $|\hat{z}(g)| = 1$, for some $g \in \mathfrak{M}(E)$, one obtains the equalities

$$|\alpha^{-1}\hat{y}(g) + 1| = 2 \quad and \quad |\alpha^{-1}\hat{y}(g)| = 1,$$

yielding that $\hat{y}(g) = \alpha$, hence $g \in B$. Since $\|\hat{z}\|_\infty = 1$, one gets $|\hat{z}| < 1|_{B^c}$, proving that B is a relative peak set of E. ∎

Remark 3.4. In the preceding proof, one can show in a similar way that the function $\hat{y}$, determining B, peaks on it, due to the relation (2.13).

We come now to the main theorem of this section, extending the relevant one in [9, p. 123, Theorem (2.3)] and in [14, p. 193, Theorem 7.1], as well.

Theorem 3.5. *Considering the context of Theorem 3.1, the set $P_H(E)$ of relative peak points of E is dense in the relative Šilov boundary of E; that is, one has*

$$\overline{P_H(E)} = \partial_H(E). \tag{3.8}$$

Proof. By (2.17), one has only to prove that $\partial_H(E) \subseteq \overline{P_H(E)}$. Indeed, if $f \in \partial_H(E)$, then, for every open neighbourhood V of f, there exists $x \in H$, such that $M_{\hat{x}} \subseteq V$, hence (Lemma 3.3), there is $B \in \mathcal{P}_H(E)$, with $B \subseteq M_{\hat{x}} \subseteq V$. According to Theorem 3.1, V contains a $g \in P_H(E)$, that is, the Šilov point f adheres to $P_H(E)$, proving the assertion. ∎

4 On the hypotheses of Theorem 3.1

Concerning the condition (3.1), we note that, in the classical case, this is satisfied due to the more general fact that *every relative weakly peak set of E contains a relative Choquet point of E* [34, p. 27, Corollary 2.9]. In this respect, we first examine hereditary properties referring to relative peak sets. Cf. also [25, p. 160, Theorem 1] and [37, p. 162, Exercise 33.7.a].

Lemma 4.1. *Let E be a topological algebra and H a subspace of it. Then, every finite intersection of relative peak sets is a set of the same type. In particular, if $\mathfrak{M}(E)$ is compact, then,*

(4.1) *for every neighbourhood U of a relative weakly peak set F, there exists a relative peak set B, such that $F \subseteq B \subseteq U$.*

Proof. Let $F = \bigcap_{n=1}^{k} B_n$, with $B_n \in \mathcal{P}_H(E)$ and peaking functions $\hat{x}_n$, $x_n \in H$, respectively. That is, we have

$$\hat{x}_n = 1|_{B_n} = \|\hat{x}_n\|_\infty \quad \textit{and} \quad |\hat{x}_n| < 1|_{B_n^c}, \tag{4.2}$$

for every $n = 1, 2, ..., k$ (cf. Remark 2.1). By setting

$$x = \frac{1}{k} \sum_{n=1}^{k} x_n,$$

we get $x \in H$, with $\hat{x} = 1|_F$, while for $g \notin F$, there exists $n_0 \in \{1, 2, ..., k\}$, such that $g \notin B_{n_0}$. Hence $|\hat{x}_{n_0}(g)| < 1$, implying $|\hat{x}| < 1|_{F^c}$, so that $F \in \mathcal{P}_H(E)$. Now, taking $F \in \mathcal{P}_H^w(E)$, that is, $F = \bigcap_{\alpha \in I} B_\alpha$, with $B_\alpha \in \mathcal{P}_H(E)$, $\alpha \in I$, and U an open neighbourhood of F, then, by the compactness of $\mathfrak{M}(E)$, there exists a finite subcovering of it, say $\{U,\ B_{\alpha_i}^c (i = 1, ..., n)\}$, such that $\bigcap_{i=1}^{n} B_{\alpha_i} \subseteq U$. Thus,

$$F = \bigcap_{\alpha \in I} B_\alpha \subseteq \bigcap_{i=1}^{n} B_{\alpha_i} \equiv B \subseteq U,$$

with $B \in \mathcal{P}_H(E)$, as before, yielding the assertion. ∎

A countable intersection of relative peak sets becomes a relative peak set in a class of topological algebras having the "Gel'fand transforms" of the subspaces considered σ-complete. The latter is accomplished when the subspaces concerned are closed and the Gel'fand transform algebras of the algebras involved σ-complete. (In this context, see [9, p. 126, Proposition (2.6)], [26, p. 275, Lemma 5.2 or p. 276, Theorem 5.1, 5] and [10, p. 110, Example 3]). In the same class of algebras, one obtains (4.1) by considering a G_δ-set in place of the open neighbourhood U (see [12, p. 56, Lemma 12.2]).

Lemma 4.2. *Let E be a topological algebra and H a subspace of it, with $H^\wedge \subseteq E^\wedge$ σ-complete. Then, every countable intersection of relative peak sets of E is such a one. In particular, if $\mathfrak{M}(E)$ is Lindelöf, then,*

(4.3) *for every G_δ-set W, containing a relative weakly peak set F of E, there exists a relative peak set B of E, such that $F \subseteq B \subseteq W$.*

Proof. Let $F = \bigcap_{n=1}^{\infty} B_n$, with $B_n \in \mathcal{P}_H(E)$, and peaking functions $\hat{x}_n$, $x_n \in H$, $n \in \mathbb{N}$ (cf. also (4.2)). By considering $\phi = \sum_{n=1}^{\infty} 2^{-n} \hat{x}_n$, we have, by hypothesis for $H^\wedge$, that $\phi = \hat{x}$, for some $x \in H$, which peaks on F, so that F is a relative peak set. On the other hand, taking $F \in \mathcal{P}_H^w(E)$ contained in a G_δ-set W, then, $F = \bigcap_{\alpha \in I} B_\alpha$, with $B_\alpha \in \mathcal{P}_H(E)$, and $W = \bigcap_{n=1}^{\infty} W_n$, W_n open in $\mathfrak{M}(E)$. By the assumption for $\mathfrak{M}(E)$, there is, for its open covering $\{W_n, B_\alpha^c,\ \alpha \in I\}$, a countable subcovering $\{W_n, B_{\alpha_{i_n}}^c,\ i \in \mathbb{N}\}$. Thus, setting $B_n \equiv \bigcap_{i=1}^{\infty} B_{\alpha_{i_n}} \subseteq W_n$, one gets that $B_n \in \mathcal{P}_H(E)$, and $F \subseteq B_n \subseteq W_n$, $n \in \mathbb{N}$. Then, $F \subseteq B \equiv \bigcap_{n=1}^{\infty} B_n \subseteq \bigcap_{n=1}^{\infty} W_n \equiv W$, where $B \in \mathcal{P}_H(E)$, as before. ∎

We remark that the second part of Lemma 4.1 derives from the preceding lemma, at the cost, however, of the σ-completeness of $H^\wedge$. Furthermore, in the context of Lemma 4.2, a relative weakly peak set becomes a relative peak one, if it is also G_δ, a basic property of relative peak sets (cf. [7, p. 96, Lemma 2.3.1]).

Corollary 4.3. *Let E be a topological algebra, H a subspace of E, and $B \subseteq \mathfrak{M}(E)$. Moreover, assume the following two assertions:*

1) B is a relative peak set of E.

2) B is a G_δ relative weakly peak set of E.

Then, 1) $\Rightarrow$ 2)*, while* 2) $\Rightarrow$ 1) *if, in addition, E satisfies (4.3)* (take, for instance, $\mathfrak{M}(E)$ Lindelöf and $H^\wedge \subseteq E^\wedge$ σ-complete).

Proof. 1)$\Rightarrow$2): A relative peak set is also a relative weakly one, and if $\hat{x}$, $x \in H$, is the peaking function, then, $B = \hat{x}^{-1}(\{\alpha\})$, that is, a G_δ-set. 2)$\Rightarrow$1): It is obvious by Lemma 4.2. ∎

Concerning G_δ-property, we recall that *in a Hausdorff topological space, satisfying the first axiom of countability, every singleton is a G_δ*-point. On the other hand, in a topological space *every zero set of a continuous complex-valued function f, and more generally every set of the form $f^{-1}(c)$, with c a constant, is a G_δ-set, while in a perfectly normal space every closed subset is*, by definition, G_δ (cf. [30, p. 33, Definition 4.15]). On the basis of the preceding and of Corollary 4.3, we get at the following.

Corollary 4.4. *In a topological algebra E, satisfying (4.3) with respect to a subspace H, and having spectrum $\mathfrak{M}(E)$ first countable, relative peak points and relative weakly peak points coincide.*

Corollary 4.5. *Let E be a topological algebra satisfying (4.3) relative to a subspace H, with spectrum $\mathfrak{M}(E)$ perfectly normal. Then, the notions of relative peak sets and relative weakly peak sets are identical.*

In the sequel we give conditions guaranteeing relation (3.1) in Theorem 3.1. First, we need the following extension of a relevant result in [7, p. 102, Theorem 2.4.1 or p. 104, Theoerm 2.4.2], [13, p. 428, Lemma 4.5], [33, p. 52, Lemma 7.22] for function algebras on a compact space.

Lemma 4.6. *Let E be a topological algebra and H a subalgebra of E, with $H^\wedge \subseteq E^\wedge$ σ-complete. Moreover, assume that, for every G_δ-set W, containing an $F \in \mathcal{P}^w_H(E)$, there exists $B \in \mathcal{P}_H(E)$, such that $F \subseteq B \subseteq W$* (cf. Lemma 4.2). *Then, for every $x \in H$, with $\|\hat{x}\|_\infty = \beta \in \mathbb{R}_+$, there exists $y \in H$, such that*

$$\hat{x} = \hat{y}|_F \quad and \quad \|\hat{y}\|_\infty = \|\hat{x}\|_F. \tag{4.4}$$

Proof. Let $x \in H$ with $\|\hat{x}\|_F = \alpha$. (Without loss of generality, we assume that $\alpha > 1$; otherwise, we consider $y = \frac{4}{3\alpha}x \in H$). Then, the sets

$$U_n = \Big\{ f \in \mathfrak{M}(E) : |\hat{x}(f)| < \alpha + \frac{1}{2^n} \Big\},\ n \in \mathbb{N},$$

provide a decreasing sequence of open neighbourhoods of F, so, by hypothesis, for every $n \in \mathbb{N}$, there exists $B_n \in \mathcal{P}_H(E)$, such that $F \subseteq B_n \subseteq U_n$. Thus, for every $n \in \mathbb{N}$, there is $x_n \in H$, with

$$\hat{x}_n = 1|_{F \subseteq B_n} = \|\hat{x}_n\|_\infty \quad and \quad |\hat{x}_n| < 1|_{U_n^c \subseteq B_n^c}.$$

Now, for every $n \in \mathbb{N}$, we choose a positive integer $k(n)$, such that

$$|\hat{x}_n|^{k(n)} < \frac{1}{2^n \beta} \, |_{U_n^c},$$

and we define

$$\phi = \hat{x} \sum_{n=1}^{\infty} 2^{-n} \hat{x}_n^{k(n)}.$$

Since $\sum_{n=1}^{\infty} 2^{-n} \|\hat{x}_n\|_\infty^{k(n)} = 1 < +\infty$, by the hypothesis for $H^\wedge$, there exists $y \in H$, such that $\phi = \hat{y}$, hence $\hat{x} = \hat{y}|_F$. If $f \in \bigcap_{n=1}^{\infty} U_n$, then, $|\hat{x}(f)| \leq \alpha$, implying that

$$|\hat{y}(f)| \leq \alpha \sum_{n=1}^{\infty} 2^{-n} |\hat{x}_n(f)|^{k(n)} \leq \alpha \sum_{n=1}^{\infty} 2^{-n} \|\hat{x}_n\|_\infty^{k(n)} = \alpha. \tag{4.5}$$

Moreover, if $f \notin \bigcap_{n=1}^{\infty} U_n$, then, we have two cases:

i) There exists $n_0 \in \mathbb{N}$, with $f \in U_{n_0}$ and $f \notin U_n$ for every $n > n_0$, so that

$$|\hat{x}_n(f)|^{k(n)} |\hat{x}(f)| < \|\hat{x}_n\|_\infty^{k(n)} (\alpha + 2^{-n_0}) = \alpha + 2^{-n_0}, \quad n = 1, ..., n_0,$$

and

$$|\hat{x}_n(f)|^{k(n)} |\hat{x}(f)| < \frac{1}{2^n \beta} \|\hat{x}\|_\infty = 2^{-n} < 2^{-n_0}, \quad n > n_0.$$

Hence,

$$\begin{aligned} |\hat{y}(f)| &\leq |\hat{x}(f)| \Big(\sum_{n=1}^{n_0} 2^{-n} |\hat{x}_n(f)|^{k(n)} + \sum_{n=n_0+1}^{\infty} 2^{-n} |\hat{x}_n(f)|^{k(n)} \Big) \\ &< (\alpha + 2^{-n_0}) \sum_{n=1}^{n_0} 2^{-n} + 2^{-n_0} \sum_{n=n_0+1}^{\infty} 2^{-n} \\ &= (\alpha + 2^{-n_0})(1 - 2^{-n_0}) + 2^{-n_0} 2^{-n_0} \\ &= \alpha - (\alpha - 1)\, 2^{-n_0} < \alpha. \end{aligned} \tag{4.6}$$

ii) If $f \notin U_n$, for every $n \in \mathbb{N}$, then,

$$|\hat{x}_n(f)|^{k(n)} |\hat{x}(f)| < \frac{1}{2^n \beta} \|\hat{x}\|_\infty = 2^{-n}, \;\; n \in \mathbb{N},$$

implying that

$$|\hat{y}(f)| \leq |\hat{x}(f)| \sum_{n=1}^{\infty} 2^{-n} |\hat{x}_n(f)|^{k(n)} \leq \sum_{n=1}^{\infty} 2^{-2n} = \frac{1}{3} < \alpha. \tag{4.7}$$

From (4.5), (4.6) and (4.7) one derives the desired relation $\|\hat{y}\|_\infty = \alpha = \|\hat{x}\|_F$. ■

We come now to the proof of the condition (3.1) in Theorem 3.1 (cf. also [33, p. 52, "proof of Theorem"]).

Theorem 4.7. *Let E be a topological algebra, satisfying (4.3) relative to a unital separating subalgebra H of E, having $H^\wedge \subseteq E^\wedge$ σ-complete. Then,*

$$B \cap Ch_H(E) \neq \emptyset, \tag{4.8}$$

for every $B \in \mathcal{P}_H(E)$.

Proof. Taking $B \in \mathcal{P}_H(E)$, *the family*

$$\mathcal{A} = \{F \in \mathcal{P}_H^w(E) : F \subseteq B\} \tag{4.9}$$

is inductively ordered, hence (Zorn's Lemma) *$\mathcal{A}$ has a minimal element*, say F_0. Assuming that F_0 is not a singleton, say $F_0 = \{f_0, g_0\}$, then, by the hypothesis for H, there exists $x \in H$, such that $\hat{x}(g_0) = 1$ and $\hat{x}(f_0) = 0$, hence $\|\hat{x}\|_{F_0} = 1$. By applying Lemma 4.6 for F_0 and x, there exists $y \in H$, with

$$\hat{y} = \hat{x}|_{F_0} \quad \textit{and} \quad \|\hat{y}\|_\infty = \|\hat{x}\|_{F_0} = 1.$$

Setting now

$$K = \{f \in \mathfrak{M}(E) : \hat{y}(f) = 1\}, \tag{4.10}$$

we have $F_0 \cap K = \{g_0\}$ (since $f_0 \notin K$), and $K \in \mathcal{P}_H(E)$, with peaking function

$$\hat{z} = \frac{1}{2}(1_{E^\wedge} + \hat{y}) \in H^\wedge. \tag{4.11}$$

Indeed, $\hat{z} = 1|_K$, while by assuming that $|\hat{z}(g)| = 1$, for $g \in K^c$, we get $|\hat{y}(g) + 1| = 2$ and $|\hat{y}(g)| = 1$, yielding that $\hat{y}(g) = 1$, hence $g \in K$, a contradiction. So, $|\hat{z}| < 1|_{K^c}$, proving that $K \in \mathcal{P}_H(E)$. This implies that $F_0 \cap K = \{g_0\} \in \mathcal{A}$, with $\{g_0\} \subsetneqq F_0$, a contradiction to the minimality of F_0. Thus, *the minimal elements of $\mathcal{A}$ are singletons*. Finally, let $\{f'\} \subseteq B$ be a minimal element of $\mathcal{A}$. By hypothesis, for every neighbourhood U of f', there exists $B' \in \mathcal{P}_H(E)$, such that $f' \in B' \subseteq U$, hence, by 5)⇒1) in Theorem 5.5 below, $f' \in Ch_H(E)$, which completes the proof. ∎

Remark 4.8. Based on the preceding proof, we note that the empty set might be considered as a minimal relative weakly peak set, while if such a set is non-empty, the same is reduced to a singleton. Furthermore, if the spectrum of the algebra involved satisfies the first axiom of countability, one concludes that *the relative weakly peak point $f' \in B$ becomes a relative peak point* (cf. Corollary 4.4).

Thus, we obtain the following extension of [7, p. 105, Corollary 2.4.6].

Corollary 4.9. *Let E be a unital H-bounded topological algebra, with H a unital separating subalgebra of it, and $H^\wedge \subseteq E^\wedge$ σ-complete, such that (4.3) be satisfied. Then, every relative peak set of E contains a relative weakly peak point of E. In particular, if $\mathfrak{M}(E)$ fulfils the first axiom of countability, then, every relative peak set of E contains a relative peak point.*

Remark 4.10. Based on Corollary 4.9, one can repeat the proof in Theorem 3.5, thus getting at a variant of the same result.

A consequence of Lemma 4.6 and Theorem 4.7 is the next result [25, p. 161, Lemma 1].

Corollary 4.11. *Let E be a unital topological algebra and H a unital subalgebra, with $H^\wedge$ a σ-complete subalgebra of $E^\wedge$. Moreover, suppose that, for every G_δ-set W containing an $F \in \mathcal{P}^w_H(E)$, there exists $B \in \mathcal{P}_H(E)$, with $F \subseteq B \subseteq W$* (cf. Lemma 4.2). *Finally, let $x \in H$, with*

$$\|\hat{x}\|_F = |\hat{x}(f_0)| \equiv \alpha = \|\hat{x}\|_\infty \in \mathbb{R}, \tag{4.12}$$

for some $f_0 \in F$ (take, for instance, F compact). *Then, the set*

$$N = \{f \in F : \hat{x}(f) = \hat{x}(f_0)\} \subseteq F \tag{4.13}$$

is a relative weakly peak set of E. In particular, if $F \in \mathcal{P}_H(E)$, then, $N \in \mathcal{P}_H(E)$, as well.

Proof. If $\hat{x}(f_0) = 0$, then, $\|\hat{x}\|_F = 0$, so that $N = F$. Now, if $\hat{x}(f_0) \neq 0$, by taking

$$\hat{y} = \frac{1}{2}\Big(1_{E^\wedge} + \frac{\hat{x}}{\hat{x}(f_0)}\Big) \in H^\wedge, \tag{4.14}$$

we get that $\|\hat{y}\|_F = \hat{y}(f_0) = 1 = \|\hat{y}\|_\infty$. So,

$$N = \{f \in F : \hat{y}(f) = 1\} \subseteq F,$$

and showing that

$$N = S \cap F, \tag{4.15}$$

where $S \in \mathcal{P}_H(E)$, we have the assertion. By Lemma 4.6, there exists $z \in H$, such that $\hat{y} = \hat{z}|_F$ and $\|\hat{z}\|_\infty = \|\hat{y}\|_F = 1$. Hence, the set

$$S = \{f \in \mathfrak{M}(E) : \hat{z}(f) = 1\}$$

is a relative peak one (cf. proof of (4.10)), and this implies (4.15). In particular, if F is a relative peak set, then, $N = S \cap F$ is also a relative peak one (cf. Lemma 4.1). ∎

Scholium 4.12. Concerning the previous corollary, we remark that *the set N*, defined by (4.13), *is a peak set of the algebra $E^\wedge|_F$, relative to its subalgebra $H^\wedge|_F$*: Our claim results from an analogous argument to that in proof of (4.10), since $N \subseteq F \subseteq \mathfrak{M}(E^\wedge|_F)$, given that $E^\wedge|_F$ separates points of $\mathfrak{M}(\mathcal{C}(F)) = F$, with (4.14) as a peaking function. Thus, without assuming (4.12) and (4.13), Corollary 4.11 is restated as follows.

Corollary 4.13. *Let E be a unital H-bounded topological algebra, where H a unital subalgebra of E, with $H^\wedge \subseteq E^\wedge$ σ-complete, such that (4.3) be fulfilled. If $F \in \mathcal{P}^w_H(E)$, then, every $N \in \mathcal{P}_{H^\wedge|_F}(E^\wedge|_F)$, with $N \subseteq F$, becomes a member of $\mathcal{P}^w_H(E)$.*

An immediate consequence of Corollary 4.13 is the following extension of Bishop's Lemma [7, p. 104, Corollary 2.4.4].

Corollary 4.14 (Bishop's Lemma). *Let E be a unital H-bounded topological algebra, where H a unital subalgebra, with $H^\wedge \subseteq E^\wedge$ σ-complete. Moreover, suppose that, for every G_δ-set W containing an $F \in \mathcal{P}_H^w(E)$, there exists $B \in \mathcal{P}_H(E)$, with $F \subseteq B \subseteq W$* (cf. Lemma 4.2). *Finally, let $G \in \mathcal{P}^w_{H^\wedge|_F}(E^\wedge|_F)$, with $G \subseteq F$, such that*

$$G = \bigcap_{i \in I} B_i, \quad B_i \in \mathcal{P}_{H^\wedge|_F}(E^\wedge|_F), \text{ with } B_i \subseteq F, \ i \in I. \tag{4.16}$$

Then, $G \in \mathcal{P}_H^w(E)$.

Scholium 4.15. Referring to *the relation $B_i \subseteq F$, $i \in I$*, in (4.16), we remark that it *is fulfilled when $\mathfrak{M}(E^\wedge|_F) = F$*. The latter is *accomplished if E has continuous Gel'fand map*, given that *F is an E-convex subset of $\mathfrak{M}(E)$ relative to H* (cf. after (7.3) below), since, by hypothesis, $F \in \mathcal{P}_H^w(E)$. Thus, one has

$$\mathfrak{M}(E^\wedge|_F) \cong F = (F)_E = (F)_E^H.$$

Now, concerning our assumption in Theorem 3.1 for the exponential map, we remark that *in a unital σ-complete locally m-convex algebra E, with continuous Gel'fand map $\mathcal{G}_E$, the exponential map acts on the Gel'fand transform algebra $E^\wedge$ of E* (cf. also [27, p. 146, proof of Lemma 2.1]). In the same vein of ideas, *the continuity of the Gel'fand map implies actually the convergence in $E^\wedge$ of every convergent series in E*. On the other hand, *the σ-completeness of $H^\wedge$ implies that $H^\wedge$ is closed under exponentiation*, in the case H is subalgebra of E. So, one concludes that:

(4.17) in the context of Theorem 4.7, the assumptions of Theorem 3.1, referring to (3.1), and the action of the exponential map on $H^\wedge$, are fulfilled.

Finally, the normed subspace X of $\mathbb{C}^I$, considered in Theorem 3.1, is constructed according to the following result ([26, p. 25, Lemma 4.3])

(4.18) **Lemma 4.16.** *Let E be a topological vector space and B an absolutely convex, closed and bounded, subset of E. Then, the vector subspace of E generated by B,*

$$E_B = \bigcup_{n \in \mathbb{N}} nB,$$

is a normed space, whose norm is defined by the gauge function of B.

Now, by taking $E = \mathbb{C}^I$ and $B = \overline{co}(\mu(\mathfrak{M}(E)))$, then, B is closed, absolutely convex and complete subspace of $\mathbb{C}^I$. *If E has bounded spectrum $\mathfrak{M}(E)$*, then, B is bounded (cf. [23, p. 240, (1)]), so by Lemma 4.16, *the set*

$$X = \bigcup_{n \in \mathbb{N}} n \cdot \overline{co}(\mu(\mathfrak{M}(E)))$$

is a Banach subspace of $\mathbb{C}^I$.

5 Relative peak and Choquet points

Relative peak points are related to the relative Choquet points according to the following result, including, in effect, an extension of [10, p. 118, Proposition 3.1.9] for function algebras on a completely regular space and [37, p. 153, Corollary 31.8] for function algebras on a compact space.

Lemma 5.1. *Let E be a unital topological algebra and H a subspace of E, containing the constants. Then,*

$$P_H(E) \subseteq Ch_H(E). \tag{5.1}$$

Proof. Considering $f \in P_H(E)$, there exists $x \in H$, such that $\hat{x}(f) = 1$ and $|\hat{x}| < 1|_{\{f\}^c}$. If $\mu_f \in r_H(f)$, then, $0 = 1 - \hat{x}(f) = 1 - (Re\hat{x})(f) = \mu_f(1_{E^\wedge}) - \mu_f(Re\hat{x}) = \mu_f(1_{E^\wedge} - Re\hat{x})$, with $Re\hat{x} \leq |\hat{x}| \leq 1$, and since $\mu_f \in \mathcal{M}_c^+(\mathfrak{M}(E))$, one gets (cf. [6, p. 69, Proposition 9]) $1_{E^\wedge} - Re\hat{x} = 0|_{Supp(\mu_f)}$. Hence, $Supp(\mu_f) \subseteq Z(1_{E^\wedge} - Re\hat{x}) = \{f\}$, where $Z(1_{E^\wedge} - Re\hat{x})$ stands for the zero set of $1_{E^\wedge} - Re\hat{x}$, so that $\mu_f = \delta_f$, that is, $f \in Ch_H(E)$. ∎

Regarding (5.1), we shall see that, in an appropriate class of topological algebras, it becomes equality (cf. Corollary 5.8 below). As a matter of fact, a more general result is valid, identifying relative weakly peak points with relative Choquet points (cf. Theorem 5.5 below). For this we need some basic results and definitions.

Given a subspace H of a topological algebra E, we consider the vector subspace of $\mathcal{C}(\mathfrak{M}(E), \mathbb{R})$,

$$ReH^\wedge = \{u \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R}) : u = Re\hat{x},\ \ x \in H\}, \tag{5.2}$$

and for a given $f \in \mathfrak{M}(E)$, we define the functions

$$\overline{Q}_f, \underline{Q}_f : \mathcal{C}(\mathfrak{M}(E), \mathbb{R}) \longrightarrow \mathbb{R}, \tag{5.3}$$

by the relations

$$\overline{Q}_f(h) := \inf\{Re\hat{x}(f) : Re\hat{x} \geq h,\ \ x \in H\}, \tag{5.4}$$

and

$$\underline{Q}_f(h) := \sup\{Re\hat{x}(f) : Re\hat{x} \leq h,\ \ x \in H\}, \tag{5.5}$$

for every $h \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R})$. The functions (5.3) fulfill the following properties (cf. [2, p. 100, (2.6)–(2.12)] and [34, p. 20 and 21]):

1) $\overline{Q}_f(h_1 + h_2) \leq \overline{Q}_f(h_1) + \overline{Q}_f(h_2),\ \ h_1, h_2 \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R})$.

2) $\overline{Q}_f(\lambda h) = \lambda \overline{Q}_f(h),\ h \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R}),\ \ \lambda > 0$.

3) If $h_1 \leq h_2$, then, $\overline{Q}_f(h_1) \leq \overline{Q}_f(h_2),\ \ h_1, h_2 \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R})$.

4) $\overline{Q}_f(Re\hat{x}) = Re\hat{x}(f) = \underline{Q}_f(Re\hat{x}),\ \ x \in H$.

5) $-\overline{Q}_f(-h) = \underline{Q}_f(h),\ \ h \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R})$.

A direct consequence of 5) is that

$$\overline{Q}_f(\lambda h) = \lambda \underline{Q}_f(h),\ \ \lambda < 0. \tag{5.6}$$

Furthermore, the functions $\overline{Q}_f, \underline{Q}_f$, $f \in \mathfrak{M}(E)$, are related to the representing measures of f relative to H. As a matter of fact, for every $h \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R})$, the functions $\overline{Q}_f, \underline{Q}_f$ realize the infimum and supremum, respectively, of the values at h of all representing measures of f. This is accomplished by adapting to our context a relevant argument of [24, p. 259, Lemma 9.7.2] or [33, p. 49, Lemma 7.19]. Cf. also [2], [7], [12], [34]. In the proof of the latter, we use that *every positive linear form of $\mathcal{C}_c(X)$, with X completely regular Q-space is continuous*. This is derived from the fact that every positive linear form on $\mathcal{C}_c(X)$, X completely regular, preserves bounded subsets of X, and so becomes continuous, when X is still a Q-space, since then *$\mathcal{C}_c(X)$ is bornological* (cf. [21, p. 284, Theorem 1], [20, p. 220, Proposition 1, (a)], [14, p. 199, Lemma 7.4 and p. 201, Corollary 7.1]). In this respect, we say that a completely regular space X is a *Q-space*, if every character of the algebra $\mathcal{C}_c(X)$ is continuous. (The previous definition of a Q-space, by means of the algebra $\mathcal{C}_c(X)$, is used for convenience, although there is a relevant one for a topological space (*"Hewitt space"*; cf. [35, p. 206, (Q4)])). For example, *every completely regular Lindelöf space is a Q-space* (cf. [29, p. 141, and 142]).

On the basis of the preceding discussion, one obtains the next.

Proposition 5.2. *Let E be a topological algebra, whose spectrum $\mathfrak{M}(E)$ is a Q-space, H a subspace of E, and $f \in \mathfrak{M}(E)$. If $\mu_f \in r_H(f)$, then,*

$$\underline{Q}_f(h) \leq \mu_f(h) \leq \overline{Q}_f(h), \tag{5.7}$$

for every $h \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R})$. Moreover, if $\alpha \in \mathbb{R}$ satisfies

$$\underline{Q}_f(h) \leq \alpha \leq \overline{Q}_f(h), \tag{5.8}$$

then, there exists $\mu_f \in r_H(f)$, such that

$$\alpha = \mu_f(h) = \int_{\mathfrak{M}(E)} h d\mu_f.$$

In particular,

$$\underline{Q}_f(h) = \inf\{\mu_f(h) : \mu_f \in r_H(f)\} \tag{5.9}$$

and

$$\overline{Q}_f(h) = \sup\{\mu_f(h) : \mu_f \in r_H(f)\}. \tag{5.10}$$

We note that the assumption of being $\mathfrak{M}(E)$ a Q-space is not needed for the proof of (5.7). An immediate consequence of (5.9) and (5.10) is the following.

Corollary 5.3. *Let E be a topological algebra with spectrum $\mathfrak{M}(E)$ a Q-space, and H a subspace of E. Then, the following two assertions are equivalent:*

1) $f \in Ch_H(E)$.

2) $\underline{Q}_f(h) = \overline{Q}_f(h)$, for every $h \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R})$ and a given $f \in \mathfrak{M}(E)$.

The next theorem characterizes the relative Choquet boundary $Ch_H(E)$, in terms of weakly peak points and strong boundary points of E, relative to H, generalizing a relevant result of [5, p. 325, Theorem 6.5] (cf. also [7], [24], [25], [33]). We previously need the next result, which completes Lemma 3.3, including, in effect, an extension of [19]; see also [32, p. 138, Lemma (3.3.10)].

Lemma 5.4. *Let E be a unital topological algebra with a subspace H containing the constants. Then, for $f \in M_{\hat{x}}$, with $x \in H$, there exists $B \in \mathcal{P}_H(E)$, such that $f \in B \subseteq M_{\hat{x}}$.*

Proof. Taking $f \in M_{\hat{x}}$, $x \in H$, then, $f(x) \neq 0$ (unless $\hat{x} = 0$), so one can define

$$y = \frac{1}{2}\Big(\frac{x}{f(x)} + 1_E\Big) \in H.$$

By setting

$$B = \{g \in \mathfrak{M}(E) : \hat{y}(g) = 1\},$$

one gets $f \in B \subseteq M_{\hat{x}}$, with $B \in \mathcal{P}_H(E)$ (cf. (4.10), (4.11)). ■

The previous lemma, in conjunction with the fact that every relative peak point is of the form $M_{\hat{x}}$, $\hat{x} \in H^\wedge$ the corresponding peaking function, leads us to the conclusion:

(5.11) *whenever H contains the constants, singletons of the form $M_{\hat{x}}$, $x \in H$, are precisely the relative peak points of E.*

Theorem 5.5. *Let E be a unital topological algebra with spectrum $\mathfrak{M}(E)$ a Q-space, $f \in \mathfrak{M}(E)$, and H a unital subalgebra of E, with $H^\wedge \subseteq E^\wedge$ σ-complete. Moreover, consider the following six assertions:*

1) $f \in Ch_H(E)$.

2) For every open neighbourhood U of f and $0 < \varepsilon < 1$, there exists $x \in H$, such that

$$\|\hat{x}\|_\infty \leq 1, \ \ |\hat{x}(f)| > 1 - \varepsilon \ \ \textit{and} \ \ |\hat{x}| < \varepsilon|_{U^c}. \tag{5.12}$$

3) $f \in s_H(E)$.

4) For every open neighbourhood U of f, there exists $x \in H$, with $f \in M_{\hat{x}} \subseteq U$.

5) For every open neighbourhood U of f, there exists $B \in \mathcal{P}_H(E)$, such that $f \in B \subseteq U$.

6) $f \in P_H^w(E)$.

Then

$$1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 5) \Rightarrow 1) \quad \textit{and} \quad 3) \Rightarrow 6).$$

If, in addition, E satisfies (4.3), then 6)⇒5), as well, so that all the previous assertions are equivalent.

Note. We can equivalently, consider an arbitrary (not necessarily open) neighbourhood of f.

Proof. 1)⇒2): Let $f \in Ch_H(E)$, $0 < \varepsilon < 1$ and U an open neighbourhood of f. Then (cf. [36, p. 55, Urysohn Lemma]), there exists $0 \geq h \in \mathcal{C}(\mathfrak{M}(E), \mathbb{R})$, with $h(f) = 0$ and $h < \log \varepsilon|_{U^c}$, while (Corollary 5.3)

$$0 = h(f) = \delta_f(h) = \underline{Q}_f(h) = \sup\{Re\hat{x}(f) : Re\hat{x} \leq h, \;\; x \in H\}.$$

Since $\log(1 - \varepsilon) < 0 = \underline{Q}_f(h)$, there is $y \in H$, such that $Re\hat{y} \leq h$ and $Re\hat{y}(f) > \log(1 - \varepsilon)$. Then, by the hypothesis for $H^\wedge$, $e^{\hat{y}} = \hat{x} \in H^\wedge$, for some $x \in H$, which satisfies (5.12).

2)⇒3): Assuming 2), and taking an open neighbourhood U of f, we shall find $x \in H$, such that

$$\hat{x}(f) = 1 \;\; \textit{and} \;\; |\hat{x}| < 1|_{U^c}.$$

For this, we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ in H and a sequence $\{U_n\}$ of open neighbourhoods of f, such that

$$\begin{aligned} &(i) \quad U_n \supset U_{n+1} \\ &(ii) \quad \hat{x}_n(f) = 1 \\ &(iii) \quad \|\hat{x}_n\|_\infty < \frac{4}{3} \\ &(iv) \quad |\hat{x}_n| < \frac{1}{3}|_{U_n^c} \\ &(v) \quad \|\hat{x}_j\|_{U_n} \equiv \sup_{g\in U_n} |\hat{x}_j(g)| \leq 1 + \frac{1}{3\cdot 2^n}, \;\; if \;\; j < n. \end{aligned} \tag{5.13}$$

Setting $U_1 = U$, for $\varepsilon = \frac{1}{4}$, there is, by hypothesis, $y_1 \in H$, with $\|\hat{y}_1\|_\infty \leq 1$, $|\hat{y}_1(f)| > \frac{3}{4}$ and $|\hat{y}| < \frac{1}{4}$ on U_1^c. By taking $x_1 = (\hat{y}_1(f))^{-1}y_1 \in H$, we get $\|\hat{x}_1\|_\infty < \frac{4}{3}$, $\hat{x}_1(f) = 1$ and $|\hat{x}_1| < \frac{1}{3}$ on U_1^c. Assuming that $U_1, ..., U_n$ and $x_1, ..., x_n$ have been constructed and satisfy (5.13), we define

$$U_{n+1} = \bigcap_{k=1}^{n} \Big\{g \in \bigcap_{j=1}^{n} U_j : |\hat{x}_k(g)| < 1 + \frac{1}{3\cdot 2^{n+1}}\Big\},$$

which is an open neighbourhood of f, with $U_{n+1} \subseteq U_n$. By 2), there is $y_{n+1} \in H$, with $\|\hat{y}_{n+1}\|_\infty \leq 1$, $|\hat{y}_{n+1}(f)| > \frac{3}{4}$ and $|\hat{y}_{n+1}| < \frac{1}{4}$ on U_{n+1}^c. The function $\hat{x}_{n+1}$, with $x_{n+1} = (\hat{y}_{n+1}(f))^{-1}y_{n+1} \in H$, satisfies (ii)-(v), so we inductively obtain the desired sequences $\{x_n\}$ and $\{U_n\}$. Now, by the σ-completeness of $H^\wedge$, we have

$$\phi = \sum_{n=1}^{\infty} 2^{-n}\hat{x}_n \in H^\wedge, \tag{5.14}$$

so that $\phi = \hat{x}$, for some $x \in H$. By (ii), $\hat{x}(f) = 1$, and if $g \in U^c$, since $\{U_n\}$ is decreasing, $g \in \bigcap_{n=1}^\infty U_n^c$, hence by (iv), $|\hat{x}(g)| < 1$. On the other hand, if $g \in U$, then, for $g \in \bigcap_{n=1}^\infty U_n$, one has by (v), $|\hat{x}_j(g)| \leq 1$, for every $j \in \mathbb{N}$, hence $|\hat{x}(g)| \leq 1$. If $g \notin \bigcap_{n=1}^\infty U_n$, then, $g \in U_m$ and $g \notin U_{m+1}$, for some $m \geq 1$, hence $g \notin U_{m+j}$, $j = 1, 2, ...$ Thus, by (iii)-(v), one gets

$$
\begin{aligned}
|\hat{x}(g)| &\leq \sum_{n=1}^{m-1} 2^{-n}|\hat{x}_n(g)| + 2^{-m}|\hat{x}_m(g)| + \sum_{n=m+1}^{\infty} 2^{-n}|\hat{x}_n(g)| \\
&< \Big(1+\frac{1}{3\cdot 2^m}\Big)\Big(1-\frac{1}{2^{m-1}}\Big) + \frac{4}{3\cdot 2^m} + \frac{1}{3\cdot 2^m} \\
&= 1 - \frac{1}{3\cdot 2^{2m-1}} < 1,
\end{aligned} \tag{5.15}
$$

implying that $f \in s_H(E)$.

3)⇒4): See (2.19).

4)⇒5): Obvious by Lemma 5.4.

5)⇒1): If 5) holds true, there exists $x \in H$, with

$$\hat{x} = 1|_{B\subseteq U} = \|\hat{x}\|_\infty \quad and \quad |\hat{x}| < 1|_{B^c \supseteq U^c}. \tag{5.16}$$

By considering $\mu_f \in r_H(f)$ and applying the same argument as in the proof of Lemma 5.1, one gets (cf. (5.16)) $Supp(\mu_f) \subseteq Z(1_{E^\wedge} - Re\hat{x}) \subseteq U$. Hence $\mu_f(U^c) = 0$, and since μ_f is a probability measure, $\mu_f(U) = \mu_f(\mathfrak{M}(E)) = 1$, thus $\mu_f = \delta_f$, implying $f \in Ch_H(E)$.

3)⇒6): Let $f \in s_H(E)$ and $\{U_\alpha\}_{\alpha\in I}$ a family of open neighbourhoods of f, such that $\bigcap_{\alpha\in I} U_\alpha = \{f\}$. Then, for every $\alpha \in I$, there exists $x_\alpha \in H$, with $\hat{x}_\alpha(f) = 1 = \|\hat{x}_\alpha\|_\infty$ and $|\hat{x}_\alpha| < 1$ on U_α^c, providing the relative peak set $B_\alpha = \{g \in \mathfrak{M}(E) : \hat{x}_\alpha(g) = 1\}$, with peaking function $\hat{y}_\alpha = \frac{1}{2}(1_{E^\wedge} + \hat{x}_\alpha) \in H^\wedge$, so that $f \in B_\alpha \subseteq U_\alpha$. Hence, $f \in \bigcap_{\alpha\in I} B_\alpha \subseteq \bigcap_{\alpha\in I} U_\alpha = \{f\}$, that is $\bigcap_{\alpha\in I} B_\alpha = \{f\}$, yielding that $f \in P_H^w(E)$.

6)⇒5): Obvious by the assumption (4.3). ■

Remark 5.6. In the previous theorem, the assumption for H to be a subalgebra is needed only for the proof 1)⇒2), as well as the property of Q-space for $\mathfrak{M}(E)$. The σ-completeness of $H^\wedge \subseteq E^\wedge$ is used only for the proof of 1)⇒2)⇒3), while the unit is needed for 4)⇒5)⇒1)⇒2) and 3)⇒6).

Corollary 5.7. *Let E be a unital topological algebra, H a subspace of E containing the constants, and $f \in \mathfrak{M}(E)$. Then, the following three assertions are equivalent:*

1) $f \in s_H(E)$.

2) For every neighbourhood U of f, there exists $x \in H$, with $f \in M_{\hat{x}} \subseteq U$.

3) For every neighbourhood U of f, there exists $B \in \mathcal{P}_H(E)$, such that $f \in B \subseteq U$.

Moreover consider the following statement:

4) $f \in P_H^w(E)$.

Then, 1)⇒4), while 4)⇒1) if, for every G_δ-set W, containing an $F \in \mathcal{P}_H^w(E)$, there exists $B \in \mathcal{P}_H(E)$, such that $F \subseteq B \subseteq W$ (in other words, in the case that E satisfies (4.3)).

A direct consequence of Theorem 5.5, along with Corollary 4.4, is the next. See also [32, p. 141, Theorem (3.3.16)] and [33, p. 54, Lemma 7.25] for function algebras on a compact space.

Corollary 5.8. *Let E be a unital topological algebra with spectrum $\mathfrak{M}(E)$ a Q-space, and H a unital subalgebra of E, with $H^\wedge \subseteq E^\wedge$ σ-complete, such that (4.3) be satisfied. Then, one has*

$$Ch_H(E) = s_H(E) = P_H^w(E). \tag{5.17}$$

In particular, if, in addition, $\mathfrak{M}(E)$ satisfies the first axiom of countability, then,

$$Ch_H(E) = s_H(E) = P_H(E). \tag{5.18}$$

The following result provides two more characterizations of the relative Choquet points. See also [31, p. 49, Theorem and p. 53, Corollary 8.3, (3)].

Corollary 5.9. *Let E be a unital topological algebra with spectrum $\mathfrak{M}(E)$ a Lindelöf space, $f \in \mathfrak{M}(E)$, and H a unital subalgebra of E, with $H^\wedge \subseteq E^\wedge$ σ-complete. Then, the following three assertions are equivalent:*

1) $f \in Ch_H(E)$.

2) For every $g \neq f$, there exists $x \in H$, such that $|\hat{x}(g)| < \hat{x}(f) = \|\hat{x}\|_\infty$.

3) For every G_δ-set N, containing f, there is $B \in \mathcal{P}_H(E)$, with

$$f \in B \subseteq N. \tag{5.19}$$

Proof. Assertion 1) is equivalent to 5) according to Theorem 5.5, so we prove that statements 2) and 3) are equivalent to 5).

5)⇒2): If $g \neq f$, there is a neighbourhood U of f, with $g \notin U$, so by 5) one has $f \in B \subseteq U$, for some $B \in \mathcal{P}_H(E)$. Hence, there is $x \in H$, with $\hat{x} = 1|_B = \|\hat{x}\|_\infty$ and $|\hat{x}| < 1|_{B^c \supseteq U^c}$, providing 2).

2)⇒3): Let N be a G_δ-set containing f. Then, $N = \bigcap_{n=1}^\infty U_n$, where $\{U_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of open sets. For each $n \in \mathbb{N}$, we shall find $x_n \in H$, with

$$\hat{x}_n(f) = 1 = \|\hat{x}_n\|_\infty \quad \textit{and} \quad |\hat{x}_n| < 1|_{U_n^c}, \tag{5.20}$$

whence, by setting (cf. (5.14))

$$\phi = \sum_{n=1}^\infty 2^{-n}\hat{x}_n \in H^\wedge \;\; \textit{and} \;\; B = \{g \in \mathfrak{M}(E) : \phi(g) = 1\},$$

we obtain a relative peak set satisfying (5.19). Now, for $n \geq 1$, let $g \in U_n^c$. By hypothesis, there is $y_g \in H$, such that

$$\hat{y}_g(f) = \|\hat{y}_g\|_\infty \quad \textit{and} \quad |\hat{y}_g| < \|\hat{y}_g\|_\infty|_{V_g}, \tag{5.21}$$

where V_g is an open neighbourhood of g. Since $\hat{y}_g(f) \neq 0$, the element

$$x_g = \frac{y_g}{\hat{y}_g(f)} \in H \tag{5.22}$$

fulfills the conditions

$$\hat{x}_g(f) = 1 = \|\hat{x}_g\|_\infty \quad \textit{and} \quad |\hat{x}_g| < 1|_{V_g}. \tag{5.23}$$

The closed subspace U_n^c of $\mathfrak{M}(E)$ is Lindelöf [11, p. 175, Theorem 6.6, (2)], so for its open covering $\{V_g\}_{g\in U_n^c}$, there exists a countable subcovering $\{V_{g_i}\}_{i\in I}$, providing countable many elements $\{x_{g_i}\}_{i\in I}$ as in (5.22). Thus, the element

$$\hat{x}_n = \sum_{i=1}^\infty 2^{-i}\hat{x}_{g_i} \in H^\wedge$$

fulfills the desired condition (5.20), in view also of (5.23).

3)⇒5): Assuming 3) and taking a neighbourhood U of f, there exists an open neighbourhood, that is a G_δ-set, V in U. Hence, by hypothesis, there is $B \in \mathcal{P}_H(E)$, with $f \in B \subseteq V \subseteq U$, yielding 5). ∎

6 Relative Bishop boundary

Given a subspace H of a topological algebra E, a subset A of $\mathfrak{M}(E)$, on which every $\hat{x}$, $x \in H$, attains its maximum absolute value, is called *weakly boundary set of E relative to H*, or *relative weakly boundary set of E*. The set of relative weakly boundary sets of E is denoted by $\mathfrak{A}_H^w(E)$ and its elements are characterized by the property (cf. also (2.12))

$$A \cap M_{\hat{x}} \neq \emptyset, \quad for \;\; every \;\; x \in H. \tag{6.1}$$

It is clear that E is Weierstrass relative to H, whenever $\mathfrak{A}_H^w(E) \neq \emptyset$. The least set of the family $\mathfrak{A}_H^w(E)$ is said to be the *Bishop boundary of E relative to H*, or *relative Bishop boundary of E*, denoted by

$$\mathcal{B}_H(E) = \bigcap_{A \in \mathfrak{A}_H^w(E)} A. \tag{6.2}$$

By adopting the notion of a minimal boundary in [9, p. 131, Defition (4.4)], we say that a relative weakly boundary set is *minimal*, if it contains no proper relative weakly boundary set. Denoting by $\mathfrak{A}_H^{mw}(E)$, the set of minimal relative weakly boundary sets of E, we remark that, if $\mathcal{B}_H(E)$ exists, then,

$$\mathfrak{A}_H^{mw}(E) = \{\mathcal{B}_H(E)\}. \tag{6.3}$$

If $\mathcal{P}_H^m(E)$ stands for *the set of minimal relative peak sets and* $\mathcal{P}_H^{mw}(E)$ for *the family of minimal relative weakly peak sets*, we show that the latter is related to the one of the minimal relative weakly boundary sets, by adapting to our framework a relevant argument of [9, p. 131, Proposition (4.5)] for Banach function algebras on a metrizable space.

Lemma 6.1. *Let E be a unital topological algebra and H a subspace of it, containing the constants, such that $M_{\hat{x}} \neq \emptyset$, for every $x \in H$* (equivalently, take E a Weierstrass algebra relative to H). *Considering the family of (non-empty) minimal relative weakly peak sets of E, $\mathcal{P}_H^{mw}(E)$, then, a collection of exactly one point from each member of $\mathcal{P}_H^{mw}(E)$ constitutes a relative weakly boundary set of E; that is, a member of $\mathfrak{A}_H^w(E)$.*

Proof. Let A be a subset of $\mathfrak{M}(E)$ formed by choosing exactly one point from each member of $\mathcal{P}_H^{mw}(E)$. Taking $x \in H$, then, $M_{\hat{x}}$ contains a relative peak set (cf. Lemma 3.3), hence a minimal relative weakly peak set (cf. (4.9)). Thus, $M_{\hat{x}} \cap A \neq \emptyset$, for every $x \in H$, that is $A \in \mathfrak{A}_H^w(E)$ (cf. (6.1)). ■

A relative weakly boundary set, obtained in the way described by the previous lemma, becomes, under suitable conditions, a minimal one and it is only of this type, as the following result proves.

Proposition 6.2. *Let E be a unital topological algebra and H a subspace of it, containing the constants, such that*

$$\mathcal{P}_H^w(E) = \mathcal{P}_H(E); \tag{6.4}$$

that is, we assume that relative weakly peak sets and peak sets coincide (cf. Corollary 4.5). *Then, the family of minimal weakly boundary sets of E, $\mathfrak{A}_H^{mw}(E)$, consists of subsets of $\mathfrak{M}(E)$, formed by choosing exactly one point from each member of $\mathcal{P}_H^{mw}(E)$.*

Proof. A subset A of $\mathfrak{M}(E)$ of the above form belongs to $\mathfrak{A}^w_H(E)$ by Lemma 6.1. Assuming that A_0 is a proper subset of A, there exists $f \in A \setminus A_0$, so that $f \in B$, for some $B \in \mathcal{P}^{mw}_H(E)$ and $B \cap A_0 = \emptyset$. By hypothesis, $B \in \mathcal{P}^m_H(E)$, thus (cf. (2.15)), B is of the form $M_{\hat{x}}$, $x \in H$, with $M_{\hat{x}} \cap A_0 = \emptyset$, yielding that $A_0 \notin \mathfrak{A}^w_H(E)$, so that $A \in \mathfrak{A}^{mw}_H(E)$.

Conversely, let $A_m \in \mathfrak{A}^{mw}_H(E)$. Based on (6.4), (6.1) and (2.15), A_m intersects every $B \in \mathcal{P}^w_H(E)$, hence every $B \in \mathcal{P}^{mw}_H(E)$. Now, every collection of single points from these intersections gives, due to Lemma 6.1, a relative weakly boundary set contained in A_m, hence, it coincides with A_m. ∎

Based on the preceding, we get at the following *characterization of the existence of the relative Bishop boundary*. See [4, p. 630, Theorem 1] and [9, p. 132, Corollary (4.6)].

Theorem 6.3. *Let E be a topological algebra with subspace H and consider the following assertions:*

1) $P_H(E) = \{f \in \mathfrak{M}(E) : \{f\} = M_{\hat{x}},\ x \in H\} \in \mathfrak{A}^w_H(E)$.

2) The relative Bishop boundary of E, $\mathcal{B}_H(E)$*, exists.*

Then, 1)⇒2), while 2)⇒1) if, in addition, E has an identity, contained in H, and satisfies (6.4).

Proof. Assuming 1), $P_H(E)$ is a relative weakly boundary set contained in, and not containing, any such set (cf. (6.1)), hence $P_H(E) = \mathcal{B}_H(E)$, proving 2). Considering 2), one has $\mathfrak{A}^{mw}_H(E) = \{\mathcal{B}_H(E)\}$, hence, by Proposition 6.2,

$$\mathcal{P}^{mw}_H(E) = P^w_H(E). \tag{6.5}$$

Thus, by Lemma 6.1 and (6.4), we get 1). ∎

In the previous theorem, the condition 1) is fulfilled in the context of Lemma 6.1, when $\mathcal{P}^{mw}_H(E) = P_H(E)$ is valid. We thus obtain the coincidence of the relative Bishop boundary with the relative peak points in view also of Corollary 4.4 and Theorem 4.7.

Corollary 6.4. *Let E be a unital Weierstrass algebra relative to a unital separating subalgebra H, satisfying (4.3), with spectrum* $\mathfrak{M}(E)$ *first countable space, and* $H^\wedge \subseteq E^\wedge$ σ*-complete. Then, the relative Bishop boundary* $\mathcal{B}_H(E)$ *of E exists and consists precisely of the relative peak points* $P_H(E)$ *of E; that is*

$$\mathcal{B}_H(E) = P_H(E). \tag{6.6}$$

Scholium 6.5. By considering $H = E$, since E is automatically separating, we obtain, due to Corollary 6.4, that the Bishop boundary $\mathcal{B}(E)$ of E consists exactly of the peak points $P(E)$ of E, equivalently, of the G_δ-points in $\mathfrak{M}(E)$ of the form $M_{\hat{x}}$, $x \in E$ (cf. (5.11)). That is, we get the relation

$$\mathcal{B}(E) = P(E) \subseteq G_\delta(\mathfrak{M}(E)), \tag{6.7}$$

where $G_\delta(\mathfrak{M}(E))$ stands for the set of G_δ-points in $\mathfrak{M}(E)$. The Bishop boundary we obtain in this way is smaller than the one we get *in the class of Urysohn topological algebras with Gel'fand transform algebra* σ*-complete,* where the G_δ-points of $\mathfrak{M}(E)$

are characterized by the singletons $M_{\hat{x}},\ x \in E$; that is (cf. [15, p. 353, Theorem 3.1 and p. 354, Corollary 3.2])

$$\mathcal{B}(E) = P(E) = G_\delta(\mathfrak{M}(E)). \tag{6.8}$$

On the other hand, given a subspace H of E, we have that

$$\mathcal{B}_H(E) \subseteq \mathcal{B}(E), \tag{6.9}$$

and, of course, by the very definitions

$$\overline{\mathcal{B}_H(E)} = \partial_H(E). \tag{6.10}$$

The following result extends, in our framework, an analogous one due to *H. Bauer* [1], for function algebras; cf. also [5].

Theorem 6.6. *Let E be a unital topological algebra and H a subspace of E, containing the constants, and satisfying the next two assertions:*

1) $M_{\hat{x}} \neq \emptyset$, for every $x \in H$.

2) Every relative peak set of E contains a relative Choquet point; that is, every $B \in \mathcal{P}_H(E)$ contains an $f \in Ch_H(E)$ (cf. Theorem 4.7).

Then, every element of $H^\wedge \subseteq E^\wedge$ attains its maximum absolute value on $Ch_H(E)$; in other words, the relative Choquet boundary of E is a relative weakly boundary set of E. In particular, the closure of $Ch_H(E)$ is a relative boundary set of E.

Proof. By 1) and Lemma 3.3, for every $x \in H$, there is $B \in \mathcal{P}_H(E)$ contained in $M_{\hat{x}}$, with peaking function $\hat{y} = \frac{1}{2}(e^{it}\hat{x} + \alpha)$, where $\alpha = \|\hat{x}\|_\infty$. Hence $\hat{y} = \alpha|_B$, implying that $|\hat{x}| = \alpha|_B$, thus, by 2), the assertion. ∎

An immediate consequence from Theorem 6.6, in conjunction with Corollaries 5.8 and 6.4, is the next result, providing the context into which all the aforesaid boundaries, along with the relative peak points, coincide and become relative weakly boundary sets.

Corollary 6.7. *Let E be a unital Weierstrass algebra relative to a unital separating subalgebra H, with $H^\wedge \subseteq E^\wedge$ σ-complete, satisfying (4.3). Moreover, assume that $\mathfrak{M}(E)$ is a Q-space. Then, the sets*

$$Ch_H(E) = s_H(E) = P_H^w(E) \tag{6.11}$$

are relative weakly boundary sets, so their closures are relative boundary sets. If, moreover, $\mathfrak{M}(E)$ satisfies the first axiom of countability, then,

$$\mathcal{B}_H(E) = P_H(E) = P_H^w(E) = s_H(E) = Ch_H(E), \tag{6.12}$$

hence, their closures are identical with the relative Šilov boundary $\partial_H(E)$ of E.

7 Relative peak sets and boundaries of restriction algebras

Given a subspace H of a topological algebra E and a subset B of its spectrum $\mathfrak{M}(E)$, we examine the peak sets and points of the restriction on B of the Gel'fand transform algebra of E, $E^\wedge|_B \subseteq \mathcal{C}_c(B)$, relative to its vector subspace $H^\wedge|_B$. Concerning the spectrum of the restriction algebra $E^\wedge|_B$, we know that it contains B (up to a canonical injection), while, as we shall see, it is imbedded in the spectrum of E, by means of the *"geometric hull"* of B; that is we have

$$B \cong \mathfrak{M}(\mathcal{C}_c(B)) \underset{\rightarrow}{\subsetneq} \mathfrak{M}(E^\wedge|_B) \underset{\rightarrow}{\subsetneq} (B)_E^H \subseteq \mathfrak{M}(E).$$

In this respect, we define the *geometric* (or *E-convex*) *hull of* $B \subseteq \mathfrak{M}(E)$ *relative to* H, as the set

$$(B)_E^H \equiv E - hull_H(B) = \{f \in \mathfrak{M}(E) : |\hat{x}(f)| \leq p_B(\hat{x}) \equiv \sup_{h \in B} |\hat{x}(h)|,\ x \in H\}, \quad (7.1)$$

being closed, by the continuity of $\hat{x}$, $x \in H$, and satisfying

$$(B)_E^H = (\overline{B})_E^H. \quad (7.2)$$

For $H = E$, the set $(B)_E^E \equiv (B)_E$ is called the *geometric*, or *E-convex, hull of B*, such that one has

$$B \subseteq (B)_E \subseteq (B)_E^H. \quad (7.3)$$

If $(B)_E = B$, B is called *E-convex*, while B is named *E-convex relative to H*, if $(B)_E^H = B$. Thus, in view of (7.3), *an E-convex set relative to H is E-convex*. We note that *every relative peak or relative weakly peak set is E-convex relative to H*: indeed, since every zero set $\hat{x}^{-1}\{0\}$, $x \in H$, is E-convex relative to H, one concludes that every relative peak set of E, being of the form $(\hat{x} - \alpha)^{-1}\{0\}$, $x \in H$, is E-convex relative to H, too. Accordingly, in view also of the relation

$$\Big(\bigcap_{i \in I} A_i\Big)_E^H \subseteq \bigcap_{i \in I} (A_i)_E^H,$$

where $A_i \subseteq \mathfrak{M}(E)$, $i \in I$, one concludes that (*A. Mallios*) *intersections of E-convex sets relative to H are E-convex relative to H, as well.* Thus, *every relative weakly peak set of E is E-convex relative to H*, too.

Now, *the continuity of the Gel'fand map $\mathcal{G}_E$ of E implies the following homeomorphism into* ([14, p. 283, Theorem 1.2]

$$\mathfrak{M}(E^\wedge|_B) \overset{\theta}{\underset{\text{homeo}}{\underset{\rightarrow}{\subsetneq}}} (B)_E \subseteq (B)_E^H \subseteq \mathfrak{M}(E), \quad (7.4)$$

defined by

$$\theta \equiv {}^t(r \circ \mathcal{G}_E), \quad (7.5)$$

where $r : E^\wedge \longrightarrow E^\wedge|_B$, denotes the restriction map and ${}^t(r \circ \mathcal{G}_E)$ stands for the transpose of the map $r \circ \mathcal{G}_E$. Since $(im\theta)_E = (B)_E$, we say that $\mathfrak{M}(E^\wedge|_B)$ *is "E-convex"* (viz. $\theta(\mathfrak{M}(E^\wedge|_B))$ is so) *iff θ is onto. The surjectivity of θ is also attained if B is closed and equicontinuous* [14, p. 283, Theorem 1.2], or *when $E^\wedge|_B$ is a*

Q'-algebra, in the sense that *every maximal regular 2-sided ideal is closed* (see [18] for the terminology applied). *If*, moreover, *H is dense in E, then,*

$$\mathfrak{M}(E^\wedge|_B) \underset{\text{homeo}}{\cong} (B)_E = (B)_E^H. \tag{7.6}$$

In this respect, we note that *the algebras $E^\wedge$, $E^\wedge|_B$ are semi-simple*, in the sense that their respective Gel'fand maps are one-to-one [14, p. 114, Lemma 2.2 and p. 293, Proposition 1.2].

On the basis of (7.4), the relative peak sets of $E^\wedge|_B$ are expected to be related with the respective ones of E. In fact, we have the following (cf. also (2.15)).

Lemma 7.1. *Let E be a topological algebra with continuous Gel'fand map, H a subspace of E, $B \subseteq \mathfrak{M}(E)$, and $\partial_H(E)$ the relative Šilov boundary of E. If $\partial_H(E) \subseteq B$, then* (cf. also (2.12)),

$$M_{\widehat{\hat{x}|_B}} = M_{\hat{x}} \cap \mathfrak{M}(E^\wedge|_B), \;\; x \in H. \tag{7.7}$$

Proof. Since $\partial_H(E) \subseteq B \subsetneqq \mathfrak{M}(E^\wedge|_B) \subsetneqq \mathfrak{M}(E)$, one gets

$$p_{\mathfrak{M}(E)}(\hat{x}) = p_{\partial_H(E)}(\hat{x}) = p_B(\hat{x}|_B) = p_{\mathfrak{M}(E^\wedge|_B)}(\widehat{\hat{x}|_B}), \;\; x \in H,$$

hence the assertion. ∎

The first part of the following result constitutes, in effect, a strengthened version of an inverse of Bishop's Lemma.

Proposition 7.2. *Let E be a topological algebra with continuous Gel'fand map, H a subspace of E, and $B \subseteq \mathfrak{M}(E)$. Then, the trace on $\mathfrak{M}(E^\wedge|_B)$ of every relative peak set of E is a peak set of $E^\wedge|_B$, relative to $H^\wedge|_B$; that is,*

$$A \cap \mathfrak{M}(E^\wedge|_B) \in \mathcal{P}_{H^\wedge|_B}(E^\wedge|_B), \;\; A \in \mathcal{P}_H(E). \tag{7.8}$$

If, in addition, E has an identity contained in H, and $\partial_H(E) \subseteq B$, then, every peak set of $E^\wedge|_B$, relative to $H^\wedge|_B$, contains another one, being the trace on $\mathfrak{M}(E^\wedge|_B)$ of a relative peak set of E.

Proof. Considering $A \in \mathcal{P}_H(E)$, there exists $x \in H$, with

$$\hat{x} = 1|_A = \|\hat{x}\|_\infty \;\; \textit{and} \;\; |\hat{x}| < 1|_{A^c}. \tag{7.9}$$

Then, one has $A \cap \mathfrak{M}(E^\wedge|_B) \subseteq \mathfrak{M}(E^\wedge|_B) \subsetneqq \mathfrak{M}(E)$, with $(A \cap \mathfrak{M}(E^\wedge|_B))^{c'} \equiv \mathfrak{M}(E^\wedge|_B) \setminus (A \cap \mathfrak{M}(E^\wedge|_B)) = A^c \cap \mathfrak{M}(E^\wedge|_B) \subseteq A^c$. Hence, by (7.9) and the semi-simplicity of $E^\wedge|_B$, one obtains

$$\widehat{\hat{x}|_B} = \hat{x}|_B = 1|_{A \cap \mathfrak{M}(E^\wedge|_B)} \;\; \textit{and} \;\; |\hat{x}|_B| < 1|_{(A \cap \mathfrak{M}(E^\wedge|_B))^{c'}},$$

so that (7.8) holds true. Now, if $A \in \mathcal{P}_{H^\wedge|_B}(E^\wedge|_B)$, then (Lemma 7.1 and (2.15)), $A = M_{\widehat{\hat{x}|_B}} = M_{\hat{x}} \cap \mathfrak{M}(E^\wedge|_B)$. Thus (Lemma 3.3), there exists $A' \in \mathcal{P}_H(E)$, such that $A' \subseteq M_{\hat{x}}$. By the hypothesis for $\partial_H(E)$ and (2.16), $\emptyset \neq A' \cap \mathfrak{M}(E^\wedge|_B) \subseteq M_{\hat{x}} \cap \mathfrak{M}(E^\wedge|_B) = A$, where, according to (7.8), $A' \cap \mathfrak{M}(E^\wedge|_B) \in \mathcal{P}_{H^\wedge|_B}(E^\wedge|_B)$, proving the assertion. ∎

By taking relative weakly peak sets, we obtain an analogous result with Proposition 7.2, while for minimal relative weakly peak sets one gets at the following characterization.

Corollary 7.3. *Let E be a unital topological algebra with continuous Gel'fand map, H a subspace of E, containing the constants, $B \subseteq \mathfrak{M}(E)$, and $\partial_H(E)$ the relative Šilov boundary of E, such that $\partial_H(E) \subseteq B$. Then, the minimal weakly peak sets of $E^\wedge|_B$, relative to $H^\wedge|_B$, are exactly the traces on $\mathfrak{M}(E^\wedge|_B)$ of the minimal relative weakly peak sets of E. That is,*

$$\mathcal{P}^{mw}_{H^\wedge|_B}(E^\wedge|_B) = \mathcal{P}^{mw}_H(E) \cap \mathfrak{M}(E^\wedge|_B). \tag{7.10}$$

By employing the argument of [9, p. 135, Proposition (5.2)], suitably adapted to the present framework, we prove the invariance of the boundaries considered, when restricting the Gel'fand transform algebra of the topological algebra involved, to a subspace of the spectrum of the latter algebra, containing its Šilov boundary. Thus, we obtain the next.

Theorem 7.4. *Let E be a topological algebra with continuous Gel'fand map, H a subspace of E, $B \subseteq \mathfrak{M}(E)$, and $\partial_H(E)$ the Šilov boundary of E, such that $\partial_H(E) \subseteq B$. Then, if the relative Šilov and Bishop boundaries for E and $E^\wedge|_B$ exist, one has*

$$\partial_{H^\wedge|_B}(E^\wedge|_B) = \partial_H(E), \tag{7.11}$$

and

$$\mathcal{B}_{H^\wedge|_B}(E^\wedge|_B) = \mathcal{B}_H(E). \tag{7.12}$$

Moreover, if the analogous relation to (6.4) holds true for both E and $E^\wedge|_B$, which also have identities contained in the respective subspaces, then,

$$P_{H^\wedge|_B}(E^\wedge|_B) = P_H(E). \tag{7.13}$$

Proof. By (6.1), the hypothesis for B and Lemma 7.1, we get that $\partial_H(E)$ is a boundary set of $E^\wedge|_B$ relative to $H^\wedge|_B$, therefore $\partial_{H^\wedge|_B}(E^\wedge|_B) \subseteq \partial_H(E)$, which implies that $\partial_{H^\wedge|_B}(E^\wedge|_B)$ is a relative boundary set of E, so that $\partial_H(E) \subseteq \partial_{H^\wedge|_B}(E^\wedge|_B)$. Finally, since $\mathcal{B}_H(E) \subseteq \partial_H(E) \subseteq B \subseteq \mathfrak{M}(E^\wedge|_B)$, by a similar argument, we obtain (7.12), equivalently (7.13), according to our hypothesis (see also Theorem 6.3). ∎

Remark 7.5. Concerning Choquet boundaries in the point of view of the above Theorem 7.4, one can have an analogous information for such boundaries, as an application of Corollary 5.8, provided the framework of the initial algebra E is still valid for the restriction algebra $E^\wedge|_B$ [8]. The same argument holds for the strong boundaries too; however, in this concern see Theorems 7.6 and 7.9 below.

Theorem 7.6. *Let E be a topological algebra with continuous Gel'fand map, H a subspace of E, $B \subseteq \mathfrak{M}(E)$, and $Ch_{H^\wedge|_B}(E^\wedge|_B)$ the relative Choquet boundary of $E^\wedge|_B$. Then,*

$$Ch_{H^\wedge|_B}(E^\wedge|_B) \subseteq Ch_H(E). \tag{7.14}$$

If, moreover, the map θ, as in (7.4), is proper (:the inverse image of a compact set is compact)*, and $Ch_H(E) \subseteq \mathfrak{M}(E^\wedge|_B)$, then, one gets*

$$Ch_{H^\wedge|_B}(E^\wedge|_B) = Ch_H(E). \tag{7.15}$$

Proof. It follows from [17, Theorem 3.4]. ■

For a similar result, referring to the relative strong boundary, we need the following characterization about relative strong boundary points (cf. also Corollary 5.7).

Lemma 7.7. *Let E be a unital topological algebra, H a subspace containing the constants, and* $f \in \mathfrak{M}(E)$. *Moreover, consider the next two assertions:*

1) $f \in s_H(E)$.

2) $\{f\} \in \mathcal{P}_H^{mw}(E)$.

Then, 1)⇒2), while 2)⇒1) if the following holds true:

(7.16) *for every* G_δ*-set W containing an* $F \in \mathcal{P}_H^{mw}(E)$, *there exists* $B \in \mathcal{P}_H(E)$, *with* $F \subseteq B \subseteq W$.

Scholium 7.8. On the basis of Lemma 7.7, we can apply condition (7.16) in place of (4.3), to gain an appropriate extension of Theorem 5.5. As a consequence, along with a similar extension of Lemma 4.6, we also obtain a strengthening of Theorem 4.7. Thus, by still applying (7.16), in place of (4.3), Corollaries 4.11 and 4.13 assert that a minimal weakly peak set F of E, relative to H, can not genuinely contain any peak set of $E^\wedge|_F$, relative to $H^\wedge|_F$. Accordingly, *Bishop's Lemma*, under the same modification, *yields that a minimal weakly peak set F of E relative to H, does not genuinely contain any weakly peak set G of* $E^\wedge|_F$, *relative to* $H^\wedge|_F$, *such that* $G = \bigcap_{i \in I} B_i$, *with* $B_i \subseteq F,\ i \in I$.

By the preceding discussion we get at the following.

Theorem 7.9. *Let E be a unital topological algebra with continuous Gel'fand map, H a unital subalgebra with* $H^\wedge$ σ*-complete subalgebra of* $E^\wedge$, $B \subseteq \mathfrak{M}(E)$, *and* $\partial_H(E)$ *the relative Šilov boundary of E, such that* $\partial_H(E) \subseteq B$, *while we further assume (7.16). Then, one has*

$$s_H(E) = s_{H^\wedge|_B}(E^\wedge|_B). \tag{7.17}$$

Proof. By hypothesis and (2.19) we have $s_H(E) \subseteq \partial_H(E) \subseteq B \subseteq \mathfrak{M}(E^\wedge|_B)$. Now, let $f \in s_H(E)$ and U an open neighbourhood of f in $\mathfrak{M}(E^\wedge|_B)$. Then, $U = V \cap \mathfrak{M}(E^\wedge|_B)$, with V open neighbourhood of f in $\mathfrak{M}(E)$, and there exists (Corollary 5.7) $B \in \mathcal{P}_H(E)$, such that $f \in B \subseteq V$. Hence, $f \in B \cap \mathfrak{M}(E^\wedge|_B) \subseteq V \cap \mathfrak{M}(E^\wedge|_B) = U$, where (Proposition 7.2) $B \cap \mathfrak{M}(E^\wedge|_B) \in \mathcal{P}_{H^\wedge|_B}(E^\wedge|_B)$, so that $f \in s_{H^\wedge|_B}(E^\wedge|_B)$.

Conversely, if $f \in s_{H^\wedge|_B}(E^\wedge|_B)$, equivalently (Lemma 7.7) $\{f\} \in \mathcal{P}^{mw}_{H^\wedge|_B}(E^\wedge|_B) = \mathcal{P}_H^{mw}(E) \cap \mathfrak{M}(E^\wedge|_B)$ (Corollary 7.3). Thus, $\{f\} = A \cap \mathfrak{M}(E^\wedge|_B)$, with $A \in \mathcal{P}_H^{mw}(E)$, hence, by Bishop's Lemma (Corollary 4.14) and Scholium 4.15, $\{f\} \in \mathcal{P}_H^{w}(E)$. So $\{f\} = A$, therefore $f \in s_H(E)$ in view of Lemma 7.7. ■

Remark 7.10. In the previous theorem, the assumptions of H being a subalgebra with $H^\wedge \subseteq E^\wedge$ σ-complete and of the validity of (7.16) are needed only for the proof of the implication $s_{H^\wedge|_B}(E^\wedge|_B) \subseteq s_H(E)$.

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