Slant helices in the Euclidean 3-space revisited

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Abstract

In this paper, we study the surfaces whose geodesics are slant curves. We show that a unit speed curve γ in the 3-dimensional Euclidean space is a slant helix if and only if it is a geodesic of a helix surface. We prove that the striction line of a helix surface is a general helix; as a consequence, slant helices are characterized as geodesics of the tangent surface of a general helix. Finally, we provide two methods for constructing slant helices in helix surfaces.

1 Introduction

The study of curves of constant slope is a well established topic in differential geometry. Two families of curves of constant slope stand out above the rest: general helices and slant helices. Recall that general helices are defined by the property that their tangent makes a constant angle with a fixed direction in every point. In a similar way, slant helices are defined by the property that their principal normals make a constant angle with a fixed direction. The term *slant helix* was introduced by Izumiya and Takeuchi, [13]; however, examples of this kind of curves were studied in the past (see e.g. Salkowski curves, [22, 19], or curves of constant precession, [23]). General helices and slant helices are deeply interrelated; observe that the principal normal lines of a general helix are perpendicular to a fixed direction, so that a general helix is also a slant helix. On the other hand, it is well known that the tangent indicatrix and the binormal indicatrix of a slant helix are spherical helices, [13, 12]. Another beautiful relation between

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general and slant helices is given in [17], by using the concept of the successor transformation of Frenet curves: in this transformation, the tangent vector of the first curve plays the role of principal normal vector of the second curve. General helices are the successor curves of plane curves, and slant helices are the successor curves of general helices.

A classical result stated by M.A. Lancret in 1802 and first proved by B. De Saint Venant in 1845 (see [24]) is that a unit speed curve α is a general helix if and only if $\tau_\alpha/\kappa_\alpha$ is constant, where κ_α and τ_α stand for the curvature and torsion of α. It is shown in [13] a similar result for slant helices: a necessary and sufficient condition for a curve α with $\kappa_\alpha > 0$ to be a slant helix is that the function

$$\sigma = \frac{\kappa_\alpha^2}{(\kappa_\alpha^2 + \tau_\alpha^2)^{3/2}} \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'$$

be constant. Observe that $\sigma \equiv 0$ implies that α is a general helix. Somehow, $|\sigma|$ measures how much the curve moves away from being a general helix, and $|\sigma'|$ measures how much the curve moves away from being a slant helix.

General helices have a nice characterization as geodesics of cylinders (and for this reason they are also called cylindrical helices). In fact, it can be proved that every geodesic of a cylinder shaped over a plane curve is a general helix, and that every general helix can be obtained in this way. In other words, cylinders are the surfaces whose geodesics are general helices.

A topic that has attracted the interest of many mathematicians in recent years is the surfaces of constant slope or helix surfaces. They were introduced in [3] as the surfaces in $\mathbb{S}^2 \times \mathbb{R}$ for which the unit normal makes a constant angle with the tangent direction to $\mathbb{R}$. Later, this kind of surfaces have been extended to other ambient spaces (see e.g. [5, 18, 6, 9, 21, 20, 4]). Although these surfaces are interesting from a mathematical point of view, this kind of surfaces have also been studied in other areas. For example, the applications of these surfaces in the theory of liquid crystals and of layered fluids were considered in [2]; and these surfaces are of interest in the shape-from-shading problem (see [14]), which arises when recovering a shape from a single image (this is a classical problem in computer vision, [25, 8]). Related with this, the isophotic curves play an important role in visual psychophysics and vision theory, [11, 15]. Isophotic curves on a surface can be identified with the curves such that the surface normals along the curve make a constant angle with a fixed direction, [16]. Then isophotic curves are related with slant helices, [10]. The concept of constant angle surface has been extended to higher dimensions in [7], where the authors define the helix submanifolds as those whose tangent spaces make a constant angle with a fixed direction.

This paper is organized as follows. In section 2 we give some preliminaries and basic facts about Frenet curves and helix surfaces in $\mathbb{R}^3$; in particular, every helix surface is a ruled surface. In section 3 we study the relation between slant helices and helix surfaces, and prove that a curve γ in $\mathbb{R}^3$ is a slant helix if and only if γ is a geodesic of a helix surface (Theorem 3). As an application of this result we obtain the following characterization of the curves of constant precession: a curve γ in $\mathbb{R}^3$ is a curve of constant precession if and only if it is a geodesic with linear slope of a helix surface generated by an epicycloid or hypocycloid. In section 4 we prove that the striction line of a helix surface is a general helix, and the helix surface is nothing but the tangent surface of this general helix. Consequently, a curve γ in $\mathbb{R}^3$ is a slant helix if and only if γ is a geodesic of the tangent surface of a general helix (Theorem 7). We finish this section giving two methods to construct slant helices in helix surfaces.

2 Preliminaries

Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve parametrized by an arbitrary parameter t. At each point of α where $\alpha'(t) \times \alpha''(t) \neq 0$, the Frenet frame $\{T_\alpha(t), N_\alpha(t), B_\alpha(t)\}$ is defined by

$$T_\alpha = \frac{\alpha'}{||\alpha'||}, \qquad N_\alpha = \frac{\alpha' \times \alpha''}{||\alpha' \times \alpha''||} \times T_\alpha, \qquad B_\alpha = \frac{\alpha' \times \alpha''}{||\alpha' \times \alpha''||}.$$

The geometry of the curve α is essentially encoded in its invariants, curvature κ_α and torsion τ_α , which can be computed by

$$\kappa_\alpha = \frac{||\alpha' \times \alpha''||}{||\alpha'||^3} \qquad \text{and} \qquad \tau_\alpha = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{||\alpha' \times \alpha''||^2},$$

where $\langle , \rangle$ denotes the usual inner product of $\mathbb{R}^3$. These invariants represent basically the rate of change of the tangent vector and the osculating plane, respectively. When α is parametrized by the arclength s, the variation of the Frenet frame is described by the usual Frenet-Serret equations:

$$\begin{aligned} T'_\alpha(s) &= \nabla^0_{T_\alpha} T_\alpha(s) = \kappa_\alpha(s)\, N_\alpha(s), \\ N'_\alpha(s) &= \nabla^0_{T_\alpha} N_\alpha(s) = -\kappa_\alpha(s)\, T_\alpha(s) + \tau_\alpha(s)\, B_\alpha(s), \\ B'_\alpha(s) &= \nabla^0_{T_\alpha} B_\alpha(s) = -\tau_\alpha(s)\, N_\alpha(s), \end{aligned} \tag{1}$$

where ∇^0 denotes the Levi-Civita connection on $\mathbb{R}^3$.

A submanifold M of a Euclidean space $\mathbb{R}^n$ is said to be a *helix submanifold* if there is a fixed direction $\mathbf{u} \in \mathbb{R}^n$ such that the tangent space of the submanifold makes a constant angle with $\mathbf{u}$, [7]. In the case of hypersurfaces with unit normal vector N, this definition is equivalent to the fact that the angle φ between N and $\mathbf{u}$ is a constant function along M. A helix hypersurface $M \subset \mathbb{R}^n$ of angle φ will be denoted by M_φ. Without loss of generality, we can assume $\varphi \in [0, \pi/2]$.

The construction of the Euclidean helix hypersurfaces is made in [7]. Let $H \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ be an orientable hypersurface and let η be a unit normal vector field. Let $\mathbf{u} = (0, \ldots, 0, 1)$ and define the vector field $T(x) = \cos\varphi\ \eta(x) + \sin\varphi\ \mathbf{u}$, where $x \in H$ and $\varphi \in \mathbb{R}$ is constant. For an appropriate small ε, define an immersion $f_\varphi : M = H \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ by $f_\varphi(x, z) = x + z\, T(x)$. Then we have the following result (theorems 2.4 and 2.7 of [7]): *The immersed hypersurface $f_\varphi(M)$ is a helix hypersurface of angle φ, and each helix hypersurface $M_\varphi \subset \mathbb{R}^n$ is locally obtained in this way.*

Take a unit speed plane curve $\beta : I \subset \mathbb{R} \to \mathbb{R}^2 \subset \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, with curvature $\kappa_\beta(t)$ and Frenet frame $\{T_\beta(t), N_\beta(t)\}$. Then we have the Frenet equations of the curve β,

$$\begin{aligned} T'_\beta(t) &= \kappa_\beta(t)\, N_\beta(t), \\ N'_\beta(t) &= -\kappa_\beta(t)\, T_\beta(t). \end{aligned}$$

The helix surface $M = M_\varphi$ built on the curve β is parametrized by

$$X(t,z) = \beta(t) + z\,(\cos\varphi\, N_\beta(t) + \sin\varphi\, \mathbf{u}).$$

To emphasize the dependence of both the angle φ and the curve β, sometimes we will use the notation $M_{\beta,\varphi}$.

The tangent frame is given by

$$\begin{aligned} X_t(t,z) &= (1 - \cos\varphi\, \kappa_\beta(t)\, z)T_\beta(t), \\ X_z(t,z) &= \cos\varphi\, N_\beta(t) + \sin\varphi\, \mathbf{u}, \end{aligned}$$

and then the unit normal vector field is given by

$$N(t,z) = \frac{X_t \times X_z}{||X_t \times X_z||} = -\sin\varphi\, N_\beta(t) + \cos\varphi\, \mathbf{u}. \tag{2}$$

The shape operator $A(t,z) = -dN(t,z)$ at a point $X(t,z)$ is given by

$$A(t,z) = \begin{pmatrix} \dfrac{\sin\varphi\, \kappa_\beta(t)}{1 - \cos\varphi\, \kappa_\beta(t)\, z} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the Gaussian and mean curvatures (K and H) of the helix surface are given by

$$K = 0 \qquad \text{and} \qquad H = \frac{\sin\varphi\, \kappa_\beta(t)}{2(1 - \cos\varphi\, \kappa_\beta(t)\, z)}.$$

3 Slant helices and helix surfaces

In this section we study the geodesics $\gamma(s)$ of a helix surface M_φ. We have two distinguished cases according to the angle φ:

(i) $\varphi = 0$. Then M_φ is a plane (or an open piece of a plane), and γ is a straight line.

(ii) $\varphi = \pi/2$. Then M_φ is a cylinder, and γ is a general helix (or cylindrical helix).

Without loss of generality, from now on we assume that $\varphi \in (0, \pi/2)$.

Let $\gamma(s) = X(t(s), z(s))$ be a unit speed geodesic in M_φ, with $\kappa_\gamma > 0$, then the unit tangent is given by

$$T_\gamma(s) = t'(s)(1 - \cos\varphi\, \kappa_\beta(t(s))\, z(s))T_\beta(t(s)) + z'(s)(\cos\varphi\, N_\beta(t(s)) + \sin\varphi\, \mathbf{u}).$$

Then there is a differentiable function $\theta(s)$ such that

$$\begin{aligned} t'(s)(1 - \cos\varphi\, \kappa_\beta(t(s))\, z(s)) &= \sin\theta(s), \\ z'(s) &= \cos\theta(s). \end{aligned}$$

The function $\theta(s)$, called the slope function of the geodesic, is nothing but the angle function between the geodesic and the director vector at $\gamma(s)$. From the Frenet equations of γ, (1), we deduce

$$\begin{aligned} \kappa_\gamma(s) N_\gamma(s) = \cos\theta(s)\, &(\theta'(s) - \cos\varphi\, t'(s)\, \kappa_\beta(t(s)))\, T_\beta(t(s)) \\ &+ \sin\theta(s)\, (t'(s)\, \kappa_\beta(t(s)) - \cos\varphi\, \theta'(s))\, N_\beta(t(s)) \\ &- \sin\varphi\, \theta'(s)\, \sin\theta(s)\, \mathbf{u}. \end{aligned} \tag{3}$$

Since $\gamma(s)$ is a geodesic, then $N_\gamma(s)$ is orthogonal to the surface M, so we can assume

$$N_\gamma(s) = N(t(s), z(s)). \tag{4}$$

(The case $N_\gamma(s) = -N(t(s), z(s))$ is similar.) From the equations (2) and (3) we get

$$\cos\theta(s)\, (\theta'(s) - \cos\varphi\, t'(s)\, \kappa_\beta(t(s))) = 0, \tag{5}$$

$$\sin\theta(s)\, (t'(s)\, \kappa_\beta(t(s)) - \cos\varphi\, \theta'(s)) = -\sin\varphi\, \kappa_\gamma(s), \tag{6}$$

$$-\sin\varphi\, \theta'(s)\, \sin\theta(s) = \cos\varphi\, \kappa_\gamma(s). \tag{7}$$

From (5) we deduce

$$\theta'(s) - \cos\varphi\, t'(s)\, \kappa_\beta(t(s)) = 0; \tag{8}$$

otherwise, $\cos\theta(s) = 0$ and then the function $\theta(s)$ is constant. Hence (7) yields $\cos\varphi\, \kappa_\gamma(s) = 0$, a contradiction.

It is easy to see that the binormal vector $B_\gamma(s)$ is given by

$$\begin{aligned} B_\gamma(s) &= T_\gamma(s) \times N_\gamma(s) = \\ &\cos\theta(s)\, T_\beta(t(s)) - \cos\varphi \sin\theta(s)\, N_\beta(t(s)) - \sin\varphi \sin\theta(s)\, \mathbf{u}. \end{aligned} \tag{9}$$

By taking covariant derivative in (4) we get

$$N'_\gamma(s) = \sin\varphi\, t'(s)\, \kappa_\beta(t(s))\, T_\beta(t(s)). \tag{10}$$

Now, again from the Frenet equations of γ, and bearing (9) in mind, we obtain

$$\begin{aligned} N'_\gamma(s) &= -\kappa_\gamma(s)\, T_\gamma(s) + \tau_\gamma(s)\, B_\gamma(s) \\ &= (-\kappa_\gamma(s)\, \sin\theta(s) + \tau_\gamma(s)\, \cos\theta(s))\, T_\beta(t(s)) \\ &\quad - \cos\varphi\, (\kappa_\gamma(s)\, \cos\theta(s) + \tau_\gamma(s)\, \sin\theta(s))\, N_\beta(t(s)) \\ &\quad - \sin\varphi\, (\kappa_\gamma(s)\, \cos\theta(s) + \tau_\gamma(s)\, \sin\theta(s))\, \mathbf{u}. \end{aligned} \tag{11}$$

From (10) and (11) we get

$$-\kappa_\gamma(s)\, \sin\theta(s) + \tau_\gamma(s)\, \cos\theta(s) = \sin\varphi\, t'(s)\, \kappa_\beta(t(s)), \tag{12}$$

$$\kappa_\gamma(s)\, \cos\theta(s) + \tau_\gamma(s)\, \sin\theta(s) = 0. \tag{13}$$

Finally, from (7), (8) and (13) we deduce that the curvature and torsion of the geodesic $\gamma(s)$ are given by

$$\kappa_\gamma(s) = -t'(s)\,\sin\varphi\,\sin\theta(s)\,\kappa_\beta(t(s)) = -\tan\varphi\,\theta'(s)\,\sin\theta(s), \tag{14}$$

$$\tau_\gamma(s) = t'(s)\,\sin\varphi\,\cos\theta(s)\,\kappa_\beta(t(s)) = \tan\varphi\,\theta'(s)\,\cos\theta(s). \tag{15}$$

We have proved the following result.

Proposition 1 *A unit speed curve $\gamma(s) = X(t(s), z(s))$, with $\kappa_\gamma > 0$, is a geodesic in the helix surface M_φ if and only if there is a differentiable function $\theta(s)$ such that*

$$t'(s)(1 - \cos\varphi\,\kappa_\beta(t(s))\,z(s)) = \sin\theta(s), \tag{16}$$

$$z'(s) = \cos\theta(s), \tag{17}$$

$$\cos\varphi\,t'(s)\,\kappa_\beta(t(s)) = \theta'(s). \tag{18}$$

Moreover, the curvature and torsion of γ are given by (14) and (15), respectively.

Example 1 (Geodesics of a circular cone) If $\kappa_\beta(t)$ is a nonzero constant κ_0, then β is a circle of radius $1/\kappa_0$, M_φ is a circular cone, and the geodesics γ of M_φ are rectifying curves, [1, 13]. We obtain the exact parametrizations of these curves by using Proposition 1.

Since $\beta(t) = \frac{1}{\kappa_0}(\cos(\kappa_0 t), \sin(\kappa_0 t), 0)$, the parametrization of the cone M_φ is given by

$$X(t,z) = \Big(\Big(\frac{1}{\kappa_0} - z\cos\varphi\Big)\cos(\kappa_0 t), \Big(\frac{1}{\kappa_0} - z\cos\varphi\Big)\sin(\kappa_0 t), z\sin\varphi\Big).$$

By using Proposition 1, $\gamma(s) = X(t(s), z(s))$ is a geodesic of the cone M_φ if and only

$$t(s) = \frac{1}{\kappa_0\cos\varphi}\big(\operatorname{arccot}(as+b) - t_0\big),$$

$$z(s) = \frac{1}{a}\sqrt{(as+b)^2+1} + \frac{1}{\kappa_0\cos\varphi},$$

where a, b and t_0 are constants. Therefore, the coordinates of the curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ are given by

$$\gamma_1(s) = -\frac{1}{a}\cos\varphi\,\sqrt{(as+b)^2+1}\,\cos\big(\sec\varphi\,(\operatorname{arccot}(as+b) - t_0)\big),$$

$$\gamma_2(s) = -\frac{1}{a}\cos\varphi\,\sqrt{(as+b)^2+1}\,\sin\big(\sec\varphi\,(\operatorname{arccot}(as+b) - t_0)\big),$$

$$\gamma_3(s) = \frac{1}{a}\sin\varphi\,\sqrt{(as+b)^2+1} + \frac{\tan\varphi}{\kappa_0}.$$

From (14) and (15), the curvature and torsion of γ are given by

$$\kappa_\gamma(s) = a\tan\varphi\,((as+b)^2+1)^{-3/2},$$

$$\tau_\gamma(s) = -a\tan\varphi\,(as+b)\,((as+b)^2+1)^{-3/2},$$

and therefore $\tau_\gamma/\kappa_\gamma = -(as+b)$, i.e. $\gamma(s)$ is a rectifying curve, [1].

The curve γ is a slant helix if the principal normal lines of γ make a constant angle φ with a fixed direction, [13]. The case $\varphi = \pi/2$ corresponds with the general helices. In Example 1, we provide examples of geodesics in a circular cone, that are at the same time rectifying curves and slant helices.

The curvature and torsion of a slant helix (not a general helix) can be described in a nice way.

Proposition 2 (cf. [17]) *Let $\gamma(s)$ be a unit speed curve fully in $\mathbb{R}^3$ and suppose it is not a general helix. Then γ is a slant helix if and only if, up to reflections, there exist a nonzero constant λ and a differentiable non-constant function $\theta(s)$ such that*

$$\kappa_\gamma(s) = -\lambda\,\theta'(s)\,\sin\theta(s),$$
$$\tau_\gamma(s) = \lambda\,\theta'(s)\,\cos\theta(s).$$

It is well known that the geodesics $\gamma(s)$ of M_φ are slant helices, [21]. In fact, from (4) and (2) we obtain that $\langle N_\gamma(s), \mathbf{u}\rangle = \cos\varphi$ is a nonzero constant, where $\mathbf{u}$ is a unit vector normal to $\mathbb{R}^2$. This shows that γ is a slant helix.

We now prove the converse.

Theorem 3 *Let $\gamma(s)$ be a unit speed curve fully in $\mathbb{R}^3$ and suppose it is not a general helix. If γ is a slant helix, then there is a plane curve β and an angle $\varphi \in (0, \pi/2)$ such that γ is (congruent to) a geodesic of the helix surface $M_{\beta,\varphi}$.*

Proof. Let $\gamma(s)$ be an arclength parametrized slant helix (not a general helix), with $\kappa_\gamma > 0$. Let λ and $\theta(s)$ be the nonzero constant and the non-constant differentiable function given in Proposition 2. Define $\varphi = \arctan(\lambda)$ and let $z(s)$ be a solution of (17). We distinguish two cases:
Case 1: $\theta'\,z + \sin\theta = 0$. By using (17) we deduce

$$\theta''\,\sin\theta - 2\,\theta'^2\,\cos\theta = 0,$$

and then $\theta(s) = -\operatorname{arccot}(a(s+b))$, for certain constants a and b. This implies that $\tau_\gamma/\kappa_\gamma$ is a linear function on the arclength parameter s, i.e. γ is a geodesic of a cone, [13].
Case 2: $\theta'\,z + \sin\theta \neq 0$. Let $t(s)$ be a solution of the differential equation

$$t'(s) = \theta'(s)\,z(s) + \sin\theta(s) \neq 0.$$

Define the function

$$\kappa(u) = \frac{1}{\cos\varphi}\left(\frac{\theta'}{t'}(t^{-1}(u))\right),$$

and consider a plane curve β with curvature κ. Now we can construct a helix surface $M_{\beta,\varphi}$ parametrized by $X(t,z) = \beta(t) + z\,(\cos\varphi\,N_\beta(t) + \sin\varphi\,\mathbf{u})$. It is a straightforward computation to show that the curve $\tilde{\gamma}(s) = X(t(s), z(s))$ is a geodesic in $M_{\beta,\varphi}$ whose curvature and torsion are precisely $\kappa_\gamma(s)$ and $\tau_\gamma(s)$. This shows that γ and $\tilde{\gamma}$ are congruent. ∎

Example 2 (Curves of constant precession) Let $\beta(u) = (x(u), y(u))$ be the curve in $\mathbb{R}^2$ defined by

$$x(u) = (a+b)\cos u - b\cos\left(\frac{a+b}{b}u\right),$$
$$y(u) = (a+b)\sin u - b\sin\left(\frac{a+b}{b}u\right),$$

for two nonzero constants a and b, [24, pp. 27–28]. If $ab > 0$, then β is an epicycloid; otherwise, β is a hypocycloid. The curve is closed when a/b is rational.

The arc-length parameter t is given by

$$t = t(u) = \frac{4b(a+b)}{a}\cos\left(\frac{au}{2b}\right),$$

and the curvature of β is computed as

$$\kappa_\beta(t) = \frac{a+2b}{4b(a+b)}\frac{1}{\sqrt{1-\left(\frac{at}{4b(a+b)}\right)^2}}.$$

To find geodesics $\gamma(s)$ in a helix surface $M_{\beta,\varphi}$ (and so slant helices) we need to solve the ODE system given in Proposition 1. It is not difficult to check that if we define

$$\mu = \frac{a}{2b(a+b)} \qquad \text{and} \qquad \varphi = \arccos\left(\frac{a}{a+2b}\right),$$

then a solution of (16)–(18) is given by the following functions:

$$\theta(s) = \mu s, \qquad t(s) = -\frac{2}{\mu}\cos(\mu s), \qquad z(s) = \frac{1}{\mu}\sin(\mu s). \tag{19}$$

In this case, the curvature and torsion of these slant helices can be computed by using (14) and (15):

$$\kappa_\gamma(s) = -\mu\tan(\varphi)\,\sin(\mu s), \qquad \tau_\gamma(s) = \mu\tan(\varphi)\,\cos(\mu s).$$

Hence, $\gamma(s)$ is a curve of constant precession, [23]. See Figure 1.

Conversely, every curve of constant precession is a geodesic of a helix surface $M_{\beta,\varphi}$, β being an epicycloid or a hypocycloid. In fact, let $\gamma(s)$ be a curve of constant precession; then there exist two nonzero constants ω and μ such that $\kappa_\gamma(s) = -\omega\,\sin(\mu s)$ and $\tau_\gamma(s) = \omega\,\cos(\mu s)$, [23]. Take $\varphi = \arctan(\omega/\mu)$ and define a and b by

$$a = \frac{2}{\mu}\cot^2(\varphi), \qquad b = \frac{\cos\varphi}{\mu(1+\cos\varphi)}.$$

Let β be the corresponding epicycloid or hypocycloid, according to the sign of ab. It is a straightforward computation to check that $\gamma(s)$ is congruent to the geodesic of the surface $M_{\beta,\varphi}$ determined by (19). Hence we have proved the following characterization of curves of constant precession.

Proposition 4 *Let $\gamma(s)$ be a unit speed curve fully in $\mathbb{R}^3$. Then γ is a curve of constant precession if and only if there is an epicycloid or hypocycloid β and an angle $\varphi \in (0, \pi/2)$, such that γ is (congruent to) a geodesic with linear slope of the helix surface $M_{\beta,\varphi}$.*

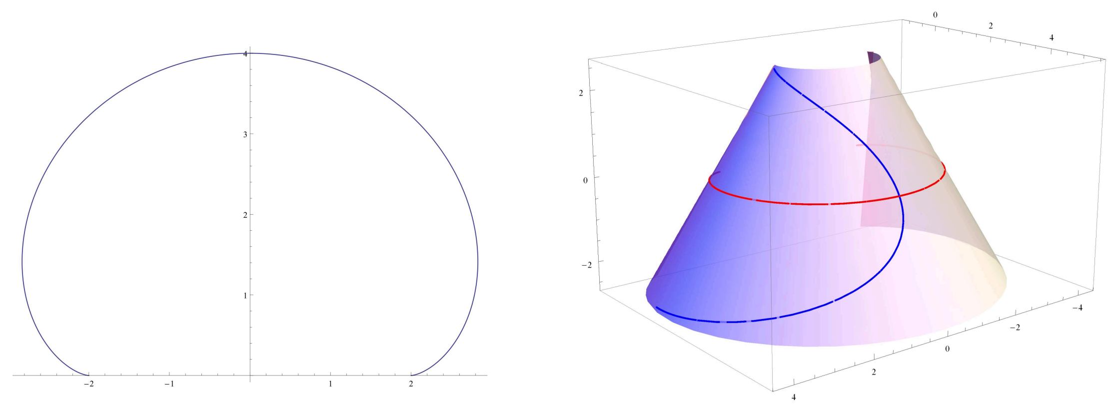


Figure 1: Curve of constant precession as geodesic of a helix surface constructed on an epicycloid. At the left, the epicycloid of radii $a = 2, b = 1$; at the right, the epicycloid in red, the helix surface of angle $\varphi = \pi/3$ in pink, and the curve of constant precession with $\omega = \sqrt{3}/3$ and $\mu = 1/3$ in blue.

4 A natural parametrization for the helix surfaces

When a ruled surface is noncylindrical (i.e. its rulings are always changing direction), one may find a natural parametrization:

$$Y(u, v) = \alpha(u) + v\,\mathbf{d}(u),$$

for which $\mathbf{d} \times \mathbf{d}' \neq 0$, $||\mathbf{d}|| = 1$ and $\langle \alpha', \mathbf{d}' \rangle = 0$. The curve α is called the striction curve. What is the natural parametrization of a helix surface $M_{\beta,\varphi}$?

Take a unit speed plane curve $\beta : I \subset \mathbb{R} \to \mathbb{R}^2 \subset \mathbb{R}^3$, an angle $\varphi \in (0, \pi/2)$, and consider the ruled surface $M = M_{\beta,\varphi}$ parametrized by

$$X(t, z) = \beta(t) + z\,(\cos\varphi\, N_\beta(t) + \sin\varphi\,\mathbf{u}).$$

When κ_β is constant (i.e. β is a circle), M is a cone and its striction line reduces to a single point (the vertex of the cone). In this case, as we have seen above, the geodesics of M are conical geodesics or rectifying curves (see [1], [13]). Assume without loss of generality that $\kappa'_\beta \neq 0$.

A straightforward computation shows that the striction line of $M_{\beta,\varphi}$ is given by

$$\alpha(t) = \beta(t) + \frac{1}{\cos\varphi\,\kappa_\beta(t)}\,(\cos\varphi\, N_\beta(t) + \sin\varphi\,\mathbf{u}),$$

with velocity vector

$$\alpha'(t) = -\frac{\kappa'_\beta(t)}{\cos\varphi\,\kappa^2_\beta(t)}\,(\cos\varphi\, N_\beta(t) + \sin\varphi\,\mathbf{u}).$$

Hence the arclength parameter of α is $u(t) = 1/(\cos\varphi\,\kappa_\beta(t))$ and the unit tangent vector is $T_\alpha(t) = \cos\varphi\, N_\beta(t) + \sin\varphi\,\mathbf{u}$. Then $\langle T_\alpha(t), \mathbf{u}\rangle = \sin\varphi$ is constant, and this shows that $\alpha(t)$ is a general helix with axis $\mathbf{u}$ and $\tau_\alpha/\kappa_\alpha = \lambda$, for a nonzero constant λ. Hence, the natural parametrization of M is

$$Y(u,v) = \alpha(u) + v\,T_\alpha(u), \quad v \neq 0,$$

that is, $M_{\beta,\varphi}$ is the tangent surface of α (also called the tangential developable), [24, p. 64]. An alternative (and equivalent) parametrization of the tangent surface, sometimes more convenient, is the following:

$$\widetilde{Y}(u,v) = \alpha(u) + (v-u)\,T_\alpha(u), \quad v \neq u.$$

We have shown in Theorem 3 that slant helices are the geodesics of helix surfaces. How are these curves represented in (u,v)-coordinates?

Let $\gamma(s) = \widetilde{Y}(u(s),v(s))$ be a unit speed geodesic of the ruled surface M. Then

$$T_\gamma(s) = u'(s)\,(v(s)-u(s))\,\kappa_\alpha(u(s))\,N_\alpha(u(s)) + v'(s)\,T_\alpha(u(s)),$$

so that there is a differentiable function $\omega(s)$ such that

$$\begin{aligned} u'(s)\,(v(s)-u(s))\,\kappa_\alpha(u(s)) &= \sin\omega(s), \\ v'(s) &= \cos\omega(s). \end{aligned}$$

The function $\omega(s)$ represents the angle between the geodesic γ and the base curve α. Bearing in mind the Frenet equations of γ, (1), we obtain

$$\begin{aligned} \kappa_\gamma(s)\,N_\gamma(s) = &-\sin\omega(s)\,(u'(s)\,\kappa_\alpha(u(s)) + \omega'(s))\,T_\alpha(u(s)) \\ &+ \cos\omega(s)\,(\omega'(s) + u'(s)\,\kappa_\alpha(u(s))\,N_\alpha(u(s)) \\ &+ \sin\omega(s)\,u'(s)\,\tau_\alpha(u(s))\,B_\alpha(u(s)), \end{aligned}$$

and by using, without loss of generality, that $N_\gamma(s) = N(u(s),v(s)) = -B_\alpha(u(s))$, we deduce

$$\omega'(s) = -u'(s)\,\kappa_\alpha(u(s)).$$

This yields the following equations for the curvature and torsion of the geodesic:

$$\kappa_\gamma(s) = -\sin\omega(s)\,u'(s)\,\tau_\alpha(u(s)) = \lambda\,\omega'(s)\ \sin\omega(s), \tag{20}$$

$$\tau_\gamma(s) = \cos\omega(s)\,u'(s)\,\tau_\alpha(u(s)) = -\lambda\,\omega'(s)\ \cos\omega(s). \tag{21}$$

Hence, we have shown the following result.

Proposition 5 *A unit speed curve $\gamma(s) = \widetilde{Y}(u(s),v(s))$, with $\kappa_\gamma > 0$, is a geodesic of M if and only if there is a differentiable function $\omega(s)$ such that*

$$u'(s)\,(v(s)-u(s))\,\kappa_\alpha(u(s)) = \sin\omega(s), \tag{22}$$

$$v'(s) = \cos\omega(s), \tag{23}$$

$$u'(s)\,\kappa_\alpha(u(s)) = -\omega'(s). \tag{24}$$

Moreover, the curvature and torsion of γ are given by (20) and (21), respectively.

Eliminating the function $\omega(s)$ in equations (22)–(24), the Proposition 5 can be rewritten as follows. The proof is straightforward and is left to the reader.

Proposition 6 *Let $\gamma(s) = \widetilde{Y}(u(s), v(s))$, with $\kappa_\gamma > 0$, be a unit speed curve in the tangent surface M, and suppose it is not a general helix nor a rectifying curve. Then γ is a geodesic of M if and only if the following conditions hold:*

$$a)\ u(s) = \frac{(v^2)''(s) - 2}{2\,v''(s)},$$

$$b)\ |v'(s)| < 1,$$

$$c)\ \frac{v'(s)}{\sqrt{1 - v'(s)^2}} \quad \textit{is not a linear function,}$$

$$d)\ \kappa_\alpha(t) = \frac{v''}{u'\sqrt{1 - v'^2}}\big(u^{-1}(t)\big).$$

In [13], the authors show that if S is a developable surface and $\gamma \subset S$ is a slant helix and a geodesic transversal to rulings, which is not a general helix nor a conical geodesic (rectifying curve), then necessarily S is the tangent surface of a general helix. The following result states that all slant helices, except general helices and rectifying curves, can be obtained in this way. It can be proved similarly to Theorem 3.

Theorem 7 *Let $\gamma(s)$ be a unit speed curve fully in $\mathbb{R}^3$, and suppose it is not a general helix nor a rectifying curve. Then γ is a slant helix if and only if there is a general helix α such that γ is (congruent to) a geodesic of the tangent surface of α.*

This result was shown by Salkowski [22] for a special class of slant helices. We present this family of curves in the next example.

Example 3 (The Salkowski curves) Salkowski introduced in [22, p. 538] a special family of curves, with constant curvature but non-constant torsion. Recently, Monterde [19] has shown that these curves are characterized as the only slant curves with constant curvature and non-constant torsion. The family of Salkowski curves can be described as follows:

$$\begin{aligned}\gamma_m(t) = \frac{1}{\sqrt{1+m^2}}\bigg(&-\frac{1-n}{4(1+2n)}\sin((1+2n)t) - \frac{1+n}{4(1-2n)}\sin((1-2n)t) - \frac{1}{2}\sin t,\\ &\frac{1-n}{4(1+2n)}\cos((1+2n)t) + \frac{1+n}{4(1-2n)}\cos((1-2n)t) + \frac{1}{2}\cos t,\\ &\frac{1}{4m}\cos(2nt)\bigg),\end{aligned}$$

for $m \neq 0, \pm\sqrt{3}/3$, and $n = \frac{m}{\sqrt{1+m^2}}$, $n \neq 0, \pm 1/2$. Salkowski showed that these curves are the geodesics of the tangent surfaces to the following general helices, [22, p. 539]:

$$\alpha_m(u) = \frac{3n}{4\sqrt{1+m^2}} \left(\frac{\sin(1+2n)u}{1+2n} - \frac{\sin(1-2n)u}{1-2n}, \right.$$
$$- \frac{\cos(1+2n)u}{1+2n} + \frac{\cos(1-2n)u}{1-2n},$$
$$\left. \frac{\cos 2nu}{mn} + \frac{2}{3mn} \right).$$

These general helices are characterized by the equation $\kappa = m\tau$. Note that the projection of α_m into the xy-plane is either an epicycloid or a hypocycloid (see Example 2), according to the constants a and b given by

$$a = \frac{3n^2\sqrt{1-n^2}}{1-4n^2} \qquad \text{and} \qquad b = \frac{3n\sqrt{1-n^2}}{4(1+2n)}.$$

Note that to obtain exactly the curve defined in Example 2 it is necessary, after projecting the general helix α_m into the xy-plane, to make a positive rotation of angle $\pi/2$.

Salkowski observed that the curves $\gamma_m(t)$ lie in a quadratic surface $Q_1(n)$ given by

$$A(x^2+y^2) - C(z+D)^2 = B,$$

where the constants A, B, C, D are defined by

$$A = 1+m^2 > 0, \quad B = \frac{27n^4}{4(1-n^2)(1-4n^2)^2} > 0,$$
$$C = \frac{4n^2}{(1-4n^2)(1-n^2)}, \quad D = \frac{1+2n^2}{4n} > 0,$$

and the general helices $\alpha_m(u)$ also lie in another quadratic surface $Q_2(n)$ defined by

$$\tilde{A}(x^2+y^2) - \tilde{C}(z+\tilde{D})^2 = \tilde{B},$$

where the constants $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are given by

$$\tilde{A} = \frac{16(1+m^2)}{9n^2} > 0, \quad \tilde{B} = \frac{4}{(1-4n^2)^2} > 0,$$
$$\tilde{C} = -\frac{64m^2(1+m^2)}{9(1-4n^2)}, \quad \tilde{D} = -\frac{1-n^2}{2n} < 0.$$

When $n < 1/2$, then $Q_1(n)$ is a hyperboloid and $Q_2(n)$ is an ellipsoid; otherwise, i.e. when $n > 1/2$, $Q_1(n)$ is an ellipsoid and $Q_2(n)$ is a hyperboloid.

5 Examples

5.1 A method for constructing slant helices

We now present a method for constructing slant helices γ as geodesics of a helix surface, based on Proposition 6.

Let $v(s)$, $s \in I$, be a differentiable function such that

$$|v'(s)| < 1 \quad \text{and} \quad \frac{v'(s)}{\sqrt{1 - v'(s)^2}} \quad \text{is not a linear function.}$$

Define the following functions

$$u(s) = \frac{(v^2)''(s) - 2}{2v''(s)} = \frac{v(s)v''(s) + v'(s)^2 - 1}{v''(s)},$$
$$\kappa(t) = \frac{v''}{u'\sqrt{1 - v'^2}}(u^{-1}(t)).$$

Given a nonzero constant λ, let α be a general helix with curvature κ and torsion $\lambda\kappa$, and consider the tangent surface of α, parametrized by $\widetilde{Y}(u, v)$. A straightforward computation shows that equations in Proposition 6 are satisfied, and so $\gamma(s) = \widetilde{Y}(u(s), v(s))$ is a slant helix with curvature and torsion given by (20) and (21), respectively.

Now, we give several examples where we use the above method to find slant helices.

Example 4 Let v be the function defined by $v(s) = cs^2 + \frac{1}{8c}$, $c \neq 0$, that we consider defined on the interval $I = (-\frac{1}{2|c|}, \frac{1}{2|c|})$. Following the above method, define the functions

$$u(s) = 3cs^2 - \frac{3}{8c},$$
$$\kappa(t) = \frac{4c}{3}\frac{1}{\sqrt{1 - \left(\frac{8ct}{3}\right)^2}}.$$

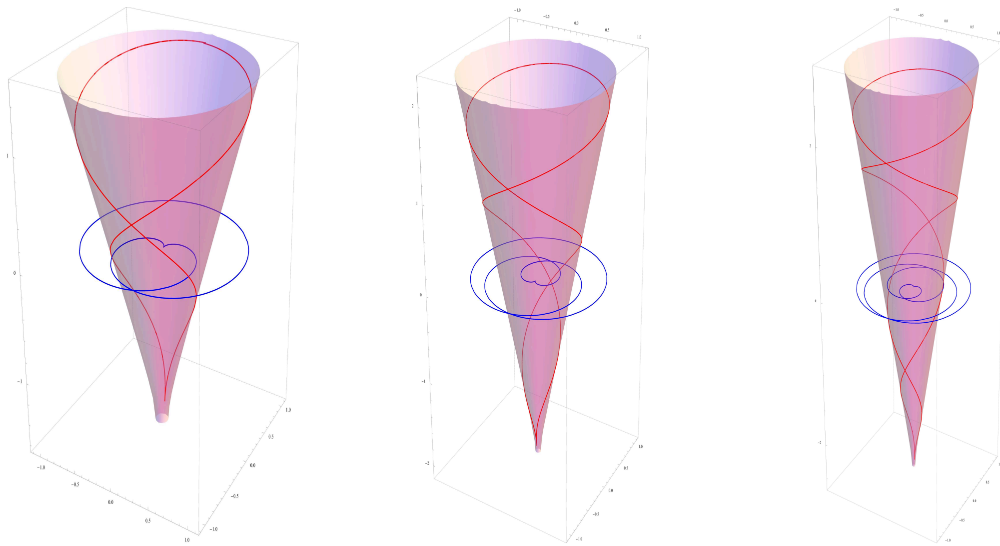
Let α be a general helix with curvature κ and torsion $\lambda\kappa$, where λ is a nonzero constant, and consider the tangent surface of α, parametrized by $\widetilde{Y}(u, v)$. Then $\gamma(s) = \widetilde{Y}(u(s), v(s))$ is a slant helix whose curvature and torsion are given by

$$\kappa_\gamma(s) = -2c\lambda \qquad \text{and} \qquad \tau_\gamma(s) = \frac{4c^2\lambda s}{\sqrt{1 - 4c^2s^2}}.$$

Note that γ is a curve with constant curvature but non-constant torsion. In particular, if we choose $\lambda = -1/(2c)$, then γ is congruent to the Salkowski curve γ_m, with $m = 2c$. The general helix α projects on an epicycloid or hypocycloid according to the following values of a and b (see Example 3),

$$a = \frac{3}{2c}\frac{1}{(\lambda^2 - 3)\sqrt{\lambda^2 + 1}} \qquad \text{and} \qquad b = \frac{3}{8c}\frac{1}{\lambda^2 + 1 + 2\sqrt{\lambda^2 + 1}}.$$

Some pictures for different values of the parameter c are given in Figure 2.



$c = 1/12$ $c = 1/16$ $c = 1/20$

Figure 2: Some slant helices (in red) for $v(s) = cs^2 + \frac{1}{8c}$. The blue curves are the projected epicycloids in the xy-plane. The pink surfaces are the quadratic surfaces where the slant helices lie (see Example 3)

Example 5 Let v be the function $v(s) = a\sqrt{s}$, $a \neq 0$, that we consider defined on the interval $I = (\frac{a^2}{4}, +\infty)$. Following the above method, define the functions

$$u(s) = \frac{4}{a}s^{3/2},$$
$$\kappa(t) = \frac{-a}{3t\sqrt{(2at)^{2/3} - a^2}}.$$

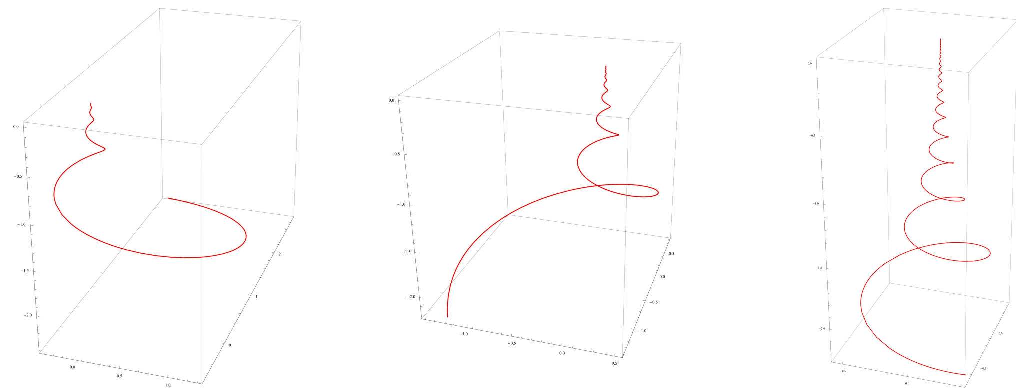
Let α be a general helix with curvature κ and torsion $\lambda\kappa$, where λ is a nonzero constant, and consider the tangent surface of α, parametrized by $\widetilde{Y}(u,v)$. Then $\gamma(s) = \widetilde{Y}(u(s), v(s))$ is a slant helix whose curvature and torsion are given by

$$\kappa_\gamma(s) = \frac{a\lambda}{4s^{3/2}} \qquad \text{and} \qquad \tau_\gamma(s) = -\frac{a^2\lambda}{4s^{3/2}\sqrt{4s - a^2}}.$$

Some pictures for $a = 1$ and different values of the parameter λ are shown in Figure 3.

Example 6 Let v be the function $v(s) = b\,e^{as}$, with $a, b > 0$, that we consider defined on the interval $I = (-\infty, -\ln(ab)/a)$. Define the following functions:

$$u(s) = 2b\,e^{as} - \frac{1}{a^2 b}\,e^{-as},$$
$$\kappa(t) = \frac{2ah(t)}{(8 + h(t))\sqrt{16 - h(t)}}, \quad \text{where } h(t) = (at + \sqrt{a^2t^2 + 8})^2.$$



$\lambda = 20$ $\lambda = 40$ $\lambda = 100$

Figure 3: Some slant helices for $v(s) = \sqrt{s}$

Let α be a general helix with curvature κ and torsion $\lambda\kappa$, where λ is a nonzero constant, and consider the tangent surface of α, parametrized by $\widetilde{Y}(u, v)$. Then $\gamma(s) = \widetilde{Y}(u(s), v(s))$ is a slant helix whose curvature and torsion are given by

$$\kappa_\gamma(s) = -\lambda a^2 b\, e^{as},$$
$$\tau_\gamma(s) = \frac{\lambda a^3 b^2\, e^{2as}}{\sqrt{1 - a^2 b^2\, e^{2as}}},$$

5.2 Another method for constructing slant helices

We now present a second method for constructing slant helices γ living in the tangent surface of a general helix.

Let $\alpha = \alpha(u)$ be a general helix with curvature function $\kappa_\alpha(u)$, and define a function g as follows:

$$g(x) = -\int_{x_0}^{x} \kappa_\alpha(u)\, du, \tag{25}$$

where x_0 is a constant. By Proposition 5 we easily deduce that $\gamma(s) = \widetilde{Y}(u(s), v(s))$, with $\kappa_\gamma > 0$, is a geodesic of the helix surface M if and only if the function $u(s)$ is a solution of the following ODE:

$$u'(s) - \left(\frac{\sin g(u(s))}{u'(s)\, g'(u(s))}\right)' = \cos g(u(s)),$$

which is equivalent to

$$u'(s)\, \sin g(u(s)) = \left(\frac{\sin^2 g(u(s))}{u'(s)\, g'(u(s))}\right)'. \tag{26}$$

Given a solution $u(s)$ of this ODE we find that

$$v(s) = u(s) - \frac{\sin g(u(s))}{u'(s)\, g'(u(s))} + c,$$

c being a constant. We illustrate this method in the following example.

Example 7 (Slant curves in the tangent surface of a helix) Let $\alpha(u)$ be the circular helix of radius r and pitch $2\pi h$, which is parameterized by

$$\alpha(u) = \left(r\cos\left(\frac{u}{R}\right), r\sin\left(\frac{u}{R}\right), \frac{hu}{R} \right), \quad R = \sqrt{r^2 + h^2}.$$

The curvature and torsion are given by

$$\kappa_\alpha = \frac{r}{R^2} \qquad \text{and} \qquad \tau_\alpha = \frac{h}{R^2},$$

and then the function g, defined in (25), is given by $g(x) = -\kappa_\alpha x + a$, a being a constant. A straightforward computation shows that the solution of (26) is given by

$$u(s) = \frac{R^2}{r}\left(a - \arccos\left(\frac{b \pm a(s+s_0)\sqrt{a^2(s+s_0)^2 + b^2 - 1}}{a^2(s+s_0)^2 + b^2} \right) \right),$$

b and s_0 being constants, and therefore the function $v(s)$ is given by

$$v(s) = \pm\frac{1}{a}\sqrt{a^2(s+s_0)^2 + b^2 - 1} + \frac{R^2}{r}\left(a - \arccos\left(\frac{b \pm a(s+s_0)\sqrt{a^2(s+s_0)^2 + b^2 - 1}}{a^2(s+s_0)^2 + b^2} \right) \right) + c.$$

The tangent surface of α can be parameterized by

$$\widetilde{Y}(u,v) = \left(r\cos\left(\frac{u}{R}\right) - \frac{r(v-u)}{R}\sin\left(\frac{u}{R}\right), r\sin\left(\frac{u}{R}\right) + \frac{r(v-u)}{R}\cos\left(\frac{u}{R}\right), \frac{hv}{R} \right).$$

By using the functions $u(s)$ and $v(s)$ computed above, we have explicit parameterizations $\widetilde{Y}(u(s), v(s))$ of the slant curves $\gamma(s)$ lying in the tangent surface of a circular helix, where the constants a, b, c and s_0 depend on the initial conditions of the geodesic γ.

The interval I, where the curve γ is defined, depends on the constant b. If $b^2 > 1$ then $I = \mathbb{R}$; otherwise, $I = (-\infty, -s_0 - \sqrt{(1-b^2)/a^2}) \cup (-s_0 + \sqrt{(1-b^2)/a^2}, +\infty)$, i.e. γ has two branches.

Some pictures for different values of the parameters a and b are shown in Figure 4.

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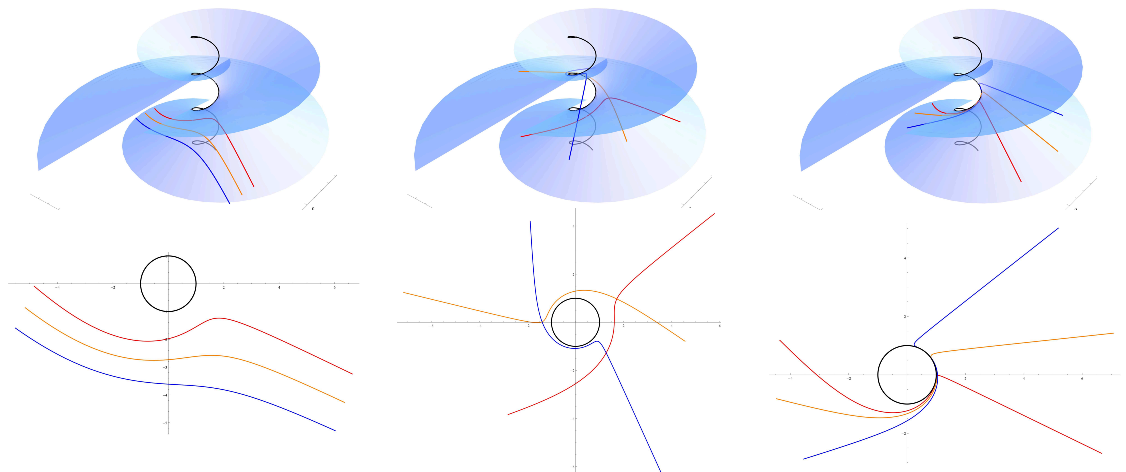

$b = 2$ (red), 3 (orange), 4 (blue) $a = 2$ (red), 4 (orange), 6 (blue) $a = 1$ (red), 1.5 (orange), 2 (blue)

$a = 1$ $b = 3$ $b = 0.5$

Figure 4: The top row shows the slant helices in the tangential surface of the circular helix of radius $r = 1$ and pitch $h = 0.5$. The second row shows the projected curves in the xy-plane. The corresponding values of the parameters a and b are shown in the last row ($c = s_0 = 0$ in all cases).

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