A complete classification of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with generalized $\xi-$parallel Jacobi structure Operator

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Abstract

The aim of the present paper is the classification of real hypersurfaces M, whose Jacobi structure Operator is *generalized* ξ*−parallel*. The notion of generalized $\xi-$parallel Jacobi structure Operator is rather new and much weaker than $\xi-$ parallel Jacobi structure Operator which has been studied so far.

1 Introduction.

An n - dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a projective space $\mathbb{C}P^n$ if $c > 0$, a hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, ξ, η, g).

Real hypersurfaces in $\mathbb{C}P^n$ which are homogeneous, were classified by R. Takagi ([15]). The same author classified real hypersurfaces in $\mathbb{C}P^n$, with constant principal curvatures in [16], but only when the number k of distinct principal curvatures satisfies $k = 3$. M. Kimura showed in [10] that if a Hopf real hypersurface M in $\mathbb{C}P^n$ has constant principal curvatures, then the number of distinct principal curvatures of M is 2, 3 or 5. J. Berndt gave the equivalent result for Hopf hypersurfaces in $\mathbb{C}H^n$ ([1]) where he divided real hypersurfaces into

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four model spaces, named A_0, A_1, A_2 and B. Analytic lists of constant principal curvatures can be found in the previously mentioned references as well as in [11], [13]. Real hypersurfaces of type A_1 and A_2 in $\mathbb{C}P^n$ and of type A_0, A_1 and A_2 in $\mathbb{C}H^n$ are said to be hypersurfaces of *type A* for simplicity and appear quite often in classification theorems. Real hypersurfaces of type A_1 in $\mathbb{C}H^n$ are divided into types $A_{1,0}$ and $A_{1,1}$ ([11]). For more information and examples on real hypersurfaces, we refer to [13].

A Jacobi field along geodesics of a given Riemannian manifold (M, g) plays an important role in the study of differential geometry. It satisfies a well known differential equation which inspires Jacobi operators. For any vector field X, the Jacobi operator is defined by R_X: $R_X(Y) = R(Y, X)X$, where R denotes the curvature tensor and Y is a vector field on M. R_X is a self - adjoint endomorphism in the tangent space of M, and is related to the Jacobi differential equation, which is given by $\nabla_{\acute{\gamma}}(\nabla_{\acute{\gamma}}Y) + R(Y, \acute{\gamma})\acute{\gamma} = 0$ along a geodesic γ on M, where $\acute{\gamma}$ denotes the velocity vector field along γ on M.

In a real hypersurface M of a complex space form $M_n(c)$, $c \neq 0$, the Jacobi operator on M with respect to the structure vector field ξ, is called the structure Jacobi operator and is denoted by $R_\xi(X) = R(X, \xi,)\xi$. Conditions including this operator, generate larger classes than the conditions including the Riemannian tensor $R(X, Y)Z$. So operator $l = R_\xi$ has been studied by quite a few authors and under several conditions.

In 2007, Ki, Perez, Santos and Suh ([8]) classified real hypersurfaces in complex space forms with ξ-parallel Ricci tensor and structure Jacobi operator. J. T. Cho and U - H. Ki in [3] classified the real hypersurfaces whose structure Jacobi operator is symmetric along the Reeb flow ξ and commutes with the shape operator A.

In the present paper we classify real hypersurfaces M satisfying the condition

$$(\nabla_\xi l)X = \omega(X)\xi, \tag{1.1}$$

where ω is 1-form and $X \in T_pM$ at a point $p \in M$. This condition is rather new ([17]) and much weaker than the condition $\nabla_\xi l = 0$ that has been used so far ([3], [6], [7], [8]). Therefore a larger class is produced.

We also mention that hypersurfaces in $M_2(c)$ have not been studied as thoroughly as the ones in $M_n(c)$, $n \geq 3$. We refer here to [4], [5], [9].

The major part of the paper is to prove M is a Hopf hypersurface, that is ξ is a principal vector field and the classification follows right after that. In particular, the following theorem is proved:

Theorem 1.1. *Let M be a real hypersurface of a complex plane $M_2(c)$, $(c \neq 0)$, satisfying (1.1) for every vector field X on M. Then M is a Hopf hypersurface and satisfies $\nabla_\xi l = 0$. Furthermore, M is pseudo-Einstein, that is, there exist constants ρ and σ such that for any tangent vector X we have $QX = \rho X + \sigma g(X, \xi)\xi$ where Q is the Ricci tensor. Conversely, every pseudo-Einstein hypersurface in $M_2(c)$ satisfies (1.1) with $\omega = 0$.*

As shown in [9] the pseudo-Einstein hypersurfaces, are precisely those that are

- For $M_2(c) = \mathbb{C}P^2$: open subsets of geodesic spheres (type A_1);

• For $M_2(c) = \mathbb{C}H^2$: open subsets of
1. horospheres (type A_0);
2. geodesic spheres (type $A_{1,0}$);
3. tubes around totally geodesic complex hyperbolic lines $\mathbb{C}H^1$ (type $A_{1,1}$);

• Hopf hypersurfaces with $\eta(A\xi) = 0$.

The form ω has no restriction in its values, so it could vanish at some point. Therefore condition (1.1) could be called generalized $\xi-$parallel Jacobi structure Operator, since it generalizes the notion of $\xi-$parallel Jacobi structure Operator ($\nabla_\xi l = 0$).

2 Preliminaries

In this section, we explain explicitly the notions that were mentioned in section 1, as well as the notions that will appear in the paper. We also give a series of equations that will be our basic tools in our calculations and conclusions.

Let M_n be a Kaehlerian manifold of real dimension $2n$, equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $M_n(c)$, the following relations hold: $J^2X = -X, \quad G(JX, JY) = G(X,Y), \quad \widetilde{\nabla}J = 0$, where $\widetilde{\nabla}$ denotes the Riemannian connection of G of M_n.

Let M_{2n-1} be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighborhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX, $\eta(X)N$ is the normal component, and $\xi = -JN, \quad \eta(X) = g(X,\xi), \quad g = G|_M$.

By properties of the almost complex structure J and the definitions of η and g, the following relations hold ([2]):

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad g(X, \phi Y) = -g(\phi X, Y). \tag{2.2}$$

The above relations define an *almost contact metric structure* on M which is denoted by (ϕ, ξ, g, η). When an almost contact metric structure is defined on M, we can define a local orthonormal basis $\{e_1, e_2, ...e_{n-1}, \phi e_1, \phi e_2, ...\phi e_{n-1}, \xi\}$, called a $\phi-basis$. Furthermore, let A be the shape operator in the direction of N, and denote by ∇ the Riemannian connection of g on M. Then, A is symmetric and the following equations are satisfied:

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi. \tag{2.3}$$

As the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given by:

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \tag{2.4}$$

$$-2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi]. \tag{2.5}$$

The tangent space T_pM, for every point $p \in M$, is decomposed as following:

$$T_pM = \mathbb{D}^\perp \oplus \mathbb{D},$$

where $\mathbb{D} = ker(\eta) = \{X \in T_pM : \eta(X) = 0\}$.

The subspace $ker(\eta)$ is more usually referred as $\mathbb{D}$ and called the holomorphic distribution of M. Based on the decomposition of T_pM, by virtue of (2.3), we decompose the vector field $A\xi$ in the following way:

$$A\xi = \alpha\xi + \beta U, \tag{2.6}$$

where $\beta = |\phi\nabla_\xi\xi|$, α is a smooth function on M and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in ker(\eta)$, provided that $\beta \neq 0$.

If β vanishes identically, then $A\xi$ is expressed as $A\xi = \alpha\xi$, ξ is a principal vector field and M is a Hopf hypersurface.

Finally differentiation of a function f along a vector field X will be denoted by (Xf). All manifolds, vector fields, etc., of this paper are assumed to be connected and of class C^∞.

3 Auxiliary relations

Let us assume there exists a point $p \in M$, where $\beta \neq 0$. Then there exists a neighborhood $\mathcal{N}$ of p where $\beta \neq 0$. By putting $X = \xi$ in (1.1), combined with (2.3) and (2.6), we obtain $\beta l\phi U = -\omega(\xi)\xi$. The inner product of the last equation with ξ yields $l\phi U = 0$ which is analyzed from (2.4) and (2.6) giving $(4\alpha A + c)\phi U = 0$. From the last equation it follows that $\alpha \neq 0$ in $\mathcal{N}$.

Lemma 3.1. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1). Then the following relations hold on $\mathcal{N}$.*

$$AU = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)U + \beta\xi, \qquad A\phi U = -\frac{c}{4\alpha}\phi U. \tag{3.1}$$

$$\nabla_\xi\xi = \beta\phi U,\ \nabla_U\xi = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\phi U,\ \nabla_{\phi U}\xi = \frac{c}{4\alpha}U. \tag{3.2}$$

$$\nabla_\xi U = \kappa_1\phi U, \quad \nabla_U U = \kappa_2\phi U, \quad \nabla_{\phi U}U = \kappa_3\phi U - \frac{c}{4\alpha}\xi. \tag{3.3}$$

$$\nabla_\xi\phi U = -\kappa_1 U - \beta\xi,\ \nabla_U\phi U = -\kappa_2 U - \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\xi, \tag{3.4}$$

$$\nabla_{\phi U}\phi U = -\kappa_3 U.$$

where κ_1, κ_2, κ_3 are smooth functions on $\mathcal{N}$.

Proof.
By definition of the vector fields U, ϕU, ξ and due to (1.1), the set $\{U, \phi U, \xi\}$ is an orthonormal basis. From (2.4) we obtain

$$lU = \frac{c}{4}U + \alpha AU - \beta A\xi, \qquad l\phi U = \frac{c}{4}\phi U + \alpha A\phi U. \tag{3.5}$$

The inner products of lU with U and ϕU yield respectively

$$g(AU, U) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}, \qquad g(AU, \phi U) = \frac{\delta}{\alpha} \tag{3.6}$$

where $\gamma = g(lU, U)$ and $\delta = g(lU, \phi U)$.
So, (3.6) and $g(AU, \xi) = g(A\xi, U) = \beta$, yield

$$AU = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)U + \frac{\delta}{\alpha}\phi U + \beta\xi. \tag{3.7}$$

We have already shown in the beginning of this section that

$$l\phi U = 0 \Leftrightarrow A\phi U = -\frac{c}{4\alpha}U. \tag{3.8}$$

From (3.7), (3.8) and the symmetry of A, (3.1) has been proved.

From equations (2.6),(3.1) and relation (2.3) for $X = \xi$, $X = U$, $X = \phi U$, we obtain (3.2). Next we recall the rule

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{3.9}$$

By virtue of (3.9) for $X = Z = \xi$, $Y = U$ and for $X = \xi$, $Y = Z = U$, it is shown respectively $\nabla_\xi U \perp \xi$ and $\nabla_\xi U \perp U$. So $\nabla_\xi U = \kappa_1 \phi U$, where $\kappa_1 = g(\nabla_\xi U, \phi U)$. In a similar way, (3.9) for $X = Y = Z = U$ and $X = Z = U$, $Y = \xi$ yields-with the aid of (3.2)-respectively $\nabla_U U \perp U$ and $\nabla_U U \perp \xi$. This means that $\nabla_U U = \kappa_2 \phi U$, where $\kappa_2 = g(\nabla_U U, \phi U)$. Finally, (3.9) for $X = \phi U$, $Y = Z = U$ and $X = \phi U$, $Y = U$, $Z = \xi$-with the aid of (3.2)-yields respectively $\nabla_{\phi U} U \perp U$ and $g(\nabla_{\phi U} U, \xi) = -\frac{c}{4\alpha}$. Therefore $\nabla_{\phi U} U = \kappa_3 \phi U - \frac{c}{4\alpha}\xi$ where $\kappa_3 = g(\nabla_{\phi U} U, \phi U)$ and (3.3) has been proved. In order to prove (3.4) we use the second of (2.3) with the following combinations: *i)* $X = \xi$, $Y = U$, *ii)* $X = Y = U$, *iii)* $X = \phi U$, $Y = U$, and make use of (2.6), (3.1), (3.3). ■

By putting $X = U$, $Y = \xi$ in (2.5) we obtain $\nabla_U A\xi - A\nabla_U \xi - \nabla_\xi AU + A\nabla_\xi U = -\frac{c}{4}\phi U$, which is expanded by Lemma 3.1, to give

$$[(U\alpha) - (\xi\beta)]\xi + [(U\beta) - \xi(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha})]U +$$

$$[\kappa_2\beta + \gamma + \frac{c}{4\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) - (\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha})\kappa_1]\phi U = 0.$$

Since the vector fields $U, \phi U$ and ξ are linearly independent, the above equation gives

$$(U\alpha) = (\xi\beta), \tag{3.10}$$

$$(U\beta) = \xi(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}), \tag{3.11}$$

$$\kappa_2\beta + \gamma + \frac{c}{4\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) - (\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha})\kappa_1 = 0. \tag{3.12}$$

In a similar way, from (2.5) we get $\nabla_{\phi U}A\xi - A\nabla_{\phi U}\xi - \nabla_{\xi}A\phi U + A\nabla_{\xi}\phi U = \frac{c}{4}U$, which is expanded by Lemma 3.1, to give

$$[(\phi U\alpha) - \frac{3\beta c}{4\alpha} - \kappa_1\beta - \alpha\beta]\xi +$$

$$[(\phi U\beta) - \frac{c}{4\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) - \kappa_1(\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha}) - \beta^2]U +$$

$$[\kappa_3\beta - (\frac{c}{4\alpha^2})(\xi\alpha)]\phi U = 0,$$

which leads to

$$(\phi U\alpha) - \frac{3\beta c}{4\alpha} - \kappa_1\beta - \alpha\beta = 0, \tag{3.13}$$

$$(\phi U\beta) - \frac{c}{4\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) - \kappa_1(\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha}) - \beta^2 = 0, \tag{3.14}$$

$$(\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3. \tag{3.15}$$

Finally, (2.5) yields $\nabla_U A\phi U - A\nabla_U\phi U - \nabla_{\phi U}AU + A\nabla_{\phi U}U = -\frac{c}{2}\xi$, which is expanded by Lemma 3.1, to give

$$[-\phi U\beta + \gamma + \frac{c}{2\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \kappa_2\beta + \beta^2]\xi +$$

$$[\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \kappa_2(\frac{\beta^2}{\alpha} + \frac{\gamma}{\alpha}) - \phi U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) - \frac{\beta c}{2\alpha}]U +$$

$$[\frac{c}{4\alpha^2}(U\alpha) - \kappa_3(\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha})]\phi U = 0.$$

The above relation leads to

$$-\phi U\beta + \gamma + \frac{c}{2\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \kappa_2\beta + \beta^2 = 0, \tag{3.16}$$

$$\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \kappa_2(\frac{\beta^2}{\alpha} + \frac{\gamma}{\alpha}) - \phi U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) - \frac{\beta c}{2\alpha} = 0, \tag{3.17}$$

$$(U\alpha) = \kappa_3\frac{4\alpha}{c}(\gamma + \beta^2). \tag{3.18}$$

From (2.4) we calculate $R(U,\xi)U$, using Lemma 3.1. The result is $R(U,\xi)U = -\gamma\xi$. However, the vector field $R(U,\xi)U$ is also calculated from $R(U,\xi)U = \nabla_U\nabla_\xi U - \nabla_\xi\nabla_U U - \nabla_{[U,\xi]}U$ using also Lemma 3.1, giving

$R(U,\xi)U = [(U\kappa_1) - (\xi\kappa_2) + \kappa_3\kappa_1 - \kappa_3(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha})]\phi U + [\kappa_2\beta + \frac{c}{4\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) - (\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha})\kappa_1]\xi$. Comparing the two expressions of $R(U,\xi)U$ we get

$$(U\kappa_1) - (\xi\kappa_2) = \kappa_3(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} - \kappa_1). \tag{3.19}$$

By making use of (1.1) for $X = U$, we obtain $(\nabla_\xi l)U = \omega(U)\xi$, which is expanded with the aid of Lemma 3.1 and (3.5), giving $(\xi\gamma)U + \gamma\kappa_1\phi U = \omega(U)\xi$. Since $U, \phi U, \xi$ are linearly independent, we obtain

$$\gamma\kappa_1 = 0, \quad (\xi\gamma) = 0. \tag{3.20}$$

4 The case $\gamma \neq 0$.

Let us assume there exists a point $p_1 \in \mathcal{N}$ such that $\gamma \neq 0$ in a neighborhood W_1 of p_1. Then (3.20) yields $\kappa_1 = 0 = (\xi\gamma)$. So, by differentiating (3.12) along ξ, with the aid of (3.10), (3.15), (3.18), (3.19) we have

$$\kappa_3[-2\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \kappa_2\frac{4\alpha}{c}(\gamma + \beta^2) + \frac{\beta}{\alpha}(\gamma + \frac{c}{4} + \beta^2)] = 0. \tag{4.1}$$

If we assume that $\kappa_3 \neq 0$ in W_1 then (4.1) will give $-2\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \kappa_2\frac{4\alpha}{c}(\gamma + \beta^2) + \frac{\beta}{\alpha}(\gamma + \frac{c}{4} + \beta^2) = 0$ which is further modified giving

$$\frac{3\beta c}{4\alpha} - \frac{\beta}{\alpha}(\gamma + \beta^2) + \kappa_2\frac{4\alpha}{c}(\gamma + \beta^2) = 0. \tag{4.2}$$

Apparently, $\gamma + \beta^2 \neq 0$, otherwise relation (4.2) would yield $\frac{\beta c}{\alpha} = 0$ which is a contradiction. Therefore (4.2) yields

$$\kappa_2 = \frac{\beta c}{4\alpha^2} - \frac{3\beta c^2}{16\alpha^2(\gamma + \beta^2)}. \tag{4.3}$$

We replace the term κ_2 in (3.12), from (4.3) and then multiply the new relation with $\gamma + \beta^2$. The outcome is

$$(\gamma + \beta^2)(\gamma\alpha^2 + \frac{c\beta^2}{2} + \frac{c}{4}\gamma - \frac{c^2}{16}) - \frac{3\beta^2 c^2}{16} = 0.$$

The above equation is differentiated along ξ, combined with (3.10), (3.15), (3.18), (3.20) leading to

$$\kappa_3(\gamma + \beta^2)[\frac{8\alpha\beta}{c}(\gamma\alpha^2 + \frac{c\beta^2}{2} + \frac{c}{4}\gamma - \frac{c^2}{16}) + \frac{8\alpha^3\beta\gamma}{c} + 4\alpha\beta\gamma + 4\alpha\beta^3 - \frac{3\alpha\beta c}{2}] = 0.$$

Since we have $\kappa_3(\gamma + \beta^2) \neq 0$, the above equation yields

$$\frac{8\alpha^2\gamma}{c} + 4\beta^2 + 3\gamma - c = 0. \tag{4.4}$$

By virtue of (3.10), (3.15), (3.18), (3.20) and $\kappa_3 \neq 0$ we differentiate (4.4) to obtain

$$\frac{8\alpha^2\gamma}{c} + 4\beta^2 + 4\gamma = 0. \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$\beta^2 - 2\alpha^2 = c, \qquad \gamma = -c. \tag{4.6}$$

The differentiation of (4.6) along U, with the aid of (3.10), (3.11), (3.15), (3.18), (3.20), (4.6) and $\kappa_3 \neq 0$ leads to

$$(\beta^2 - \frac{3c}{4})\beta^2 - 2\alpha^2(\beta^2 - c) = 0.$$

The term $2\alpha^2$ is replaced from (4.6) in order to acquire $\beta^2 = \frac{4c}{5}$. So β is constant and from (4.6) we have $(\xi\alpha) = 0 \Rightarrow \kappa_3 = 0$ (due to (3.15)) which is a contradiction to our assumption $\kappa_3 \neq 0$.

This means that in W_1 we have $\kappa_3 = 0$ and the Lie brackets $[U, \xi]\alpha$, $[U, \xi]\beta$ are zero, due to (3.10), (3.11), (3.15), (3.18). The same Lie brackets are estimated from $[U, \xi] = \nabla_U \xi - \nabla_\xi U$, (3.13), (3.14), $\kappa_1 = 0$ and Lemma 3.1 as following:

$$[U, \xi]\alpha = (\gamma - \frac{c}{4} + \beta^2)(\frac{3c}{4\alpha} + \alpha), \quad [U, \xi]\beta = (\gamma - \frac{c}{4} + \beta^2)[\frac{c}{4\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \beta^2],$$

which means we have

$$(\gamma - \frac{c}{4} + \beta^2)(\frac{3c}{4\alpha} + \alpha) = 0, \quad (\gamma - \frac{c}{4} + \beta^2)[\frac{c}{4\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \beta^2] = 0. \tag{4.7}$$

The term $\gamma - \frac{c}{4} + \beta^2$ can not vanish identically, otherwise the combination of (3.12), (3.17) would imply β is constant, which would violate (3.14). Therefore $\gamma - \frac{c}{4} + \beta^2 \neq 0$ holds in W_1. Then (4.7), (3.13) and (3.14) yield

$$\alpha^2 = -\frac{3c}{4} \Rightarrow (\phi U\alpha) = 0, \quad \gamma - \frac{c}{4} = 2\beta^2 \Rightarrow (\phi U\beta) = 0. \tag{4.8}$$

Combining (4.8) with (3.17) we get

$$\kappa_2(3\beta^2 + \frac{c}{4}) + 3\beta^3 - \frac{\beta c}{2} = 0. \tag{4.9}$$

On the other hand, combining (3.12) with (4.8) we obtain $\kappa_2 = \beta - \frac{\gamma}{\beta}$. The last relation is used with (4.9) and (4.8) to remove the terms κ_2, γ leading to

$$\beta^2 = -\frac{c}{24}. \tag{4.10}$$

Next we calculate $R(\phi U, U)U$ from (2.4), (4.8), (4.10) and Lemma 3.1 to take $R(\phi U, U)U = \frac{23}{24}c\phi U$. We also have $R(\phi U, U)U = \nabla_{\phi U}\nabla_U U - \nabla_U \nabla_{\phi U} U - \nabla_{[\phi U, U]} U$, which is further developed with the help of Lemma 3.1, (3.18), (4.8), (4.9), (4.10), $\kappa_1 = \kappa_3 = 0$, resulting to $R(\phi U, U)U = \frac{13}{12}c\phi U$. Equalizing the two expressions of $R(\phi U, U)U$ we have $c = 0$ which is a contradiction in W_1.

Thus W_1 is the empty set and $\gamma = 0$ holds in $\mathcal{N}$. However, this implies $l = 0$ due to Lemma 3.1, (3.5), (3.8) and $l\xi = 0$. Such hypersurfaces do not exist ([5]) and we have a contradiction on $\mathcal{N}$. Hence M is a Hopf hypersurface.

5 Proof of Theorem 1.1

Since M is Hopf, we have $A\xi = \alpha\xi$ and α is constant ([13]). The inner product of $(\nabla_\xi l)X = \omega(X)\xi$ with ξ (because of (2.3), (3.9) and $A\xi = \alpha\xi$) yields $\omega(X) = 0$. This means that $\nabla_\xi l = 0$.

It is easy to check that $(\nabla_\xi l)\xi = 0$ for any Hopf hypersurface. Now consider a vector field $X \in \mathbb{D}$. From the Gauss equation we have $lX = (\alpha A + \frac{c}{4})X$, so that

$$(\nabla_\xi l)X = \nabla_\xi lX - l\nabla_\xi X$$

$$= \nabla_\xi(\alpha A + \frac{c}{4})X - (\alpha A + \frac{c}{4})\nabla_\xi X,$$

since $\nabla_\xi X$ is also in $\mathbb{D}$. We can simplify this, using the Codazzi equation, to get

$$(\nabla_\xi l)X = \alpha(\nabla_\xi A)X$$

$$= \alpha((\nabla_X A)\xi + \frac{c}{4}\phi X)$$

$$= \alpha((\alpha - A)\phi AX + \frac{c}{4}\phi X).$$

In particular, If X is chosen to be a principal vector field, such that $AX = \lambda_1 X$ and $A\phi X = \lambda_2\phi X$, then the condition $\nabla_\xi l = 0$ implies that

$$\alpha(\lambda_1 - \lambda_2) = 0$$

where we have used the well known relation for Hopf hypersurfaces

$$\lambda_1\lambda_2 = \frac{\lambda_1 + \lambda_2}{2}\alpha + \frac{c}{4}.$$

If $\alpha \neq 0$ then $\lambda_1 = \lambda_2$ is locally constant since it satisfies $\lambda_1^2 = \alpha\lambda_1 + \frac{c}{4}$. Therefore, M is an open subset of type A hypersurface, based on the theorems of Kimura and Berndt and the lists of principal curvatures in [15] and [11]. In case $\alpha = 0$, we have $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2$ with $\lambda_1^2 = \frac{c}{4}$ and the classification follows from [9].

Conversely let M be of type A_1 in $\mathbb{C}P^2$ or type A_0, $A_{1,0}$, $A_{1,1}$ in $\mathbb{C}H^2$. Take $X \in \mathbb{D}$ a principal vector field with principal curvature λ, and α the principal curvature of ξ. (2.4) yields $lX = (\alpha A + \frac{c}{4})X$, $\forall X \in \mathbb{D}$. Furthermore, in a real hypersurface of the previously mentioned types, we have $\lambda^2 = \alpha\lambda + \frac{c}{4}$, thus from the last two equations we have $lX = \lambda^2 X$, which is used to show $(\nabla_\xi l)X = 0$. The last equation and $(\nabla_\xi l)\xi = \nabla_\xi l\xi - l\nabla_\xi\xi = 0$ show that real hypersurfaces of type A satisfy (1.1) with $\omega = 0$.

If M is Hopf with $\alpha = 0$ then (2.4) yields $lX = \frac{c}{4}X$ for every $X \in D$. Therefore $(\nabla_\xi l)X = 0$ holds. In addition we have $(\nabla_\xi l)\xi = 0$, thus $(\nabla_\xi l)X = 0$ holds for every X, which means $\omega = 0$. ∎

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