Cauchy transformation and mutual dualities between $A^{-\infty}(\Omega)$ and $A^{\infty}(\complement\Omega)$ for Carathéodory domains

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Abstract

Let Ω be a Carathéodory domain in the complex plane $\mathbb{C}$, $A^{-\infty}(\Omega)$ the space of functions that are holomorphic in Ω with polynomial growth near the boundary $\partial\Omega$, and $A^{\infty}(\complement\Omega)$ the space of holomorphic functions in the interior of $\complement\Omega := \overline{\mathbb{C}} \setminus \Omega$, vanishing at infinity and being in $C^{\infty}(\complement\Omega)$. We prove that the Cauchy transformation of analytic functionals establishes a mutual duality between spaces $A^{-\infty}(\Omega)$ and $A^{\infty}(\complement\Omega)$. This result, together with those of [3], gives a solution to duality problem for the space $A^{-\infty}(\Omega)$ in both one and several complex variables.

1 Introduction

In the recent years the space $A^{-\infty}(\Omega)$ of holomorphic functions in a domain $\Omega \subset \mathbb{C}^n$ $(n \geq 1)$, with a polynomial growth near the boundary $\partial\Omega$, attracts a great attention of mathematicians. Among the considered topics, the duality problem has been extensively investigated for both one and several variables. Barret [5], Bell [6], Straube [18], Kiselman [11] studied whether the standard scalar product

$$\langle f, g \rangle = \int_{\Omega} f\bar{g}d\lambda \quad (\lambda \text{ is the Lebesgue measure in } \mathbb{R}^{2n})$$

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establishes the duality between $A^{-\infty}(\Omega)$ and $A^{\infty}(\bar{\Omega})$, the space of all holomorphic functions in Ω that are in $C^{\infty}(\bar{\Omega})$, or not. In particular, they proved that this is true for all smooth strictly pseudoconvex domains in $\mathbb{C}^n$ and gave some examples of domains for which this duality fails (for instance, that is so for domains which boundaries have degenerate corners).

In our recent papers [1, 2] we proved that, for a bounded convex domain Ω in $\mathbb{C}^n$, the Laplace transformation of functionals establishes a mutual duality between $A^{-\infty}(\Omega)$ and some weighted space of entire functions $A_{\Omega}^{-\infty}$ and in [3] that, for a bounded lineally convex domain Ω in $\mathbb{C}^n$ ($n \geq 2$), the Cauchy-Fantappiè transformation of functionals gives a mutual duality between $A^{-\infty}(\Omega)$ and $A^{\infty}(\widetilde{\Omega})$, where $\widetilde{\Omega}$ is the conjugate compact set for Ω.

In case of one variable it is natural to consider a duality, via the Cauchy transformation of functionals, between $A^{-\infty}(\Omega)$ and $A^{\infty}(\complement\Omega)$, the space of all functions holomorphic in the interior of the complement $\complement\Omega$, vanishing at infinity and C^{∞} on $\complement\Omega$. In this direction, Varziev and Melikhov [19] proved that if Ω is a bounded domain in $\mathbb{C}$, which is moreover strictly starlike with respect to the origin and has a piecewise smooth boundary, then the Cauchy transformation is a topological isomorphism between the strong dual $\left(A^{-\infty}(\Omega)\right)'_b$ and $A^{\infty}(\complement\Omega)$.

It should be noted that the result in [19] was obtained under rather stringent conditions on Ω. Indeed, there are convex domains in $\mathbb{C}$ whose boundaries are not piecewise smooth; also there are lineally convex domains which are not starlike with respect to any of its points.

Note, in addition, that by Köthe [14], the Cauchy transformation of functionals always establishes a mutual duality between the spaces $\mathcal{O}(\Omega)$ and $\mathcal{O}(\complement\Omega)$ of all holomorphic functions in Ω and all germs of holomorphic functions on $\complement\Omega$ that vanish at infinity.

So it is natural to ask:

(a) Can we improve for $A^{-\infty}(\Omega)$ the duality result above?

(b) What about the space $A^{\infty}(\complement\Omega)$, that is, can we have an isomorphy between $\left(A^{\infty}(\complement\Omega)\right)'_b$ and $A^{-\infty}(\Omega)$?

Here we even wish to obtain the results as close as possible to those of Köthe.

In the present paper we give positive answers to these questions for, at least, Carathéodory domains with rectifiable boundaries. (Here we recall that a bounded simply connected domain Ω in $\mathbb{C}$ is called *Carathéodory* if its boundary coincides with the boundary of the infinite component of the open set $\complement\overline{\Omega}$. In particular, every Jordan domain is Carathéodory). Namely, (a) holds for Carathéodory domains (Theorem 4.3), while (b) is true for Carathéodory domains with rectifiable boundaries (Theorem 4.5). Besides, we also provide an application to the representation problem (Theorems 5.6 – 5.7), which, to our knowledge, has never been treated.

The final remark is that it turns out from the proofs presented in this paper that the injectivity of the Cauchy transformation is much more difficult than the surjectivity, which is quite unusual for such a kind of result. Moreover, the surjectivity is valid for a class of domains which is much wider than the class of Carathéodory domains.

2 Notation and preliminaries

Let Ω be a bounded domain in $\mathbb{C}$ and $\complement\Omega := \overline{\mathbb{C}} \setminus \Omega$ its complement in the extended complex plane $\overline{\mathbb{C}}$. By $\mathcal{O}(\Omega)$ we denote the Frechét-Schwartz (briefly, (FS)-) space of all holomorphic functions in Ω. The notation $\mathcal{O}(\complement\Omega)$ is used for the dual Fréchet-Schwartz (briefly, (DFS)-) space of all germs of holomorphic functions on $\complement\Omega$ vanishing at infinity. We refer the reader to [16, 21] for information on the notions of (FS)- and (DFS)-spaces in details.

Let $d_\Omega(z) := \min_{w\in\partial\Omega} |z - w|$, the minimum Euclidean distance between $z \in \Omega$ and $\partial\Omega$. W.l.o.g., we will assume that $\Omega \subset \{z : |z| < 1\}$. Hence, $d_\Omega(z) \leq 1$ for all $z \in \Omega$. The space $A^{-\infty}(\Omega)$ of holomorphic functions in Ω with polynomial growth near $\partial\Omega$ is defined as follows:

$$A^{-\infty}(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \sup_{z\in\Omega} |f(z)|\ [d_\Omega(z)]^k < \infty, \text{ for some } k \in \mathbb{N} \right\}.$$

This space can be equipped with its natural inductive limit topology of Banach spaces

$$A^{-k}(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \|f\|_k = \sup_{z\in\Omega} |f(z)|\ [d_\Omega(z)]^k < \infty \right\},\ k = 1, 2, \ldots,$$

and hence becomes a (DFS)-space.

Denote by $A^\infty(\complement\Omega)$ the space of all holomorphic functions in the interior $\operatorname{int}\complement\Omega$ of $\complement\Omega$, vanishing at infinity and having C^∞-extensions on $\complement\Omega$. It will be endowed with its natural topology of an (FS)-space given by the system of norms

$$|g|_n := \max_{0\leq k\leq n} \max_{z\in\complement\Omega} |g^{(k)}(z)|,\quad n = 0, 1, \ldots. \tag{2.1}$$

In what follows we will assume that Ω is *regular*, that is $\operatorname{int}\overline{\Omega} = \Omega$. This excludes domains having cuts and isolated points and is natural when we consider the space $A^\infty(\complement\Omega)$. It is clear that $\operatorname{int}\complement\Omega = \complement\overline{\Omega}$ and $\partial\complement\Omega = \partial\Omega$ for a regular domain Ω.

For a locally convex space E, let E' and E'_b denote the dual and the strong dual spaces to E.

Clearly, the partial fraction $\dfrac{1}{z-\zeta}$ belongs to $A^{-\infty}(\Omega)$ for all $\zeta \in \complement\Omega$ as well as $A^\infty(\complement\Omega)$ for all $\zeta \in \Omega$. Consequently, the Cauchy transformation of functionals

$$\mathcal{F} :\ T \longmapsto \mathcal{F}_T(\zeta) := T\left(\frac{1}{z-\zeta}\right),$$

is well defined on $(A^{-\infty}(\Omega))'$ for all $\zeta \in \complement\Omega$ and on $(A^\infty(\complement\Omega))'$ for all $\zeta \in \Omega$.

3 The Cauchy transformation as a mutual epimorphism

In this section we prove that, for a regular domain Ω with rectifiable boundary, the Cauchy transformation of functionals is an epimorphism from $(A^{-\infty}(\Omega))'_b$ onto $A^\infty(\complement\Omega)$ as well as from $(A^\infty(\complement\Omega))'_b$ onto $A^{-\infty}(\Omega)$.

Proposition 3.1. *Let Ω be a bounded regular domain in $\mathbb{C}$. Then the Cauchy transformation of functionals is an epimorphism from $(A^{-\infty}(\Omega))'_b$ onto $A^{\infty}(\complement\Omega)$.*

Proof. Let $T \in \left(A^{-\infty}(\Omega)\right)'$. Then for each $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$|T(f)| \le C_k \|f\|_k, \ \forall f \in A^{-k}(\Omega).$$

Define the Banach spaces of continuous functions on Ω

$$C_0^{-k}(\Omega) := \left\{ f \in C(\Omega) : \|f\|_k < \infty, \ f(z)[d_\Omega(z)]^k \to 0 \text{ as } z \to \partial\Omega \right\}, \ k = 1, 2, \dots.$$

By the Hahn–Banach theorem, T can be extended as a continuous linear functional on $C_0^{-k}(\Omega)$ for every $k \in \mathbb{N}$. From this it follows that for each $k \in \mathbb{N}$ there exists a Borel complex measure $\mu_{T,k}$ on Ω such that

$$B_k := \int_\Omega \frac{d|\mu_{T,k}|(z)}{[d_\Omega(z)]^k} < \infty$$

and

$$T(f) = \int_\Omega f(z)\, d\mu_{T,k}(z), \ \forall f \in C_0^{-k}(\Omega).$$

Using this for the Cauchy kernel $\dfrac{1}{z-\zeta}$, we have

$$\mathcal{F}_T(\zeta) = \int_\Omega \frac{d\mu_{T,k}(z)}{z-\zeta}, \ \forall \zeta \in \complement\Omega, \ \forall k \in \mathbb{N}. \tag{3.1}$$

Given $n \in \mathbb{N}_0$, we have

$$\int_\Omega \frac{d|\mu_{T,n+1}|(z)}{|z-\zeta|^{n+1}} \le \int_\Omega \frac{d|\mu_{T,n+1}|(z)}{[d_\Omega(z)]^{n+1}} = B_{n+1} < \infty, \ \forall \zeta \in \complement\Omega.$$

Hence, by (3.1) we get

$$\mathcal{F}_T^{(n)}(\zeta) = n! \int_\Omega \frac{d\mu_{T,n+1}(z)}{(z-\zeta)^{n+1}}, \ \forall \zeta \in \complement\Omega,$$

which implies that $\mathcal{F}_T(\zeta) \in \mathcal{O}(\complement\overline{\Omega})$. In addition, for all $\zeta_1, \zeta_2 \in \complement\Omega$,

$$\begin{aligned} |\mathcal{F}_T^{(n)}(\zeta_1) - \mathcal{F}_T^{(n)}(\zeta_2)| &= \left| n! \int_\Omega \frac{d\mu_{T,n+1}(z)}{(z-\zeta_1)^{n+1}} - n! \int_\Omega \frac{d\mu_{T,n+1}(z)}{(z-\zeta_2)^{n+1}} \right| \\ &= n! |\zeta_1 - \zeta_2| \left| \sum_{\ell=0}^{n} \int_\Omega \frac{d\mu_{T,n+1}(z)}{(z-\zeta_1)^{n-\ell+1}(z-\zeta_2)^{\ell+1}} \right| \le B_{n+1}(n+1)! |\zeta_1 - \zeta_2|. \end{aligned}$$

Thus, $\mathcal{F}_T^{(n)}(\zeta)$ is continuous on $\complement\Omega$ for every $n \in \mathbb{N}_0$. Finally, we have that $\mathcal{F}_T$ belongs to $A^\infty(\complement\Omega)$ and, consequently, the corresponding Cauchy operator $\mathcal{F} : T \mapsto \mathcal{F}_T$ acts from $(A^{-\infty}(\Omega))'$ into $A^\infty(\complement\Omega)$. It is easy to see that it has a closed graph. Since $(A^{-\infty}(\Omega))'_b$ and $A^\infty(\complement\Omega)$ both are $(FS)-$spaces, the continuity of this operator follows from the closed graph theorem.

Next we prove that $\mathcal{F}_T((A^{-\infty}(\Omega))') = A^\infty(\complement\Omega)$.

Let g be a fixed function in $A^\infty(\complement\Omega)$. Since $g \in C^\infty(\complement\Omega)$, by Whitney's extension theorem for C^∞-functions [20, Theorem I], there exists its infinitely differentiable extension in $\mathbb{R}^2$, say $\widetilde{g}$. Consider the following form

$$\langle f, g\rangle := \frac{1}{2\pi i}\int_\Omega f(z)\frac{\partial\widetilde{g}}{\partial\bar{z}}(z)\,d\bar{z}\wedge dz,\ \ f \in A^{-\infty}(\Omega).$$

Since $g \in \mathcal{O}(\complement\overline{\Omega}) \cap C^1(\complement\Omega)$, we have $\dfrac{\partial\widetilde{g}}{\partial\bar{z}} = \dfrac{\partial g}{\partial\bar{z}} = 0$ on $\complement\Omega$. From this it follows that $\dfrac{\partial\widetilde{g}}{\partial\bar{z}}(\zeta) = 0$ for all $\zeta \in \partial\complement\Omega = \partial\Omega$. Then, by Taylor's formula,

$$\left|\frac{\partial\widetilde{g}}{\partial\bar{z}}(z)\right| \le C_k[d_\Omega(z)]^k,\ \forall z \in \Omega,\ \forall k \in \mathbb{N},$$

where C_k are some constants depending only on $\widetilde{g}$. Consequently, for every $k \in \mathbb{N}$ and all $f \in A^{-k}(\Omega)$,

$$|\langle f, g\rangle| \le \frac{C_k}{2\pi}\int_\Omega |f(z)|[d_\Omega(z)]^k\,d\lambda_z \le \frac{C_k\lambda(\Omega)}{2\pi}\|f\|_k,$$

where λ is the Lebesgue measure in $\mathbb{R}^2$. Therefore, $\varphi := \langle\cdot, g\rangle \in (A^{-\infty}(\Omega))'$.

It remains to check that the Cauchy transformation of φ coincides with g. To do this, let $\zeta \in \complement\overline{\Omega}$ and $R > 0$ be so large that $\overline{\Omega} \cup \{\zeta\} \subset B_R := \{z : |z| < R\}$. Applying the Cauchy–Green formula for C^1-functions and using that $g \in \mathcal{O}(\complement\overline{\Omega})$ and the equalities $\dfrac{\partial\widetilde{g}}{\partial\bar{z}}(z) = 0$ and $\widetilde{g}(z) = g(z)$ $(z \in \partial B_R)$ and $g(\infty) = 0$, we have

$$\begin{aligned}
\mathcal{F}\varphi(\zeta) &= \left\langle\frac{1}{z-\zeta}, g\right\rangle = \frac{1}{2\pi i}\int_\Omega \frac{1}{z-\zeta}\cdot\frac{\partial\widetilde{g}}{\partial\bar{z}}(z)\,d\bar{z}\wedge dz\\
&= \frac{1}{2\pi i}\int_{B_R}\frac{1}{z-\zeta}\cdot\frac{\partial\widetilde{g}}{\partial\bar{z}}(z)\,d\bar{z}\wedge dz = \widetilde{g}(\zeta) - \frac{1}{2\pi i}\int_{\partial B_R}\frac{\widetilde{g}(z)\,dz}{z-\zeta}\\
&= g(\zeta) - \frac{1}{2\pi i}\int_{\partial B_R}\frac{g(z)\,dz}{z-\zeta} = g(\zeta).
\end{aligned}$$

Thus, $\mathcal{F}_\varphi(\zeta) = g(\zeta)$ for all $\zeta \in \complement\overline{\Omega}$, and, by reasons of continuity, $\mathcal{F}_\varphi = g$ on $\complement\Omega$. This completes the proof. ∎

Next, consider the space $A^\infty(\complement\Omega)$. Recall that the topology of this space is given by the system of norms $(|\cdot|_n)_{n\in\mathbb{N}_0}$ (see (2.1)) under which $A^\infty(\complement\Omega)$ becomes an (FS)-space. By $|\cdot|'_n$ we denote the dual norms

$$|T|'_n := \sup\{|T(g)| : |g|_n \le 1, g \in A^\infty(\complement\Omega)\},\ \ T \in (A^\infty(\complement\Omega))'.$$

Notice that the strong dual $(A^\infty(\complement\Omega))'_b$ is nothing that the (DFS)-space

$$\operatorname*{ind}_n\{T \in (A^\infty(\complement\Omega))' : |T|'_n < \infty\}.$$

Proposition 3.2. *Let Ω be a bounded regular domain in $\mathbb{C}$ with rectifiable boundary. Then the Cauchy transformation of functionals is an epimorphism from $\left(A^\infty(\complement\Omega)\right)'_b$ onto $A^{-\infty}(\Omega)$.*

Proof. Let $T \in \left(A^\infty(\complement\Omega)\right)'$. Then $|T|'_n < \infty$ for some $n \in \mathbb{N}$ and, consequently,

$$|T(g)| \le |T|'_n \, |g|_n, \;\; g \in A^\infty(\complement\Omega).$$

For every $\zeta \in \Omega$, the Cauchy kernel $(z-\zeta)^{-1}$ belongs to $A^\infty(\complement\Omega)$ and

$$\left|\frac{1}{z-\zeta}\right|_n = \max_{0\le k\le n} \max_{z\in\complement\Omega} \frac{k!}{|z-\zeta|^{k+1}} \le \frac{n!}{[d_\Omega(\zeta)]^{n+1}}, \;\; \forall \zeta \in \Omega.$$

Hence,

$$\|\mathcal{F}_T\|_{n+1} = \sup_{\zeta\in\Omega} |\mathcal{F}_T(\zeta)|[d_\Omega(\zeta)]^{n+1} \le n!|T|'_n. \tag{3.2}$$

By standard arguments, one can see that $\mathcal{F}_T \in \mathcal{O}(\Omega)$. Thus, $\mathcal{F}_T \in A^{-\infty}(\Omega)$. Applying (3.2), we see that the Cauchy transformation $\mathcal{F}$ is a linear continuous operator from $\left(A^\infty(\complement\Omega)\right)'_b$ into $A^{-\infty}(\Omega)$.

Prove now that $\mathcal{F}((A^\infty(\complement\Omega)))' = A^{-\infty}(\Omega)$.

Let $f \in A^{-\infty}(\Omega)$. Then $f \in \mathcal{O}(\Omega)$ and there is $n \in \mathbb{N}$ such that

$$\|f\|_n = \sup_{z\in\Omega} |f(z)|[d_\Omega(z)]^n < \infty. \tag{3.3}$$

As above, take $R > 0$ so large that $\overline{\Omega} \subset B_R$. By Whitney's extension theorem (see, e.g., [10, Theorem 2.3.6]), there exists a linear continuous extension operator $L : C^{n+1}(\overline{B}_R \setminus \Omega) \to C^{n+1}(\overline{B}_R)$, where $C^{n+1}(\overline{B}_R \setminus \Omega)$ and $C^{n+1}(\overline{B}_R)$ are the spaces of $(n+1)$-times continuously differentiable functions on $\overline{B}_R \setminus \Omega$ and $\overline{B}_R$, respectively. By $\|L\|$ we denote the norm of the operator L.

Obviously, $A^\infty(\complement\Omega) \subset C^{n+1}(\overline{B}_R \setminus \Omega)$ and, for every $g \in A^\infty(\complement\Omega)$, $\dfrac{\partial g}{\partial \bar z} = 0$ on $\complement\Omega$. Since $Lg\Big|_{B_R\setminus\Omega} = g$, we have that $\dfrac{\partial (Lg)}{\partial \bar z} = 0$ on $\partial\Omega$. For every $z \in \Omega$, take $\zeta \in \partial\Omega$ so that $|\zeta - z| = d_\Omega(z)$. By Taylor's formula, there is $\theta = \theta(z,\zeta) \in (0,1)$ such that

$$\frac{\partial(Lg)}{\partial \bar z}(z) = \sum_{j=0}^{n} \frac{1}{(n-j)!j!} \frac{\partial^n(\partial(Lg)/\partial\bar z)}{\partial x^{n-j}\partial y^j}(z+\theta(\zeta-z))\;(x-\xi)^{n-j}(y-\eta)^j,$$

where $z = x+iy$ and $\zeta = \xi + i\eta$. Then

$$\left|\frac{\partial(Lg)}{\partial \bar z}(z)\right| \le \frac{2^n}{n!}\|L\||g|_{n+1}[d_\Omega(z)]^n$$

and, consequently, the linear form

$$\langle f, g\rangle := \frac{1}{2\pi i}\int_\Omega f(z)\frac{\partial(Lg)}{\partial \bar z}(z)\, d\bar z \wedge dz, \;\; g \in A^\infty(\complement\Omega),$$

satisfies the following estimate

$$|\langle f, g\rangle| \le C|g|_{n+1},\ g \in A^\infty(\mathbb{C}\Omega),$$

where $C := \dfrac{2^{n-1}}{\pi n!}\|L\|\|f\|_n\lambda(\Omega)$. Hence, $\varphi := \langle f, \cdot\rangle$ is a linear continuous functional on $A^\infty(\mathbb{C}\Omega)$.

We now check that for Ω with rectifiable boundary, $\mathcal{F}(\varphi) = f$. Indeed, for such a domain there exists a sequence of finitely connected domains Ω_k with smooth boundaries $\partial\Omega_k$ such that $\overline{\Omega}_k \subset \Omega_{k+1}$, $\Omega = \bigcup_k \Omega_k$ and the lengths ℓ_k of $\partial\Omega_k$ are bounded from above by some constant M independent of k. By the Cauchy-Green formula

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Omega_k} f(z)\frac{\partial(Lg)}{\partial\bar z}(z)\,d\bar z\wedge dz\\
=\ &\frac{1}{2\pi i}\int_{\Omega_k}\frac{\partial(fLg)}{\partial\bar z}(z)\,d\bar z\wedge dz = \frac{1}{2\pi i}\int_{\partial\Omega_k} f(z)(Lg)(z)\,dz.
\end{aligned}$$

Consequently,

$$\langle f, g\rangle = \lim_{k\to\infty}\frac{1}{2\pi i}\int_{\partial\Omega_k} f(z)(Lg)(z)\,dz.$$

Let now g be holomorphic in some neighborhood of $\partial\Omega$. Take k so large that $\partial\Omega_k$ lies in this neighborhood. As above, for every $z \in \partial\Omega_k$, choose $\zeta \in \partial\Omega$ so that $|\zeta - z| = d_\Omega(z)$. Then, using twice Taylor's formula, we have that

$$\begin{aligned}
&(Lg)(z)\\
&= g(\zeta) + \frac{g'(\zeta)}{1!}(z-\zeta) + \ldots + \frac{g^{(n)}(\zeta)}{n!}(z-\zeta)^n +\\
&\qquad\qquad \frac{(Lg)^{(n+1)}(z+\theta_1(z-\zeta))}{(n+1)!}(z-\zeta)^{n+1}\\
&= g(z) - \frac{(Lg)^{(n+1)}(z+\theta_2(z-\zeta))}{(n+1)!}(z-\zeta)^{n+1} +\\
&\qquad\qquad \frac{(Lg)^{(n+1)}(z+\theta_1(z-\zeta))}{(n+1)!}(z-\zeta)^{n+1},
\end{aligned}$$

where $\theta_{1,2} = \theta_{1,2}(z,\zeta) \in (0,1)$. Thus,

$$|(Lg)(z) - g(z)| \le B[d_\Omega(z)]^{n+1},\ z \in \partial\Omega_k,$$

where B is some constant depending only on $|g|_{n+1}$. From this it follows that

$$\left|\int_{\partial\Omega_k} f(z)(Lg)(z)\,dz - \int_{\partial\Omega_k} f(z)g(z)\,dz\right| \le B\|f\|_n d_k\ell_k,$$

where $d_k := \max_{z\in\partial\Omega_k} d_\Omega(z) \to 0$ as $k\to\infty$.

Thus, for g holomorphic in some neighborhood of $\partial\Omega$,

$$\langle f, g\rangle = \lim_{k\to\infty}\frac{1}{2\pi i}\int_{\partial\Omega_k} f(z)g(z)\,dz.$$

In particular, for $g(z) = \dfrac{1}{z-\zeta}$ with a fixed $\zeta \in \Omega$,

$$\mathcal{F}_{\varphi}(\zeta) = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{\partial\Omega_k} \frac{f(z)}{z-\zeta}\, dz = f(\zeta).$$

This completes the proof. ∎

4 The Cauchy transformation as a mutual isomorphism

For a subset Z in $\mathbb{C}$, let

$$\mathcal{PF}[Z] := \left\{ \frac{1}{z-\zeta} : \zeta \in Z \right\},$$

being a family of partial fractions.

As an immediate consequence of Propositions 3.1 and 3.2, the well-known Banach criterion of completeness, and the open mapping theorem, we have the following result.

Proposition 4.1. *(a) Let Ω be a bounded regular domain in $\mathbb{C}$. The Cauchy transformation of functionals is an isomorphism between $(A^{-\infty}(\Omega))'_b$ and $A^{\infty}(\complement\Omega)$ if and only if the system $\mathcal{PF}[\complement\Omega]$ is complete in $A^{-\infty}(\Omega)$.*

(b) Let Ω be a bounded regular domain in $\mathbb{C}$ with rectifiable boundary. The Cauchy transformation of functionals is an isomorphism between $\left(A^{\infty}(\complement\Omega)\right)'_b$ and $A^{-\infty}(\Omega)$ if and only if the system $\mathcal{PF}[\Omega]$ is complete in $A^{\infty}(\complement\Omega)$.

Thus, we should study the question about the completeness of the corresponding systems of partial fractions in spaces $A^{-\infty}(\Omega)$ and $A^{\infty}(\complement\Omega)$. To do this, we start with the following auxiliary, perhaps known, result.

Lemma 4.2. *Let Ω be a bounded domain in $\mathbb{C}$ and*

$$\widetilde{A}^{-k}(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \widetilde{\|f\|}_k := \int_{\Omega} |f(z)| [d_{\Omega}(z)]^k \, d\lambda_z < \infty \right\}, \ k \in \mathbb{N},$$

being normed spaces. Then $A^{-\infty}(\Omega) = \bigcup_{k=1}^{\infty} \widetilde{A}^{-k}(\Omega)$ and the original topology in $A^{-\infty}(\Omega)$ coincides with the topology of inductive limit from the sequence $\left(\widetilde{A}^{-k}(\Omega)\right)_{k=1}^{\infty}$.

Proof. Since $\widetilde{\|f\|}_k \le \lambda(\Omega)\|f\|_k$ for all $f \in A^{-k}(\Omega)$, $A^{-k}(\Omega) \hookrightarrow \widetilde{A}^{-k}(\Omega)$ for every $k \in \mathbb{N}$.

Let now $f \in \widetilde{A}^{-k}(\Omega)$. Fixing the point $z \in \Omega$, we have that the ball $B_z := \{w : |w-z| \le \frac{1}{2} d_{\Omega}(z)\}$ is contained in Ω and $d_{\Omega}(w) \ge \frac{1}{2} d_{\Omega}(z)$ for all

$w \in B_z$. Then, using also that $|f|$ is a subharmonic function in Ω, we have

$$
\begin{aligned}
|f(z)| &\leq \frac{4}{\pi[d_\Omega(z)]^2}\int_{B_z}|f(w)|\,d\lambda_w \\
&\leq \frac{4}{\pi[d_\Omega(z)]^2}\left(\sup_{w\in B_z}\frac{1}{[d_\Omega(w)]^k}\right)\int_\Omega |f(w)|[d_\Omega(w)]^k\,d\lambda_w \\
&\leq \frac{2^{k+2}}{\pi[d_\Omega(z)]^{k+2}}\widetilde{\|f\|}_k.
\end{aligned}
$$

Therefore, $\|f\|_{k+2} \leq 2^{k+2}\pi^{-1}\widetilde{\|f\|}_k$ for all $f \in \widetilde{A}^{-k}(\Omega)$. Consequently, $\widetilde{A}^{-k}(\Omega) \hookrightarrow A^{-k-2}(\Omega)$.

This completes the proof. ∎

Theorem 4.3. *Let Ω be a Carathéodory domain in $\mathbb{C}$. Then the Cauchy mapping is an isomorphism from $(A^{-\infty}(\Omega))'_b$ onto $A^\infty(\complement\Omega)$.*

Proof. By Proposition 4.1 (a), we should only prove that $\mathcal{PF}[\complement\Omega]$ is complete in $A^{-\infty}(\Omega)$.

First, we prove that the set of all polynomials is dense in $A^{-\infty}(\Omega)$.

Fix any $k \in \mathbb{N}$ and note that $d_\Omega(w) \leq 2d_\Omega(z)$ for every $z \in \Omega$ and all $w \in \Omega$ with $|w-z| \leq d_\Omega(z)$. Then

$$
\frac{1}{r^2}\int_{|w-z|\leq r}[d_\Omega(w)]^k\,d\lambda_w \leq 2^k\pi[d_\Omega(z)]^k,\ \ 0<r\leq d_\Omega(z),\ z\in\Omega.
$$

Consequently, for each $f \in \widetilde{A}^{-k}(\Omega)$,

$$
\begin{aligned}
&\int_\Omega |f(z)|\left\{\sup_{0<r\leq d_\Omega(z)}\frac{1}{r^2}\int_{|w-z|\leq r}[d_\Omega(w)]^k\,d\lambda_w\right\}d\lambda_z \\
\leq\ & 2^k\pi\int_\Omega |f(z)|[d_\Omega(z)]^k\,d\lambda_z = 2^k\pi\widetilde{\|f\|}_k < \infty.
\end{aligned}
$$

Hence, by Hedberg [9, Theorem 1] the set of all polynomials is dense in $\widetilde{A}^{-k}(\Omega)$ (this space coincides with $H^1([d_\Omega(z)]^k;\Omega)$ in the notation of [9]). Using Lemma 4.2, we obtain the desired density of polynomials in $A^{-\infty}(\Omega)$.

To complete the proof, it is sufficient to note that each polynomial can be approximated in the topology of $A^{-\infty}(\Omega)$ by partial fractions of $\mathcal{PF}[\Omega^c]$. But this is an easy consequence of the well-known fact that $\mathcal{PF}[\complement B_R]$ is a complete system in $O(B_R)$ (as above, B_R is the open disk centered at the origin of radius R). Taking R so large that $\overline{\Omega} \subset B_R$, we can approximate each polynomial by functions from span$\mathcal{PF}[\complement B_R] \subset \mathcal{PF}[\complement\Omega]$ in the topology of $O(B_R)$ and, in particular, by the norm $\|f\| := \max_{z\in\overline{\Omega}}|f(z)|$. It remains to note that $\|f\|_k \leq \|f\|$ for all $f \in \mathcal{O}(\Omega)\cap C(\overline{\Omega})$. ∎

Proposition 4.4. *Let Ω be a bounded domain in $\mathbb{C}$. Then the system $\mathcal{PF}[\Omega]$ is complete in $A^\infty(\complement\Omega)$.*

Proof. Let $g \in A^\infty(\complement\Omega)$. By Whitney's extension theorem, there exists $\widetilde{g} \in C^\infty(\mathbb{R}^2)$ such that $\widetilde{g}^{(\alpha)}|_{\complement\Omega} = g^{(\alpha)}$ for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. Fixed $z \in \complement\Omega$, let $R > 0$ be so large that $\overline{\Omega} \cup \{z\} \subset B_R$. Applying the Cauchy–Green formula and using that $g \in \mathcal{O}(\complement\Omega)$, we have

$$\begin{aligned} g(z) &= \widetilde{g}(z) = \frac{1}{2\pi i}\int_{\partial B_R} \frac{\widetilde{g}(\tau)}{\tau - z}\, d\tau + \frac{1}{2\pi i}\int_{B_R} \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{\tau - z}\, d\bar{\tau} \wedge d\tau \\ &= \frac{1}{2\pi i}\int_{\partial B_R} \frac{g(\tau)}{\tau - z}\, d\tau + \frac{1}{2\pi i}\int_{\Omega} \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{\tau - z}\, d\bar{\tau} \wedge d\tau \\ &= \frac{1}{2\pi i}\int_{\Omega} \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{\tau - z}\, d\bar{\tau} \wedge d\tau. \end{aligned}$$

Thus,

$$g(z) = \frac{1}{2\pi i}\int_{\Omega} \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{\tau - z}\, d\bar{\tau} \wedge d\tau,\ z \in \complement\Omega,$$

and, consequently,

$$g^{(k)}(z) = \frac{k!}{2\pi i}\int_{\Omega} \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{(\tau - z)^{k+1}}\, d\bar{\tau} \wedge d\tau,\ z \in \complement\Omega,\ k \in \mathbb{N}_0. \tag{4.1}$$

As above, since $\frac{\partial\widetilde{g}}{\partial\bar{z}}(\tau) = 0$ on $\partial\Omega$, $\frac{\partial\widetilde{g}}{\partial\bar{z}}(\tau) = o([d_\Omega(\tau)]^m)$ $(\tau \in \Omega,\ m \in \mathbb{N})$. Hence, the functions $\frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{(\tau - z)^{k+1}}$ are continuous and, consequently, equicontinuous on $\overline{\Omega} \times \complement\Omega$. Consequently, for a fixed $\varepsilon > 0$ and $n \in \mathbb{N}_0$, there is $\delta > 0$ such that

$$\left| \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{(\tau - z)^{k+1}} - \frac{(\partial\widetilde{g}/\partial\bar{z})(\xi)}{(\xi - z)^{k+1}} \right| \le \varepsilon, \forall z \in \complement\Omega, \forall \xi, \tau \in \overline{\Omega} \text{ with } |\xi - \tau| \le \delta,$$
$$\text{and } k = 0, \ldots, n.$$

Choosing a partition of Ω into measurable pairwise disjoint subsets $\Omega_1, \ldots, \Omega_p$, the diameters of which are less than δ and taking arbitrary points $\tau_j \in \Omega_j$ $(1 \le j \le p)$, we then have

$$\begin{aligned} &\left| \frac{1}{2\pi i}\int_{\Omega} \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{(\tau - z)^{k+1}}\, d\bar{\tau} \wedge d\tau - \sum_{j=1}^{p} 2i\lambda(\Omega_j)\frac{(\partial\widetilde{g}/\partial\bar{z})(\tau_j)}{(\tau_j - z)^{k+1}} \right| \\ \le\ & \frac{1}{2\pi}\sum_{j=1}^{p}\int_{\Omega_j} \left| \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau)}{(\tau - z)^{k+1}} - \frac{(\partial\widetilde{g}/\partial\bar{z})(\tau_j)}{(\tau_j - z)^{k+1}} \right| d\lambda \le \frac{1}{2\pi}\sum_{j=1}^{p}\varepsilon\lambda(\Omega_j) = \frac{\lambda(\Omega)}{2\pi}\varepsilon, \end{aligned}$$

for all $z \in \complement\Omega$ and $k = 0, \ldots, n$. Then, putting $c_j := \frac{2i\lambda(\Omega_j)}{\pi}\frac{\partial\widetilde{g}}{\partial\bar{z}}(\tau_j)$ and using (4.1), we obtain that the function

$$h(z) := \sum_{j=1}^{p} \frac{c_j}{\tau_j - z} \in \operatorname{span}\mathcal{PF}[\Omega]$$

satisfies the inequalities

$$|g^{(k)}(z) - h^{(k)}(z)| \le k!\frac{\lambda(\Omega)}{2\pi}\varepsilon,\ z \in \complement\Omega, k = 0, \dots, n.$$

Thus,

$$|g - h|_n \le n!\frac{\lambda(\Omega)}{2\pi}\varepsilon,$$

which completes the proof. ■

As an immediate consequence of Propositions 4.1 (b) and 4.4 we have the desired result about the strong dual of $A^\infty(\complement\Omega)$.

Theorem 4.5. *Let Ω be a bounded regular domain in $\mathbb{C}$ with rectifiable boundary. Then the Cauchy mapping is an isomorphism from $\left(A^\infty(\complement\Omega)\right)'_b$ onto $A^{-\infty}(\Omega)$.*

To this end, combining Theorems 4.3 and 4.5, we obtain

Corollary 4.6. *Let Ω be a Carathéodory domain with rectifiable boundary. Then the Cauchy transformation of functionals establishes a mutual duality between the spaces $A^{-\infty}(\Omega)$ and $A^\infty(\complement\Omega)$.*

5 A possibility of representation of functions from $A^{-\infty}(\Omega)$ and $A^\infty(\complement\Omega)$ by series of partial fractions

In this section we study the question whether functions from the two spaces $A^{-\infty}(\Omega)$ and $A^\infty(\complement\Omega)$ can be represented in the form of a series of partial fractions.

This problem can be formulated in a general setting as follows: a sequence (x_k) of non-zero elements of a locally convex space H is said to be an *absolutely representing system* in H if any element x from H can be represented in the form of the series

$$x = \sum_{k=1}^{\infty} c_k x_k,$$

which converges absolutely in the topology of H (see, e.g., [12]). This concept is more general than the concept of basis, where the uniqueness of the representation is essentially required.

There are various criteria for a countable system to be absolutely representing in (FS)- and (DFS)- spaces. Later such criteria have been formulated for more practical spaces. One of those results will be used in this section.

5.1 Weakly sufficient sets for $A^{-\infty}(\Omega)$

Recall that we consider the space $A^{-\infty}(\Omega)$ with its natural topological structure, the internal inductive limit of Banach spaces $A^{-k}(\Omega)$:

$$(A^{-\infty}(\Omega), \tau) = \liminf A^{-k}(\Omega).$$

For a subset $S \subseteq \Omega$, we define $\|f\|_{k,S} := \sup_{z\in S} |f(z)|\ [d_\Omega(z)]^k$, and $A^{-k,S}(\Omega) := \{f \in A^{-\infty}(\Omega) : \|f\|_{k,S} < +\infty\}$. In general, $\|\cdot\|_{k,S}$ is a semi-norm, and we have inclusion relations $A^{-k}(\Omega) \subset A^{-k,S}(\Omega) \subset A^{-\infty}(\Omega)$. From this it follows that $A^{-\infty}(\Omega) = \bigcup_{k\in\mathbb{N}} A^{-k}(\Omega) = \bigcup_{k\in\mathbb{N}} A^{-k,S}(\Omega)$.

Since $A^{-k}(\Omega) \hookrightarrow A^{-k,S}(\Omega)$, one can endow $A^{-\infty}(\Omega)$ with another weaker internal inductive limit topology from the sequence $(A^{-k,S}(\Omega), \|\cdot\|_{k,S})$ of vector spaces equipped with the semi-norms:

$$(A^{-\infty}(\Omega), \tau_S) = \lim\ \mathrm{ind}\ A^{-k,S}(\Omega).$$

Definition 5.1. (see, e.g., [17]) A subset $S \subseteq \Omega$ is called *weakly sufficient* for $A^{-\infty}(\Omega)$ if the two topologies τ and τ_S are equivalent.

It is clear that the domain Ω itself is weakly sufficient for the space $A^{-\infty}(\Omega)$. The question to ask is: does there exist a discrete (countable) subset S which is weakly sufficient? The answer is positive, and moreover, there are different ways to solve this problem.

In [8] it was shown, in particular, that for a rather general weighted inductive limit of spaces of holomorphic functions in any domain of $\mathbb{C}$ satisfying some natural growth conditions, there always exists a discrete weakly sufficient set. The following result is given in [3].

Lemma 5.2. *Let Ω be an arbitrary domain in $\mathbb{C}$. Then in the space $A^{-\infty}(\Omega)$ every weakly sufficient set contains a discrete, weakly sufficient subset which is closed in Ω.*

It should be noted that Lemma 5.2, by applying the result from [8], works for any domain in $\mathbb{C}$. It however guarantees only a "theoretical existence" of such a discrete set. In [7] an explicit ("algorithmic") method for construction of a countable weakly sufficient set was presented for any bounded domain with C^1 smooth boundary. Applying this algorithm we can also obtain the (explicit) existence of weakly sufficient sets for $A^{-\infty}(\Omega)$ in the present paper.

5.2 Sufficient sets for $A^\infty(\complement\Omega)$

For a set $S \subset \complement\Omega$, we put

$$|f|_{m,S} := \max_{0\le k\le n}\max_{z\in S} |f^{(k)}(z)|,\ \ f \in A^\infty(\complement\Omega),\ n = 0, 1, \ldots.$$

Then the system of seminorms $(|f|_{m,S})_{m=0}^\infty$ defines another topology τ_S in $A^\infty(\complement\Omega)$ that is weaker than the origin topology τ.

Similar to the definition of weak sufficiency above, we can introduce the following notion.

Definition 5.3. The set S is called *sufficient* for $A^\infty(\complement\Omega)$ if the two topologies τ and τ_S coincide.

It is clear that S is sufficient for $A^\infty(\complement\Omega)$ if and only if

$$\forall m\ \exists \ell\ \exists A_m > 0: \ \ |f|_m \le A_m\,|f|_{\ell,S},\ \forall f \in A^\infty(\complement\Omega).$$

The existence of sufficient sets in $A^\infty(\complement\Omega)$ is given in the following result ([3, Corollary 3.6]).

Lemma 5.4. *Let Ω be a regular domain in $\mathbb{C}$. Then there exists a sequence $(\zeta_k)_{k=1}^\infty \subset \complement\Omega$, without accumulation points in $\complement\overline{\Omega}$, which is a sufficient set for $A^\infty(\complement\Omega)$.*

5.3 Functions from $A^\infty(\complement\Omega)$ can be represented in series of partial fractions

Applying a criterion [13, Theorem F] and Theorem 4.5 to the spaces $A^\infty(\complement\Omega)$ and $A^{-\infty}(\Omega)$, we obtain the following.

Lemma 5.5. *Let Ω be a bounded regular domain in $\mathbb{C}$ with rectifiable boundary. The system $\left(\frac{1}{z_k-\zeta}\right)_{k=1}^\infty$, where $(z_k) \subset \Omega$, is absolutely representing in the space $A^\infty(\complement\Omega)$ if and only if the set (z_k) is weakly sufficient for the space $A^{-\infty}(\Omega)$, i.e.,*

$$\forall s \in \mathbb{N}\ \exists p = p(s) \in \mathbb{N},\ \exists C = C(s) > 0 \text{ such that}$$

$$\sup_{z\in\Omega} |g(z)|[d_\Omega(z)]^p \le C \sup_{k\ge 1} |g(z_k)|[d(z_k)]^s,\ \forall g \in A^{-\infty}(\Omega).$$

By Lemma 5.2, we have the following representation result for $A^\infty(\complement\Omega)$.

Theorem 5.6. *Let Ω be a bounded regular domain in $\mathbb{C}$ with rectifiable boundary. Then there is a discrete sequence $(z_k)_{k=1}^\infty \subset \Omega$, such that the system $\left(\frac{1}{z_k-\zeta}\right)_{k=1}^\infty$ is absolutely representing in the space $A^\infty(\complement\Omega)$, that is, any function $g \in A^\infty(\complement\Omega)$ can be represented in the form of a series of partial fractions*

$$g(\zeta) = \sum_{k=1}^\infty \frac{c_k}{z_k - \zeta},\ \forall \zeta \in \complement\Omega,$$

that converges absolutely in the space $A^\infty(\complement\Omega)$.

5.4 The space $A^{-\infty}(\Omega)$ has no absolutely representing system of partial fractions

We note that for the space $A^{-\infty}(\Omega)$ the "dual relationship" above would look as follows: the system

$$\left(\frac{1}{z-\zeta_k}\right)_{k=1}^\infty,\ \zeta_k \in \complement\Omega\ (k \in \mathbb{N}),$$

is absolutely representing in $A^{-\infty}(\Omega)$, that is, each function $f \in A^{-\infty}(\Omega)$ can be represented in the form of a series

$$f(z) = \sum_{k=1}^\infty \frac{c_k}{z-\zeta_k},\ \forall z \in \Omega, \tag{5.1}$$

which converges absolutely in the topology of $A^{-\infty}(\Omega)$, if and only if the set $(\zeta_k)_{k=1}^{\infty} \subset \complement\Omega$ is sufficient for the space $A^{\infty}(\complement\Omega)$.

However, despite Lemma 5.4, there does not exist any absolutely representing system of partial fractions in the space $A^{-\infty}(\Omega)$. In other words, representation (5.1) is impossible.

Theorem 5.7. *Let Ω be a Carathéodory domain with rectifiable boundary. There is no absolutely representing system of partial fractions in $A^{-\infty}(\Omega)$.*

Proof. Assume, on the contrary, that there exists a sequence $(\zeta_k)_{k=1}^{\infty} \subset \complement\Omega$ such that the corresponding system $\left(\frac{1}{z-\zeta_k}\right)_{k=1}^{\infty}$ is absolutely representing for the space $A^{-\infty}(\Omega)$. This means that for a function $f \in A^{-\infty}(\Omega)$, we have a representation (5.1), where the series converges absolutely in the topology of $A^{-\infty}(\Omega)$.

We may assume w.l.o.g. that $0 \in \Omega$. Since the topology in $A^{-\infty}(\Omega)$ is stronger than the topology of pointwise convergence on Ω,

$$\sum_{k=1}^{\infty} \frac{|c_k|}{|z-\zeta_k|} < \infty, \quad \text{for each } z \in \Omega. \tag{5.2}$$

In particular, for $z = 0 \in \Omega$, we have

$$M := \sum_{k=1}^{\infty} \frac{|c_k|}{|\zeta_k|} < \infty.$$

Let $R := \max\{|z| : z \in \partial\Omega\}$. Using the facts that $|z-\zeta| > \dfrac{|\zeta|}{2}$, $\forall z \in \Omega$, $\forall |\zeta| > 2R$, and $d_\Omega(z) \le \min\{R, |z-\zeta|\}$, $\forall z \in \Omega$, $\forall \zeta \in \complement\Omega$, we have

$$\begin{aligned}
|f(z)| &\le \sum_{|\zeta_k|\le 2R} \frac{|c_k|}{|z-\zeta_k|} + \sum_{|\zeta_k|>2R} \frac{|c_k|}{|z-\zeta_k|} \\
&\le \frac{1}{d_\Omega(z)} \sum_{|\zeta_k|\le 2R} |c_k| + \sum_{|\zeta_k|>2R} \frac{2|c_k|}{|\zeta_k|} \\
&\le \frac{1}{d_\Omega(z)} \left(\sum_{|\zeta_k|\le 2R} \frac{2R}{|\zeta_k|}|c_k| + \sum_{|\zeta_k|>2R} \frac{2R|c_k|}{|\zeta_k|} \right) \\
&= \frac{2RM}{d_\Omega(z)}, \text{ for all } z \in \Omega.
\end{aligned}$$

This shows that every function $f \in A^{-\infty}(\Omega)$, which is represented in the form (5.1), necessarily belongs to $A^{-1}(\Omega)$. On the other hand, for every $\zeta \in \partial\Omega$, the function $f_\zeta(z) := \dfrac{1}{(z-\zeta)^2}$ belongs to $A^{-2}(\Omega) \setminus A^{-1}(\Omega)$. This contradiction completes the proof. ∎

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