Double point-homogeneous spherical curves

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Abstract

A curve is, in this paper, the image of the circle S^1 under an immersion f into S^2, $\mathbb{R}^2$ or the real projective plane $P_2(\mathbb{R})$, such that every multiple point of f is an ordinary double point. Such a curve C is double point-homogeneous or DP-homogeneous when the group of diffeomorphisms (of $S^2, \mathbb{R}^2$ or $P_2(\mathbb{R})$) preserving C has a transitive action on the set of its double points. The orbits of DP-homogeneous curves in S^2 are totally determined; using combinatorial methods, we prove that they fall into five countably infinite families ; the description of every family is illustrated by drawings of some representatives with a small number of double points. As a corollary, we obtain a similar classification of the DP-homogeneous curves in $\mathbb{R}^2$. We also propose a conjecture about the classification of DP-homogeneous curves in $P_2(\mathbb{R})$.

1 Introduction

The curves considered in this paper are *generic* which means that each one is the image of an immersion f of the circle S^1 into a two-dimensional manifold M such that every multiple point of f is an ordinary double point. Such a curve C is said to be *double point-homogeneous* or *DP-homogeneous* if, for every pair (p, q) of double points of C, there is a diffeomorphism of M which preserves C and sends p onto q. The main result of this paper is the classification of the orbits of DP-homogeneous spherical curves (case $M = S^2$) under the action of the group of all diffeomorphisms of S^2. A consequence of this is the classification of DP-homogeneous plane curves (case $M = \mathbb{R}^2$) under the action of the group of

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all diffeomorphisms of $\mathbb{R}^2$. We sometimes say that two curves are *equivalent* if they belong to the same orbit.

Examples of DP-homogeneous plane curves are presented in Figures 1 and 2. Any two different curves among the eight shown there are not equivalent. But if we denote by C a curve in Figure 1 and by D the curve in Figure 2 having the same number of double points as C, and if we map $\mathbb{R}^2$ onto open subsets of S^2 by diffeomorphisms F and G, then the spherical curves $F(C)$ and $G(D)$ are equivalent. This remark suggests the existence of a first infinite family of orbits of DP-homogeneous spherical curves, the family $\mathbf{P}$, with representatives of $\mathbf{P}_1$ to $\mathbf{P}_4$ shown in Figure 3.

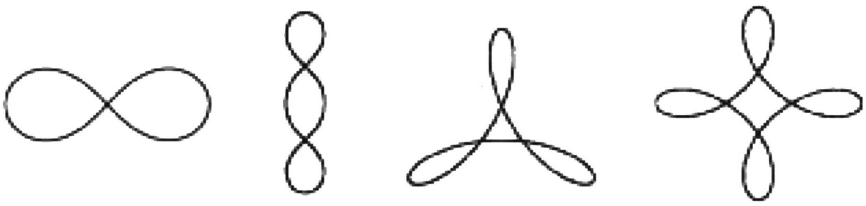

FIGURE 1: Four DP-homogeneous plane curves which are not equivalent with respect to diffeomorphisms of $\mathbb{R}^2$. They belong to Family $\mathbf{P}'$ (see Section 4).

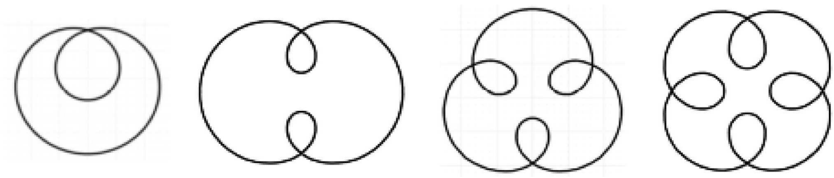

FIGURE 2: Four DP-homogeneous plane curves which are not equivalent to those of Figure 1. They belong to Family $\mathbf{P}''$ (see Section 4).

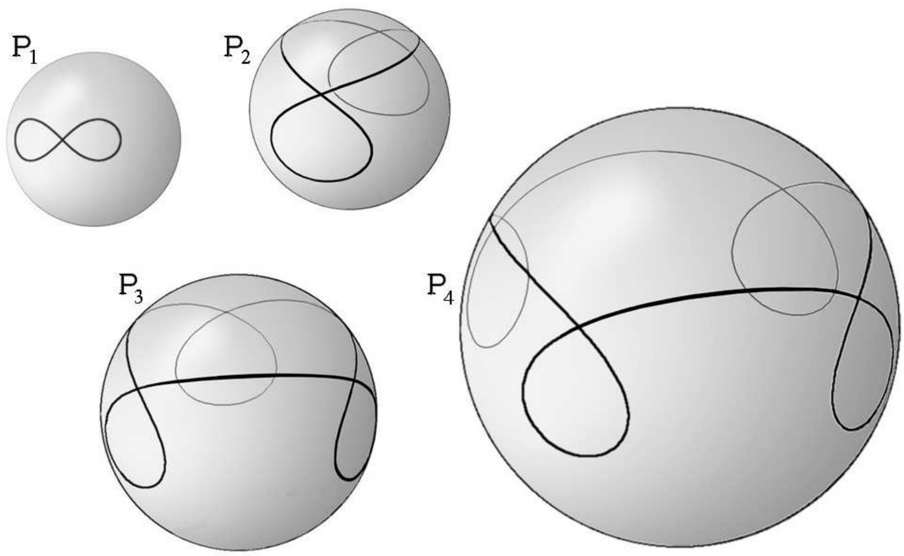

FIGURE 3: Representatives of the first four elements of the family $\mathbf{P}$ of orbits of DP-homogeneous spherical curves.

We shall prove that the other orbits of DP-homogeneous spherical curves are classified in a natural way into four families presented in Figures 4, 5, 6 and 7 by means of representatives.

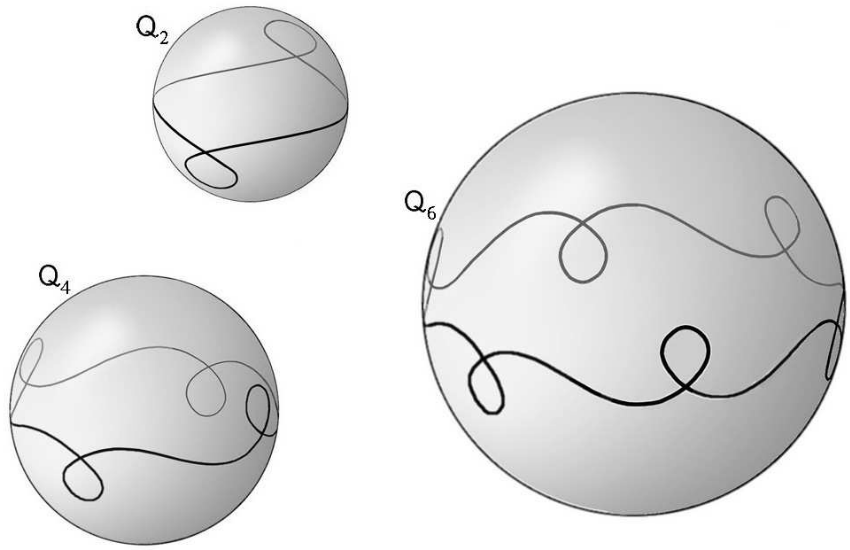

FIGURE 4: Representatives of the first three elements of the family $\mathbf{Q}$.

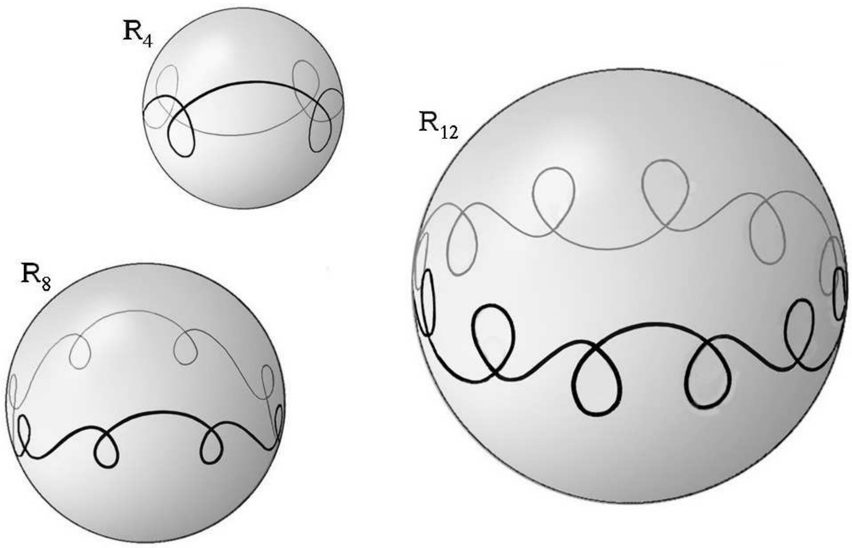

FIGURE 5: Representatives of the first three elements of the family $\mathbf{R}$.

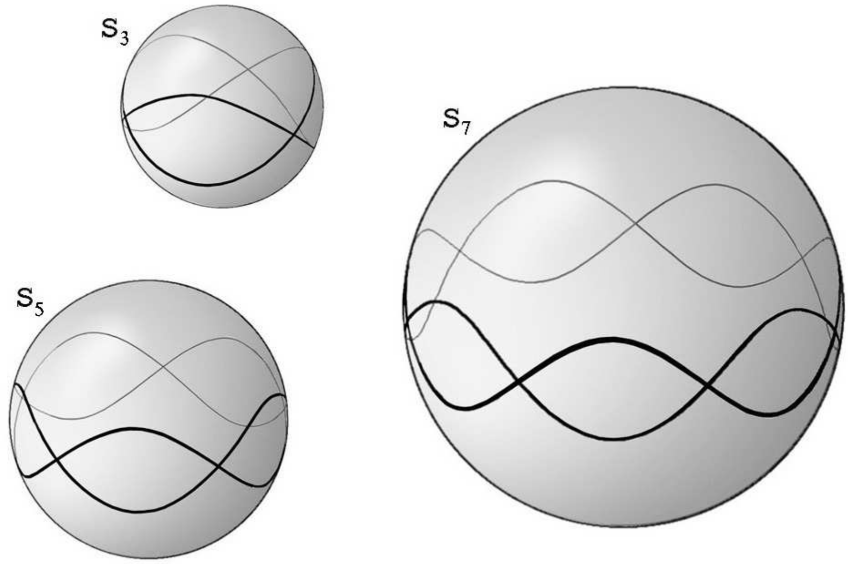

FIGURE 6: Representatives of the first three elements of the family **S**.

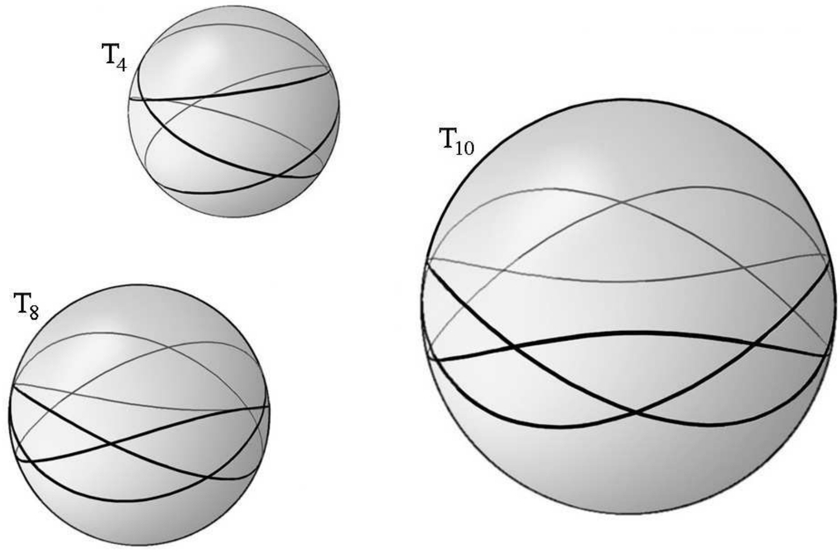

FIGURE 7: Representatives of the first three elements of the family **T**.

Let us denote by $O(n)$ the number of orbits of DP-homogeneous spherical curves having exactly n double points ($n \geq 1$); a first consequence of our classification is the fact that the function $n \rightarrow O(n)$ is completely known: its first fourteen values are 1 , 2 , 2 , 4 , 2 , 2 , 2 , 4 , 2 , 3 , 2 , 3 , 2 , 3 , and the next values satisfy the recurrence $O(n) = O(n - 12)$.

Another consequence of our classification is the analogous classification of the orbits of DP-homogeneous plane curves. Two infinite families were already presented in Figures 1 and 2. There is a third one: representatives of its first three elements are shown in Figure 8.

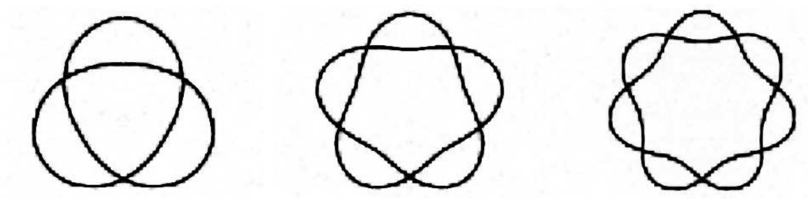

FIGURE 8: Representatives of the first elements of the third family of orbits of DP-homogeneous plane curves. They belong to Family $\mathbf{S}'$ (see Section 4).

2 Statement of the main result

The following definitions, where M denotes $\mathbb{R}^2$ or S^2, are useful for the description of DP-homogeneous (plane or spherical) curves.

DEFINITIONS: A *curvilinear m-gon* ($m \geq 1$) is any subset D of M which is homeomorphic to a closed disk and whose boundary B is a closed curve which is smooth everywhere excepted in m angular points, called *vertices*. If $m > 1$, a *side* of D is an arc of B joining neighboring vertices; if $m = 1$, it is B.

A vertex a of the curvilinear m-gon D is said to be *salient* if the measure of the interior angle of D in a is smaller than π, and is *re-entrant* if this measure is greater than π.

Let C be a curve having n double points ($n \geq 1$); a curvilinear m-gon is said to be *inscribed* in C if its sides are arcs of C joining neighboring double points if $m > 1$, the same double point if $m = 1$. An example of an inscribed 5-gon is given in Figure 9.

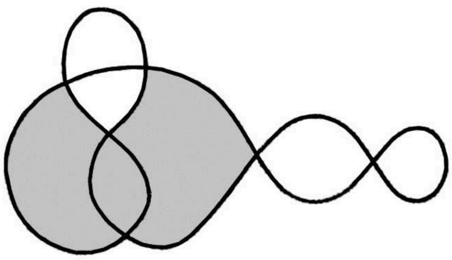

FIGURE 9: A 5-gon (coloured in grey) inscribed in a curve.

A curve C in M determines a tiling of M, whose tiles are the closures of the connected components of $M \setminus C$; for brevity's sake, we will say that the tiles of this tiling are the *tiles* of C. Such a tile is *biangular* (resp. *triangular*) if it is a curvilinear 2-gon (resp. 3-gon) with salient vertices.

THEOREM: *If a DP-homogeneous spherical curve has at least one double point, then* (under the group of all diffeomorphisms of S^2) *it belongs to one orbit of one of the following five families:*

1) *The family* $\mathbf{P}$ *is the sequence of orbits* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \dots, \mathbf{P}_n, \dots$ *where any element of* $\mathbf{P}_1$ *is a figure-eight curve and, if* $n > 1$*, any element* C *of* $\mathbf{P}_n$ *is a curve (with* n *double points) one tile of which is a curvilinear* n*-gon with salient vertices, each of these vertices being also the vertex of a curvilinear 1-gon inscribed in* C. Examples of elements of $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ are shown on the four spheres of Figure 3.

2) *The family* $\mathbf{Q}$ *is the sequence of orbits* $\mathbf{Q}_2, \mathbf{Q}_4, \mathbf{Q}_6, \dots, \mathbf{Q}_{2m}, \dots$ *where any element* C *of* $\mathbf{Q}_{2m}$ *is a curve (with* $2m$ *double points) in which a curvilinear* $2m$*-gon* D *is inscribed ; the vertices of* D *are alternately salient and re-entrant and each of them is also the vertex of a curvilinear 1-gon inscribed in* C. Examples of elements of $\mathbf{Q}_2$, $\mathbf{Q}_4$ and $\mathbf{Q}_6$ are shown on the three spheres of Figure 4.

3) *The family* $\mathbf{R}$ *is the sequence of orbits* $\mathbf{R}_4, \mathbf{R}_8, \mathbf{R}_{12}, \dots, \mathbf{R}_{4m}, \dots$ *where any element* C *of* $\mathbf{R}_{4m}$ *is a curve (with* $4m$ *double points) in which a curvilinear* $4m$*-gon* D *is inscribed; every vertex of* D *has one salient neighbour and one re-entrant neighbour, and is also the vertex of a curvilinear 1-gon incribed in* C. Examples of elements of $\mathbf{R}_4$, $\mathbf{R}_8$ and $\mathbf{R}_{12}$ are shown on the spheres of Figure 5.

4) *The family* $\mathbf{S}$ *is the sequence of orbits* $\mathbf{S}_3, \mathbf{S}_5, \mathbf{S}_7, \dots, \mathbf{S}_{2m+1}, \dots$ *where any element* C *of* $\mathbf{S}_{2m+1}$ *is a curve (with* $2m+1$ *double points) in which two curvilinear* $(2m+1)$*-gons with the same salient vertices are inscribed; they are separated by a chain of* $2m+1$ *biangular tiles with salient vertices*. Examples of elements of $\mathbf{S}_3$, $\mathbf{S}_5$ and $\mathbf{S}_7$ are shown on the three spheres of Figure 6.

5) *The family* $\mathbf{T}$ *is the sequence of orbits* $\mathbf{T}_4, \mathbf{T}_8, \mathbf{T}_{10}, \dots, \mathbf{T}_{6m-2}, \mathbf{T}_{6m+2}, \dots$ *where any element* C *of* $\mathbf{T}_{6m\pm 2}$ *is a curve (with* $6m \pm 2$ *double points) whose two tiles are curvilinear* $(3m \pm 1)$*-gons with salient vertices; they are strictly separated by a belt of* $6m \pm 2$ *triangular tiles. If* $6m \pm 2 > 4$*, then the tiling of* C *is combinatorially equivalent to the natural tiling of the boundary of an antiprism whose bases are* $(3m \pm 1)$*-gons.* Examples of elements of $\mathbf{T}_4$, $\mathbf{T}_8$ and $\mathbf{T}_{10}$ are shown on the spheres of Figure 7.

3 Gauss diagrams and proofs

Our proof of the Theorem uses diagrams introduced by Gauss [Ga]. We define them via codes of curves which are similar to the Gauss codes used in knot theory.

NOTATIONS AND DEFINITIONS: Let $C = f(S^1)$ be a spherical curve with n double points ($n > 0$). In order to define a Gauss code of C, we first give a name (letter with or without subscript) to each double point of C and then write their names following the order in which $f(u)$ meets them when u runs along S^1 ; the word (of length $2n$) so obtained is a *Gauss code* of C, which is defined (after the choice of names) up to an element of the dihedral group D_{2n}.

Every Gauss code Ω of $2n$ letters may be represented by a *Gauss diagram of order* n, i.e. a plane figure Γ consisting of

(i) a circle γ of the Euclidean plane,

(ii) the vertices of a regular $2n$-gon P inscribed in γ, also called *vertices* of Γ, denoted by the letters of Ω in such a way that neighboring vertices of P correspond to successive letters of Ω,

(iii) the n chords joining the vertices which have the same name.

Figure 10 describes an example of this representation.

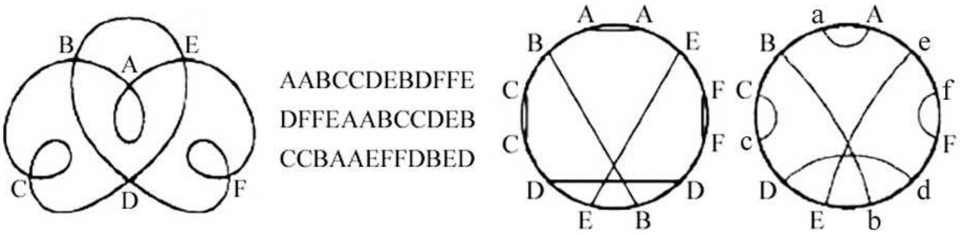

FIGURE 10: From left to right : a curve whose double points are A, B, C, D, E, F; three equivalent Gauss codes of this curve ; a Gauss diagram Γ of these codes and of the curve ; a variant to Γ with a better visibility (it is sometimes useful to take different but similar names for the endpoints of a chord).

Let K be a chord of the Gauss diagram Γ; the *step* of K is the minimum number of sides of P needed to join the endpoints of K along the boundary of P. An *s-chord* is a chord whose step is s (in the example of Fig. 10, Γ has three 1-chords, one 3-chord and two 5-chords).

One easily proves that every chord in the Gauss diagram of any plane or spherical curve has an odd step and that if a curve is DP-homogeneous, then all its chords have the same step (note that the converse is not true: for example, there is a spherical curve with three double points which is not DP-homogeneous, but whose Gauss diagram has only 1-chords).

LEMMA 1: *If a spherical curve C with n double points is DP-homogeneous, then its Gauss diagram* Γ *is invariant under the group* C_n *of rotations whose angles are multiples of* $2\pi/n$.

Proof: Let s be the common step of the chords of Γ and let [0], [1], ... , $[2n-1]$ be the vertices of the polygon P used in the definition of Γ (the sides of P are the segments $[[j],[j+1]]$, addition being done mod $2n$). As the Lemma is obvious when $s = 1$ or $s = n$, we may assume that $1 < s < n$ and prove the assertion by contradiction. Let us agree that two chords are *neighboring* if one of them is the image of the other by a rotation of π/n.

If the assertion were false, then we could find in Γ two neighboring chords, one of them being $[[a],[a+s]]$ and the other $[[a+1],[a+1+s]]$; two possibilities occur: either one of these chords is neighboring with a third chord, or not.

α) The first assumption implies that a double point of C is a vertex of two biangular tiles of C; as C is DP-homogeneous, all the double points of C have the same property; this implies that, for every vertex $[j]$, the segment $[[j],[j+s]]$ is a chord of Γ, which is only possible when $s = n$, contradicting the condition $1 < s < n$.

β) The second assumption and the DP-homogeneity imply that the set of chords of Γ can be partitioned into disjoint pairs of neighboring chords and consequently, that the set of vertices of Γ can be partitioned into disjoint pairs of neighboring vertices which are endpoints of neighboring chords. Since $[[a],[a+s]]$ and $[[a+1],[a+1+s]]$ are such chords, the number of vertices of Γ between $[a+1]$ and $[a+s]$ is equal to $s-2$, an odd number, giving the contradiction. ∎

The notation $\Gamma(n,s)$ will be used for any plane diagram consisting of
(i) a circle γ of the Euclidean plane,
(ii) the vertices of a regular $2n$-gon P inscribed in γ,
(iii) n chords of γ with odd step s joining pairs of vertices of P, whose union is invariant under the rotation group C_n.

Note that, given integers n and s with s odd and $1 \leq s \leq n$, there is essentially one diagram with these properties. Examples are drawn in Fig. 11.

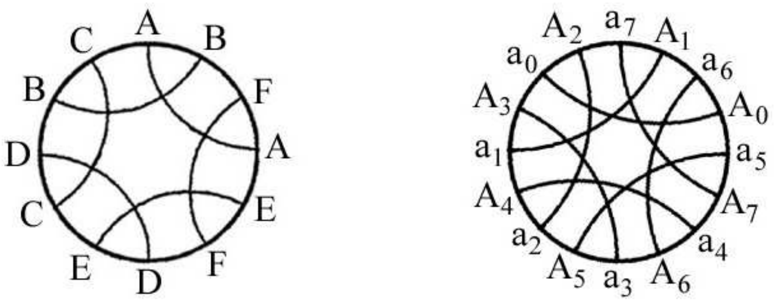

FIGURE 11: The diagrams $\Gamma(6,3)$ and $\Gamma(8,5)$; *AFBACBDCEDFE* may be a code for $\Gamma(6,3)$, and $A_0a_6A_1a_7A_2a_0A_3a_1A_4a_2A_5a_3A_6a_4A_7a_5$ for $\Gamma(8,5)$.

In the proof of Lemma 2, we use a procedure found by L. Lovasz and M.L. Marx [LM] to decide whether or not a word of $2n$ characters (n symbols occurring twice) is the Gauss code of a spherical curve. In order to increase the readability of our paper, we now recall three definitions and two properties given in [LM].

DEFINITIONS: If a word has the form $A\alpha A\beta$ where α and β are non-empty sequences, then the *vertex split at A* is the change from this word to $\alpha^{-1}\beta$ where α^{-1} has the same letters as α but in the opposite order.

The *loop removal at A* of the word $A\alpha A\beta$ is the change from this word to the one obtained from β by deleting all the letters which occur in α.

A *reduced word* of a word Ω is a non-empty word obtained from Ω after a finite number of changes (vertex splits or loop removals).

PROPERTY 1 ("biparity condition" in [LM]): If the Gauss code of a spherical curve with at least two double points A and B has the form $A\alpha A\mu B\beta B\gamma$ where α, μ, β, γ are finite (possibly empty) sequences of letters, then α and β have an even number of common letters.

PROPERTY 2 ("Theorem" in [LM]): A word Ω wherein each letter occurs twice is a Gauss code of a spherical curve if and only if no reduced word of Ω has the form $A_1A_2...A_mA_1A_2...A_m$ with m even.

LEMMA 2: *If C is a DP-homogeneous spherical curve, then its Gauss diagram belongs to one of the three families described below and shown in Fig. 12:*
a) the family **A** *consists of diagrams* $\Gamma(n,1)$ *where* $n \in \mathbb{N}_0$,

b) the family **B** *consists of diagrams* $\Gamma(n,n)$ *where* $n = 2m+1$ $(m \in \mathbb{N}_0)$,
c) the family **C** *consists of diagrams* $\Gamma(n,s)$ *where n and s depend on* $m \in \mathbb{N}_0$ *in one of the following ways:*
either $n = 6m-2$ *and* $s = 4m-1$ *or* $n = 6m+2$ *and* $s = 4m+1$.

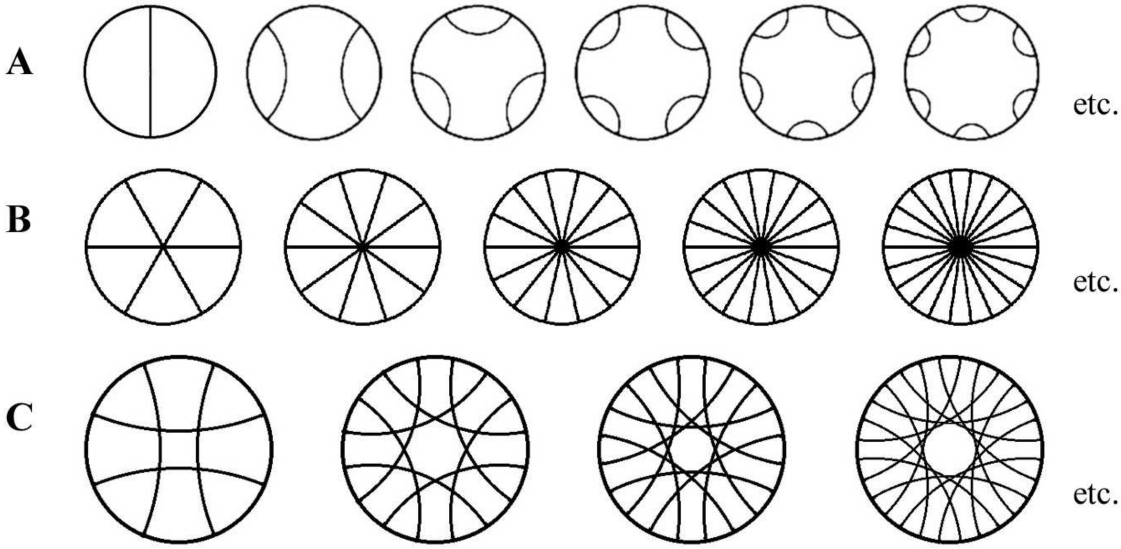

FIGURE 12: The three families of diagrams considered in Lemma 2.

Proof: The family **A** (resp. **B**) consists of all the diagrams described in Lemma 1 when the step s equals 1 (resp. n). Hence it remains to exclude, among the diagrams $\Gamma(n,s)$ such that $2 < s < n$, those which are neither of the form $n = 6m-2$ and $s = 4m-1$ nor of the form $n = 6m+2$ and $s = 4m+1$. In other words, we must exclude all diagrams $\Gamma(n,s)$ for which n is odd and, among the diagrams with n even, those for which $2n$ does not belong to $\{3s-1, 3s+1\}$. Suppose on the contrary that there is a spherical curve C whose Gauss diagram must be excluded. By Lemma 1, its Gauss code Ω may be written as $A\alpha A\mu B\beta B\gamma$ where the words μ and γ are possibly empty and $|\alpha| = |\beta| = s-1$. ($|\sigma|$ denotes the number of letters of the word σ). We distinguish three cases in order to get a contradiction.

α) <u>$2n$ is greater than $3s+1$</u>. If $s \equiv 3 \mod 4$, then we take $\mu = \varnothing$ in the notation above for Ω; if $s \equiv 1 \mod 4$, then we choose $|\mu| = 2$; in both cases, the number of common letters of α and β is odd, contradicting Property 1.

β) <u>n is odd and $2n \leq 3s+1$</u>. Since $|\alpha| = |\beta| = s-1$, we have
$$|\mu| + |\gamma| = 2n - 2(s-1) - 4 = 2n - 2s - 2 \leq 3s + 1 - 2s - 2 = s - 1,$$
which shows that a letter cannot appear twice in any word α, μ, β or γ; moreover, the distance between any letter of μ and any letter of γ is at least $s+2$; this implies that any letter of $\mu \cup \gamma$ appears also in $\alpha \cup \beta$. Hence the number of letters with two occurences in $\alpha \cup \beta$ is equal to $(2(s-1) - (2n-2s-2))/2$ i.e. $2s-n$, which implies that the number of letters common to α and β is the odd number $2s-n$, contradicting Property 1.

γ) <u>n is even and $2n < 3s+1$</u>. We use the notation introduced in the second example of Fig.11 for the Gauss code of a curve C with diagram $\Gamma(n,s)$: so

$A_k = [2k]$ and $a_k = [2k+s]$ if we identify the set of vertices of $\Gamma(n,s)$ with $\mathbb{Z}/(2n)$. By Lemma 1, a Gauss code for $\Gamma(6,5)$ can be written as

$$\Omega(6,5) = A_0a_4A_1a_5A_2a_0A_3a_1A_4a_2A_5a_3;$$

if $\Omega(6,5)$ were the Gauss code of a spherical curve, then a vertex split of $\Omega(6,5)$ at A_0 would produce the Gauss code Ω' of a spherical curve, but this is not so because Ω' does not satisfy the biparity condition, a contradiction. In the case $n > 7$, a Gauss code for $\Gamma(n,s)$ may be written as

$$\Omega(n,s) = A_0a_gA_1a_{g+1}A_2...a_{n-1}A_ha_0A_{h+1}a_1...A_ga_{g-h}A_{g+1}a_{g-h+1}...A_{n-1}a_{g-1}$$

where $g = (2n-s+1)/2$ and $h = (s-1)/2$. In this case two changes are needed to conclude: the first one is the vertex split of $\Omega(n,s)$ at A_0, giving the word

$$A_ha_{n-1}A_{h-1}...A_2a_{g+1}A_1a_gA_{h+1}a_1...A_ga_{g-h}A_{g+1}a_{g-h+1}...A_{n-1}a_{g-1}$$

or the equivalent word

$$\Omega' = A_{g+1}a_{g-h+1}...A_{n-1}a_{g-1}A_ha_{n-1}A_{h-1}...A_2a_{g+1}A_1a_gA_{h+1}a_1...A_ga_{g-h}$$

also written $\Omega' = A_{g+1}\alpha a_{g+1}\beta$ if we set

$$\alpha = a_{g-h+1}...A_{n-1}a_{g-1}A_ha_{n-1}A_{h-1}...A_2 \quad \text{and} \quad \beta = A_1a_gA_{h+1}a_1...A_ga_{g-h}$$

Finally, a loop removal of Ω' at A_{g+1} creates the reduced word $A_1a_ga_1A_g$ which means, according to Property 2, that $\Omega(n,s)$ is not the Gauss code of a spherical curve, contrary to the assumption. ∎

Proof of the Theorem: Every curve described in the Theorem is clearly DP-homogeneous; moreover, if it belongs to one of the families **P**, **Q** or **R**, then its Gauss diagram belongs to family **A** while, if it belongs to family **S** (resp. **T**), then its Gauss diagram belongs to family **B** (resp. **C**). So its remains to show that every DP-homogeneous spherical curve C with n double points belongs to one of the families **P**, **Q**, **R**, **S** or **T**. The rest of the proof has three parts, corresponding to the three possible families **A**, **B** and **C**.

a) Curves with diagram in family **A**. If the Gauss diagram of C is $\Gamma(1,1)$, then C is clearly a figure-eight curve and all such curves form the orbit $\mathbf{A}_1$. If the Gauss diagram of C is $\Gamma(n,1)$ with $n > 1$, then C is a union of n loops and n arcs connecting neighboring double points; these arcs form a Jordan curve B, which is boundary of two curvilinear n-gons.
α) If the loops of C at a double point and at its two neighbors (only one if $n = 2$) are on the same side of B, then this property is true for every double point, and so C belongs to the orbit $\mathbf{P}_n$.
β) If the loop of C at a double point is on one side of B while the loops at its neighbors are on the other side, then this property is true for every double point, which implies that n is even and that C belongs to the orbit $\mathbf{Q}_n$.
γ) If the loops of C at the neighbors of a double point are not on the same side of B, then this property is true for every double point, and so n is a multiple of 4 and C belongs to the orbit $\mathbf{R}_n$.

b) Curves with diagram in family **B**. If the Gauss diagram of C is $\Gamma(n,n)$ ($n = 2m+1, m > 0$), then any simple arc of the circle γ of $\Gamma(n,n)$ (i.e. any arc joining neighboring vertices) determines with its antipodal arc the boundary of a biangular tile of the tiling of C; these biangular tiles have the properties described in point 4 of the Theorem, and so C belongs to the orbit $\mathbf{S}_n$.

c) Curves with diagram in family **C**. A Gauss code of $\Gamma(4,3)$ is

$$\Omega(4,3) = A_0 a_3 A_1 a_0 A_2 a_1 A_3 a_2$$

The simple arcs A_0a_3, A_3a_2 and a_0A_2 of the circle γ of $\Gamma(4,3)$ determine the sides of a triangular tile Δ_0 of C. We define in the same way Δ_1 by means of A_1a_0, A_0a_3 and a_1A_3, Δ_2 by means of A_2a_1, A_1a_0 and a_2A_0, and Δ_3 by means of A_3a_2, A_2a_1 and a_3A_1; as Δ_0 and Δ_1 have a common side, as well as Δ_1 and Δ_2, Δ_2 and Δ_3, Δ_3 and Δ_0, and so these four tiles form a belt having the properties described in point 5 of the Theorem, which implies that C belongs to the orbit $\mathbf{T}_4$. In the same way, one proves that, if the Gauss diagram $\Gamma(n,s)$ of C is $\Gamma(6m-2, 4m-1)$ $(m > 1)$ or $\Gamma(6m+2, 4m+1)$ $(m > 0)$, then C belongs to the orbit $\mathbf{T}_n$. ∎

4 DP-homogeneous plane curves

COROLLARY: *If a DP-homogeneous plane curve has at least one double point, then it belongs to one orbit* (under the group of all diffeomorphisms of $\mathbb{R}^2$) *of one of the three families described below:*

1) The family $\mathbf{P}'$ *is the sequence of orbits* $\mathbf{P}'_1$, $\mathbf{P}'_2$, $\mathbf{P}'_3$,... , $\mathbf{P}'_n$, ... *where any element of* $\mathbf{P}'_1$ *is a figure-eight curve and, if* $n > 1$, *where any element* C *of* $\mathbf{P}'_n$ *is a curve (with* n *double points) one tile of which is a curvilinear* n*-gon with salient vertices, each of them being also the vertex of a curvilinear 1-gon inscribed in* C. Examples of elements of $\mathbf{P}'_1$, $\mathbf{P}'_2$, $\mathbf{P}'_3$ and $\mathbf{P}'_4$ are shown in Figure 1.

2) The family $\mathbf{P}''$ *is the sequence of orbits* $\mathbf{P}''_1$, $\mathbf{P}''_2$, $\mathbf{P}''_3$,... , $\mathbf{P}''_n$, ... *where any element of* $\mathbf{P}''_1$ *is equivalent to a Pascal snail with inner loop and, if* $n > 1$, *where any element of* $\mathbf{P}''_n$ *is a curve* C *(with* n *double points) in which a curvilinear* n*-gon* D *with re-entrant vertices is inscribed; every vertex of* D *is also the vertex of a curvilinear 1-gon inscribed in* C. Examples of elements of $\mathbf{P}''_1$, $\mathbf{P}''_2$, $\mathbf{P}''_3$ and $\mathbf{P}''_4$ are shown in Figure 2.

3) The family $\mathbf{S}'$ *is the sequence of orbits* $\mathbf{S}'_3$, $\mathbf{S}'_5$, $\mathbf{S}'_7$,... , $\mathbf{S}'_{2m+1}$, ... *where any element* C *of* $\mathbf{S}'_{2m+1}$ *is a curve (with* $2m+1$ *double points) one tile of which is a curvilinear* $(2m+1)$*-gon* D *with salient vertices;* D *is separated from the unbounded tile of* C *by a chain of* $2m+1$ *biangular tiles*. Examples of elements of $\mathbf{S}'_3$, $\mathbf{S}'_5$ and $\mathbf{S}'_7$ are shown in Figure 8.

Proof: As $\mathbb{R}^2$ is diffeomorphic to the complement of a point (denoted by ∞) in S^2, we may identify $\mathbb{R}^2$ with $S^2 \setminus \infty$. Any DP-homogeneous plane curve C may be seen as a DP-homogeneous spherical curve which, by the Theorem, belongs to one orbit of one of the families **P**, **Q**, **R**, **S** and **T**.

Suppose that the spherical curve C belongs to $\mathbf{P}_1$; if ∞ is a point of a 1-gonal tile, then C belongs (as a plane curve) to $\mathbf{P}''_1$; if not, then C belongs to $\mathbf{P}'_1$.

If C belongs to $\mathbf{P}_n$ with $n > 1$, then ∞ cannot be a point of a 1-gonal tile, which implies that C belongs to $\mathbf{P}'_n$ or $\mathbf{P}''_n$.

If C belongs to $\mathbf{S}_n$, then ∞ cannot be a point of a 2-gonal tile, which implies that C belongs to $\mathbf{S}'_n$.

If C belongs to $\mathbf{Q}_n$, $\mathbf{R}_n$ or $\mathbf{T}_n$, then every position of ∞ leads to a contradiction, which proves that, in the classification of the orbits of DP-homogeneous plane curves, there is no familiy other than $\mathbf{P}'$, $\mathbf{P}''$ and $\mathbf{S}'$. ■

5 DP-homogeneity in the real projective plane

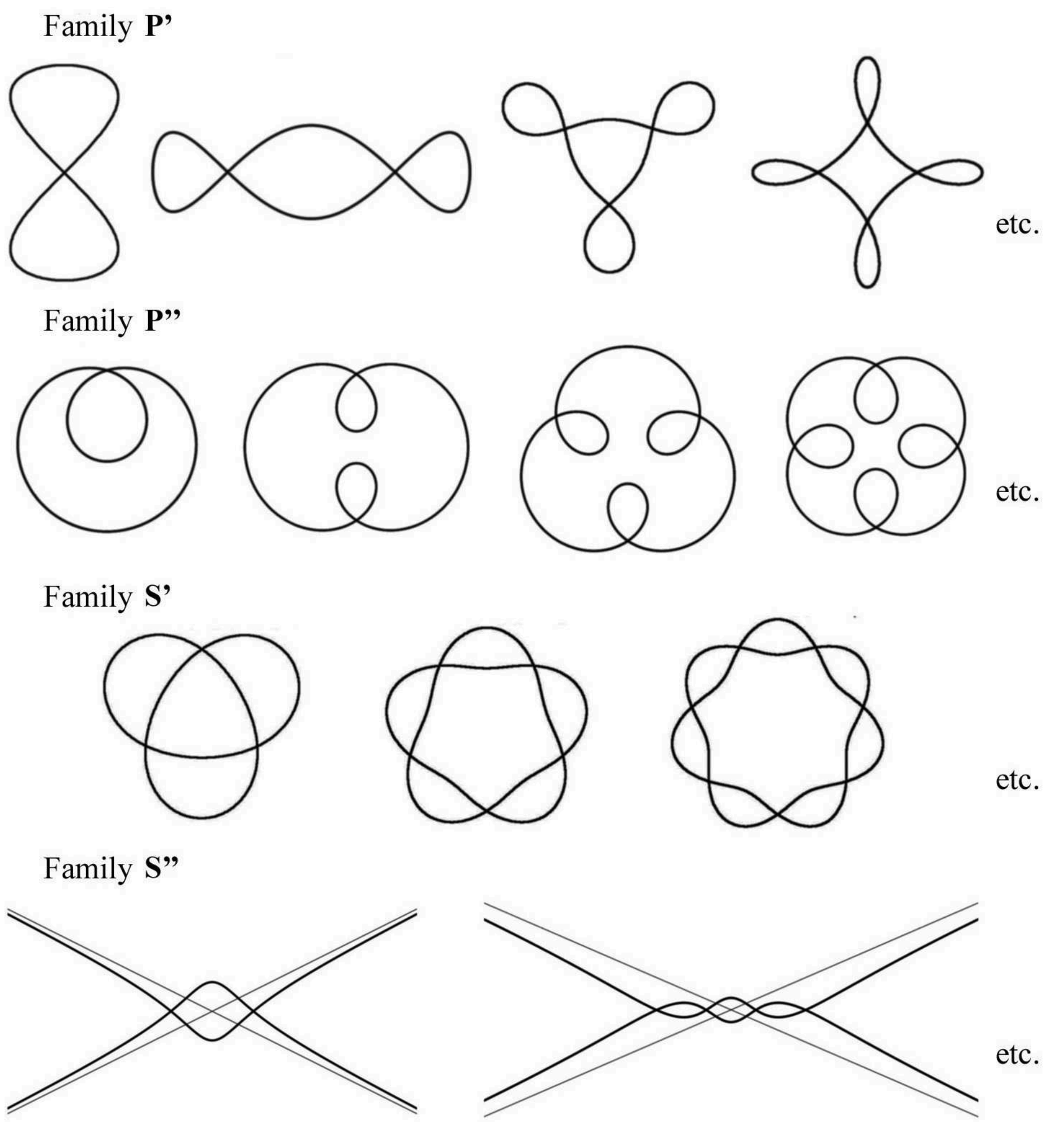

FIGURE 13: Representatives of some elements of the four families of orbits of null-homotopic DP-homogeneous curves in the real projective plane.

CONJECTURE: *Let C be a DP-homogeneous curve of $P_2(\mathbb{R})$ with at least one double point.*

1) *If C is null-homotopic, then it belongs to one orbit* (under the group of all diffeomorphisms of $P_2(\mathbb{R})$) *of one of four infinite families:*

a) *The family* $\mathbf{P}'$ *is the sequence of orbits* $\mathbf{P}'_n$ $(n \in \mathbb{N}_0)$, *whose representatives are curves with n double points sketched, for* $n < 5$, *in the first row of Fig. 13.*

b) *The family* $\mathbf{P}''$ *is the sequence of orbits* $\mathbf{P}''_n$ $(n \in \mathbb{N}_0)$, *whose representatives are curves with n double points sketched, for* $n < 5$, *in the second row of Fig. 13.*

c) *The family* $\mathbf{S}'$ *is the sequence of orbits* $\mathbf{S}'_n$ $(n = 2m + 1,\ m \in \mathbb{N}_0)$, *some representatives of which are sketched, for* $n < 8$, *in the third row of Fig. 13.*

d) *The family* $\mathbf{S}''$ *is the sequence of orbits* $\mathbf{S}''_n$ $(n = 2m,\ m \in \mathbb{N}_0)$, *some representatives of which are sketched, for* $n < 5$, *in the last row of Fig. 13.*

2) *If C is not null-homotopic, then it belongs to one orbit* (under the group of all diffeomorphisms of $P_2(\mathbb{R})$) *of one of three infinite families:*

a) *The family* $\mathbf{Q}'$ *is the sequence of orbits* $\mathbf{Q}'_n$ $(n = 2m + 1,\ m \in \mathbb{N})$, *whose representatives are sketched, for* $n < 6$, *in the upper part of Fig. 14.*

b) *The family* $\mathbf{R}'$ *is the sequence of orbits* $\mathbf{R}'_n$ $(n = 4m + 2,\ m \in \mathbb{N})$, *whose representatives are sketched, for* $n < 7$, *in the lower part of Fig. 14.*

c) *The family* $\mathbf{T}'$ *is the sequence of orbits* $\mathbf{T}'_n$ $(n = 6m - 1$ *or* $n = 6m + 1$, $m \in \mathbb{N}_0)$ *whose representatives are sketched, for* $n < 8$, *in Fig. 15.*

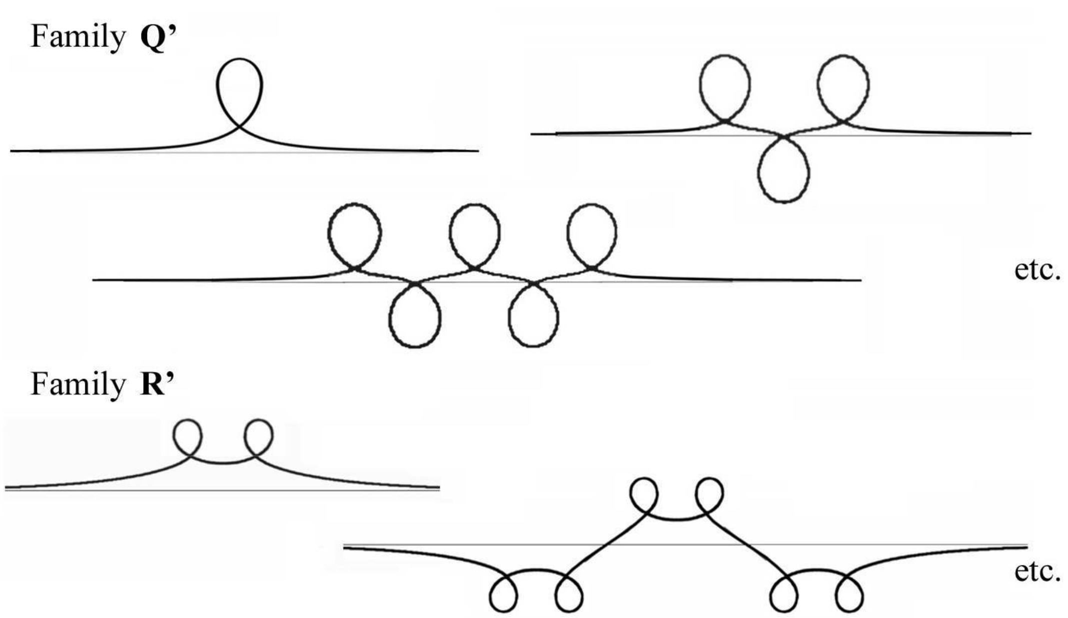

FIGURE 14: Representatives of some elements of the families of orbits $\mathbf{Q}'$ and $\mathbf{R}'$ of non null-homotopic DP-homogeneous curves in the real projective plane.

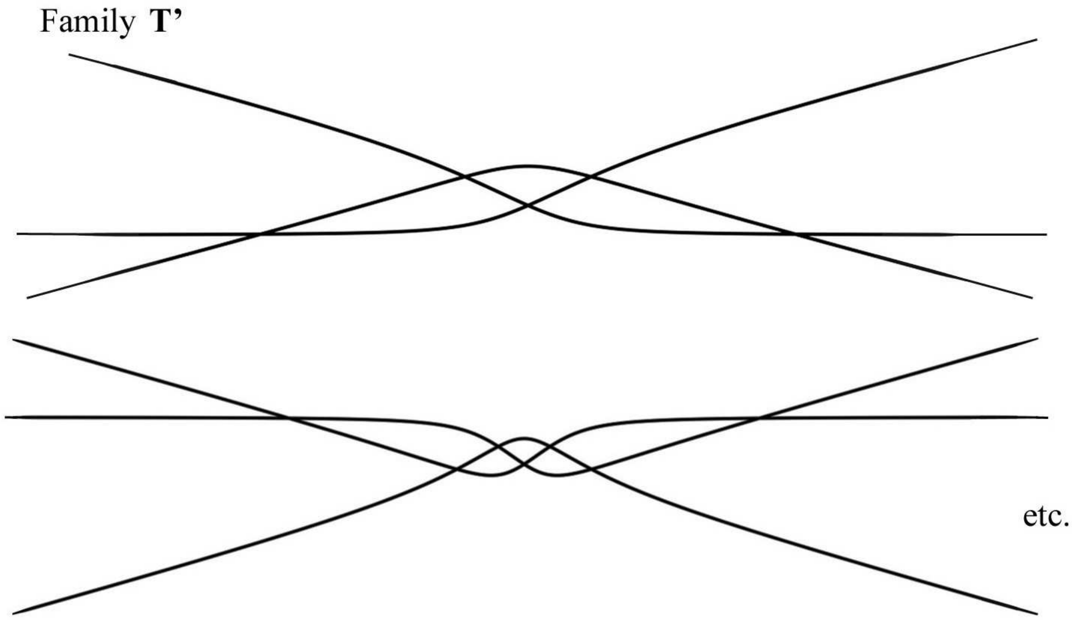

FIGURE 15: Representatives of the orbits $\mathbf{T}'_5$ and $\mathbf{T}'_7$ of non null-homotopic DP-homogeneous curves in the real projective plane.

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