On bijections, isometries and expansive maps

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Abstract

In this paper we show when a bijection on a set X can be made either an isometry or an expansive map with respect to a non-discrete metric on X. As a corollary we obtain that any bijection on an infinite set can be made biLipschitz by a non-discrete metric.

1 Introduction

In [2] Ellis raised the following question: let T be a self-map on a set X, how can we construct a non-discrete topology on X with respect to which T is continuous?

Answering the above question, de Groot and de Vries showed, among other things, that if T is a bijection on an infinite set X, then there is always a non-discrete metric d on X (i.e., (X, d) has an accumulation point) with respect to which T is a homeomorphism [6].

Moreover, it is worth noting that in [5] the authors show when a bijection on a set X can be made a homeomorphism by a compact metrizable topology on X(see also [4]).

The aim of this note is to show when a bijection on a set X can be made either an isometry or an expansive map by a non-discrete metric on X.

Let T be a self-map on a metric space (X, d). Recall that T is called an isometry if it is a distance-preserving bijection, while T is said to be expansive if it is a homeomorphism satisfying the following property: there is a $\delta > 0$, called expansivity constant for T, such that for every pair x, y of distinct points of X we have $d(T^n(x), T^n(y)) \geq \delta$ for some $n \in \mathbb{Z}$.

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Let $T : X \to X$ be a bijection and let $x \in X$. The (full) orbit of x (under T) is the set $O(x) = O(x, T) = \{T^n(x) : n \in \mathbb{Z}\}$. The cardinality of a finite orbit $O(x)$ will be called the length of $O(x)$ and x will be called a periodic point of minimal period $|O(x)|$.

The reader is referred to [3] for notations and terminology not explicitly given.

2 The results

Our first result will characterize isometries of non-discrete metric spaces.

Theorem 1. *Let T be a bijection on a set X. There is a non-discrete metric d on X with respect to which T is an isometry iff one of the following holds.*

(1) There exists an infinite orbit.

(2) For some $p \in \mathbb{N}$:

(i) there is an orbit of length p,

(ii) there exist infinitely many orbits whose length is a multiple of p.

Proof. Let us show the sufficiency. First suppose that there is an infinite orbit O. Without loss of generality we may assume that $O = \mathbb{Z}$ and $T(x) = x + 1$ for every $x \in O$.

Claim. There exists a non-discrete metric ρ on $\mathbb{Z}$ such that the map $T : (\mathbb{Z}, \rho) \to (\mathbb{Z}, \rho)$ given by $T(x) = x + 1$, for every $x \in \mathbb{Z}$, is an isometry.

Proof of the claim. Although, as noted by the referee, we may simply take the restriction of an irrational rotation of S^1 to the orbit of any point, we will give an alternative proof.

Let us define, when p is odd and $k \geq 0$, $\rho(0, 2^k p) = \|2^k p\| = \frac{1}{k+1}$ and $\rho(0, 0) = 0$. Now let us extend ρ on $\mathbb{Z}$ by $\rho(n, m) = \rho(0, m - n) = \rho(0, n - m)$. To show that ρ is a metric on $\mathbb{Z}$ it is enough to show that $\|x + y\| \leq \max(\|x\|, \|y\|)$. So let $x = 2^k p$ and $y = 2^h q$, with p and q odd. We may assume $k \geq h$.

If $k > h$, then $\|y\| > \|x\|$. So $x + y = 2^h(2^{k-h} p + q)$ with $k - h \geq 1$, hence $2^{k-h} p + q$ is odd and $\|x + y\| = \frac{1}{h+1} = \|y\|$.

If $h = k$, then $\|x\| = \|y\|$, $x + y = 2^k(p + q) = 2^m s$ with s odd and $m > k$. So $\|x + y\| = \frac{1}{m+1} < \frac{1}{k+1} = \|x\| = \|y\|$. Therefore $\|x + y\| \leq \max(\|x\|, \|y\|)$.

Clearly $T : (\mathbb{Z}, \rho) \to (\mathbb{Z}, \rho)$ is an isometry. Moreover ρ is non-discrete, in fact $(\mathbb{Z}, \rho)$ has no isolated points (it is enough to observe that 0 is an accumulation point of $(\mathbb{Z}, \rho)$). The proof of the claim is complete.

Now let d be the metric on X given by $d|O = \rho$, where ρ is the metric described in the proof of the claim above, and $d(x, y) = 1$ whenever x and y are distinct points of X not both belonging to O (observe that d is a metric because ρ is bounded by 1) . Clearly d is non-discrete and T is an isometry.

Now suppose (2) holds. Then there is an orbit $O(x)$ of length some p and, for every $n \in \mathbb{N}$, there is an orbit $O(x_n)$, with $O(x_n) \neq O(x)$, whose length is a multiple of p and $O(x_n) \neq O(x_m)$ whenever $n \neq m$.

It is not restrictive to assume that $X = O(x) \cup \bigcup_n O(x_n)$.

Case 1. ($p = 1$) In this case x is a fixed point and we define:

$d_T(x, y) = \frac{1}{n}$ whenever $y \in O(x_n)$, $d_T(y, z) = \frac{1}{n}$ whenever $y, z \in O(x_n)$ and $y \neq z$, $d_T(y, z) = \frac{1}{n} + \frac{1}{m}$ whenever $y \in O(x_n)$, $z \in O(x_m)$ and $n \neq m$.

Clearly d_T is a non-discrete metric on X (x is an accumulation point of X) with respect to which T is an isometry.

Case 2. ($p > 1$) Set $X_i = T^i(\{x\} \cup \bigcup_n O(x_n, T^p))$ for every $i \in \{0, ..., p-1\}$.

Clearly X is the disjoint union of $X_0, ..., X_{p-1}$. Since $T^i(x)$ is a fixed point of the restriction $T^p : X_i \to X_i$, we may take on each X_i the metric $d = d_{T^p}$ defined in case 1. If we define also $d(y, z) = 1$ whenever y and z do not both belong to the same X_i, we obtain a non-discrete metric on X (observe that $T^i(x)$ is an accumulation point of X_i) with respect to which T is an isometry.

Now let us show the necessity. If there are no infinite orbits, let us take an accumulation point x. Then $O(x)$ is formed by accumulation points. Let p be the length of $O(x)$. We claim that that there are infinitely many orbits whose length is a multiple of p. This is clear if $p = 1$. If $p > 1$, let η be the smallest distance between two distinct points of $O(x)$. We may assume that $\eta = 1$. Now, for every $n > 2$, let us take some $x_n \neq x$ such that $d(x_n, x) \leq \frac{1}{n}$ and $O(x_n) \neq O(x_m)$ whenever $n \neq m$. We claim that the length p_n of $O(x_n)$ is a multiple of p. Suppose not. Since T is an isometry and $d(x_n, x) \leq \frac{1}{n}$, it follows that $d(T^{p_n}(x_n), T^{p_n}(x)) \leq \frac{1}{n}$, i.e., $d(x_n, T^{p_n}(x)) \leq \frac{1}{n}$. Now $T^{p_n}(x) \neq x$ (recall that we are assuming that p_n is not a multiple of p), so $d(T^{p_n}(x), x_n) \geq d(x, T^{p_n}(x)) - d(x, x_n) \geq 1 - \frac{1}{n} > \frac{1}{n}$ (recall that $n > 2$), a contradiction.

Now let us turn our attention on expansivity. In the sequel we will make use of the following dynamical system. Let $2^{\mathbb{Z}}$, where $2 = \{0, 1\}$, be endowed with the metric given by $\rho(x, y) = 2^{-n}$ with $n = \min\{|i| : x_i \neq y_i\}$ whenever $x = (x_i)$ and $y = (y_i)$ are two distinct points of $2^{\mathbb{Z}}$, and let $\sigma : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be the map defined by $\sigma(x) = (\sigma(x)_i)$, where $x = (x_i)$ and $\sigma(x)_i = x_{i+1}$. The map σ is an expansive homeomorphism, moreover it has 2^n periodic points of minimal period $\leq n$ and points with dense orbit (see, e.g., [1] pp. 7, 33, 36). $(2^{\mathbb{Z}}, \sigma)$ (or simply the map σ) is called (full) two-sided shift, it is a classical example of a (Devaney) chaotic dynamical system (for a recent account of this topic see [7, Ch. 1]).

Theorem 2. *Let T be a bijection on a set X. There is a non-discrete metric d on X with respect to which T is expansive iff one of the following holds.*

(1) There exists an infinite orbit.

(2) All points are periodic and the set of minimal periods is infinite.

Proof. Sufficiency. Let us assume that there is a infinite orbit O. We may assume, without loss of generality, that O is a dense orbit of the two-sided shift $\sigma : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ and that $T(x) = \sigma(x)$ for every $x \in O$. Now let d be the metric on X given by $d|O = \rho$, where ρ is the metric on $2^{\mathbb{Z}}$ described above, and $d(x, y) = 1$ whenever x and y are two distinct points of X not both belonging to O. Then d is a non-discrete metric on X with respect to which T is expansive (observe that $\sigma|O$ is expansive and O is not discrete).

Now let us consider the case in which all points are periodic and the set of minimal periods is infinite. We may assume that there is a sequence $(n_k)_{k\geq 0}$ of distinct positive integers with :

(i) $n_0 > 2$,
(ii) $n_{k+1} > 2(2k+1)$ for every $k \geq 0$;

and such that n_k is the minimal period of some point of the two-sided shift σ.

Our goal is to find a sequence $(p_k)_{k\geq 0}$ in $2^{\mathbb{Z}}$ such that (p_k) converges to p_0 and p_k has minimal period n_k for every $k \geq 0$.

We define:

1) $p_0(n) = 1$ if and only if $n \equiv 0 \pmod{n_0}$;
2) $p_k(n) = p_0(n)$ whenever $|n| \leq k$, for every $k \geq 1$;
3) $p_k(n) = 0$ whenever $k < n < n_k - k$, for every $k \geq 1$;
4) $p_k(m) = p_k(n)$ if and only if $m \equiv n \pmod{n_k}$; for every $k \geq 1$.

Clearly, by 2), $\rho(p_k, p_0) < 2^{-k}$ for every k, therefore (p_k) converges to p_0.

Moreover p_k has minimal period n_k for every k. Assume not, then there is some $m_k < n_k$ which is the minimal period of p_k. By 4), n_k is a period of p_k, so n_k must be a multiple of m_k. Hence $m_k \leq \frac{1}{2}n_k$. This is a contradiction, in fact p_k has more than $\frac{1}{2}n_k$ consecutive $0's$ (p_k has $n_k - 2k - 1$ consecutive $0's$ and such number is greater than $\frac{1}{2}n_k$).

Now let us take, for every non-negative integer k, a point x_k in X of minimal period n_k. We may identify the set $Y = \bigcup_{k\geq 0} O(x_k)$ with the subset of $2^{\mathbb{Z}}$ given by $\bigcup_{k\geq 0} O(p_k)$ and we may assume that $T(x) = \sigma(x)$ for every $x \in Y$.

Now let d be the metric on X given by $d|Y = \rho$ and $d(x,y) = 1$ whenever x and y are distinct points of X not both belonging to Y.

Clearly d is a non-discrete metric on X (x_0 is an accumulation point) with respect to which T is an expansive map.

Necessity. Let us suppose that there is a non-discrete metric d on X with respect to which T is expansive, all points are periodic and the set of minimal periods is finite, i.e., bounded by a number L.

Let p be an accumulation point of (X,d) and let κ be the period of p. We may assume that $d(T^n(p), T^m(p)) \geq 1$ whenever $T^n(p) \neq T^m(p)$. Let $\delta < \frac{1}{2}$ be an expansivity constant for T.

Now let us take a neighbourhood U of p such that $T^n(U) \subset B(T^n(p), \delta)$ for every $n \in \{0, ..., L\}$.

Let $y \in U$, we claim that $d(T^n(p), T^n(y)) < \delta$ for every $n \in \mathbb{Z}$. Let m be the minimal period of y. Then $d(T^m(p), T^m(y)) = d(T^m(p), y) < \delta$. Since $B(T^n(p), \frac{1}{2}) \cap B(T^m(p), \frac{1}{2}) = \emptyset$ whenever $T^n(p) \neq T^m(p)$, it follows that $T^m(p) = p$. So m is a multiple of k. Therefore from $d(T^i(p), T^i(y)) < \delta$ for every $i \leq m$ (recall that $m \leq L$), it follows that $d(T^n(p), T^n(y)) < \delta$ for every $n \in \mathbb{Z}$.

By expansiveness of T it follows that $U = \{p\}$. Since p is an accumulation point, we reach a contradiction.

A well-known weakening of the concept of isometry is given by the following notion: a self-map map f on a metric space (X,d) is called biLipschitz if there exists some L such that $\frac{1}{L}d(x,y) \leq d(f(x), f(y)) \leq Ld(x,y)$ for every $x, y \in X$.

It is interesting to observe that, as a consequence of our results, one can obtain the following

Corollary 3. *Let T be a bijection on an infinite set X. Then there is always a non-discrete metric on X with respect to which T is biLipschitz.*

Proof. By Theorem 1 it is enough to consider the case in which all points are periodic and the set of minimal periods is infinite. Now, following the proof of Theorem 2 it can be seen that, in this case, one can take the same metric given in the part (2) of the sufficiency (observe that the shift σ is biLipschitz).

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