EXPOSITORY PAPER

The bijection between projective indecomposable and simple modules

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Abstract

For modules over a finite-dimensional algebra, there is a canonical one-to-one correspondence between the projective indecomposable modules and the simple modules. In this self-contained, purely expository note, we take a straight-line path from the basic definitions to a proof of this correspondence.

1 Introduction

In many branches of mathematics, there is a clear notion of 'atomic' or 'indivisible' object. Prime numbers, connected spaces, transitive group actions and simple groups are all examples.

In module theory, however, it is unclear which class of objects most deserves the title of 'atomic'. Certainly the simple modules have a claim. However, when we are working over a finite-dimensional algebra A, the projective indecomposable modules can also claim to be atomic. Indeed, as we shall see, the A-module A is a direct sum of projective indecomposable modules, and every projective indecomposable module appears at least once in this sum. In this sense, the projective indecomposable modules are the atoms that make up A.

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Although a simple module need not be projective indecomposable, nor vice versa, it is a remarkable fact that there is a canonical bijection

$$\{\text{projective indecomposable } A\text{-modules}\}/\cong \quad \longleftrightarrow \quad \{\text{simple } A\text{-modules}\}/\cong$$

between the isomorphism classes of projective indecomposable modules and the isomorphism classes of simple modules. The bijection is given by matching a projective indecomposable module P with a simple module S just when S is a quotient of P.

The modest purpose of this expository note is to prove this correspondence, starting from nothing. Everything here is classical; for instance, most of it can be found in Chapter 1 of Benson's book [1] or Chapter I of Skowroński and Yamagata's book [3]. The only novelty is in the arrangement.

The exposition emphasizes the fact that the bijection can be established without calling on any major theorems, or taking a detour to prove them. This is certainly clear to many algebraists, but may not always be apparent to others.

In the final section, we do allow ourselves to use the Jordan–Hölder theorem or the Krull–Schmidt theorem (either will do), but only to show that the two sets of isomorphism classes related by the bijection are finite. The bijection itself is established without it.

Two features of the exposition are worth highlighting. The first is the indispensable role played by Fitting's lemma, and especially its corollary that every endomorphism of an indecomposable module is either nilpotent or invertible (Corollary 3.2). The second is the observation that every projective indecomposable module is finitely generated (a consequence of Lemma 5.4). Although this is certainly not new, it is perhaps not quite as well-known as it should be.

2 Basic definitions

Throughout, we fix a field K, not necessarily algebraically closed. We also fix a finite-dimensional K-algebra A. Algebras are always taken to be unital, but need not be commutative.

Terms such as vector space, linear, and dimension will always mean vector space etc. over K. Module will mean left A-module (not necessarily finitely generated), and homomorphisms, endomorphisms and quotients are understood to be of modules over A. A double-headed arrow $\twoheadrightarrow$ denotes an epimorphism (surjective homomorphism).

Since A is finite-dimensional, a module is finitely generated over A if and only if it is finite-dimensional over K. A module is **cyclic** if it is generated over A by a single element, or equivalently if it is a quotient of the A-module A. It is **simple** if it is nonzero and has no nontrivial submodules. It is **indecomposable** if it is nonzero and has no nontrivial direct summands. Thus, simple implies indecomposable.

A module P is **projective** if the functor $\mathrm{Hom}_A(P, -) : A\text{-}\mathbf{Mod} \to \mathbf{Set}$ preserves epimorphisms, in the categorical sense. Concretely, this means that given module homomorphisms ϕ and π as shown, with π surjective, there exists a homomorphism ψ making the triangle commute:

$$\begin{tikzcd} & P \arrow[dl, dotted, "\psi"'] \arrow[dr, "\phi"] & \\ M \arrow[rr, two heads, "\pi"'] & & N. \end{tikzcd}$$

We record some basic facts about projective modules.

Lemma 2.1. *i. Every direct summand of a projective module is projective.*

ii. Every direct summand of a free module is projective.

iii. Let M be a module and P a projective module. If P is a quotient of M then P is a direct summand of M.

iv. Every projective module is a direct summand of a free module.

Proof. For (i), let X be a direct summand of a projective module P. We show that X is projective. Write $\sigma\colon P \twoheadrightarrow X$ and $\iota\colon X \to P$ for the projection and inclusion of the direct sum. Take homomorphisms $\pi\colon M \twoheadrightarrow N$ and $\phi\colon X \to N$. Since P is projective, there exists a homomorphism ψ such that the lower region of the diagram

$$\begin{tikzcd} X \arrow[d, "\iota"'] \arrow[dr, "1"] & & \\ P \arrow[r, two heads, "\sigma"'] \arrow[d, dotted, "\psi"'] & X \arrow[dr, "\phi"] & \\ M \arrow[rr, two heads, "\pi"'] & & N \end{tikzcd}$$

commutes. We now have a homomorphism $\psi\iota\colon X \to M$ with $\pi(\psi\iota) = \phi$.

To deduce (ii) from (i), it is enough to show that free modules are projective. Let F be free with basis $(e_s)_{s\in S}$. Take π and ϕ as shown:

$$\begin{tikzcd} & F \arrow[dl, dotted, "\psi"'] \arrow[dr, "\phi"] & \\ M \arrow[rr, two heads, "\pi"'] & & N. \end{tikzcd}$$

We may choose for each $s \in S$ an element $m_s \in M$ such that $\pi(m_s) = \phi(e_s)$. There is a unique $\psi\colon F \to M$ such that $\psi(e_s) = m_s$ for all s, and then $\pi\psi = \phi$.

For (iii), given $\pi\colon M \twoheadrightarrow P$, we may choose a homomorphism ι such that

$$\begin{tikzcd} & P \arrow[dl, dotted, "\iota"'] \arrow[dr, "\mathrm{id}_P"] & \\ M \arrow[rr, two heads, "\pi"'] & & P \end{tikzcd}$$

commutes. An easy calculation shows that $M = \ker\pi \oplus \operatorname{im}\iota$; but $\operatorname{im}\iota \cong P$, so P is a direct summand of M.

Part (iv) follows, since every module is a quotient of some free module. ∎

3 Fitting's lemma

Here we recall a very useful basic result about the dynamics of a linear operator on a finite-dimensional vector space.

Lemma 3.1 (Fitting)**.** *Let θ be a linear endomorphism of a finite-dimensional vector space X. Then $X = \ker(\theta^n) \oplus \operatorname{im}(\theta^n)$ for all $n \gg 0$.*

Proof. The chain $\ker(\theta^0) \subseteq \ker(\theta^1) \subseteq \cdots$ of subspaces of X eventually stabilizes, say at $\ker(\theta^m)$. Let $n \geq m$. If $x \in \ker(\theta^n) \cap \operatorname{im}(\theta^n)$ then $x = \theta^n(y)$ for some $y \in X$; but then $0 = \theta^n(x) = \theta^{2n}(y)$, so $y \in \ker(\theta^{2n}) = \ker(\theta^n)$, so $x = 0$. Hence $\ker(\theta^n) \cap \operatorname{im}(\theta^n) = 0$. Since $\dim \ker(\theta^n) + \dim \operatorname{im}(\theta^n) = \dim X$, the result follows. ∎

Corollary 3.2. *Every endomorphism of a finitely generated indecomposable module is either nilpotent or invertible.*

Proof. Let θ be an endomorphism of a finitely generated indecomposable module M. By Lemma 3.1, we can choose $n \geq 1$ such that $\ker(\theta^n) \oplus \operatorname{im}(\theta^n) = M$. Since M is indecomposable, $\ker(\theta^n)$ is either 0 or M. If 0 then θ^n is injective, so θ is injective; but θ is a linear endomorphism of a finite-dimensional vector space, so θ is invertible. If M then $\theta^n = 0$, so θ is nilpotent. ∎

4 Maximal submodules

We will need to know that every projective indecomposable A-module has a maximal (proper) submodule. A simple application of Zorn's lemma does not prove this, since the union of a chain of proper submodules need not be proper. (And in fact, not every module over every ring does have a maximal submodule.)

To prove it, we use two constructions. Let M be a module. We write $\operatorname{rad}(M)$ for the intersection of all the maximal submodules of M (the **Jacobson radical**). Given a left ideal I of A, we write IM for the submodule of M generated by $\{im : i \in I, m \in M\}$. Both constructions are functorial:

Lemma 4.1. *Let $f\colon M \to N$ be a homomorphism of modules. Then $f\operatorname{rad}(M) \subseteq \operatorname{rad}(N)$ and $f(IM) \subseteq IN$, for any left ideal I of A.*

Proof. The second statement is trivial. For the first, let K be a maximal submodule of N; we must prove that $f\operatorname{rad}(M) \subseteq K$. Since N/K is simple, the image of the composite $M \xrightarrow{f} N \twoheadrightarrow N/K$ is either N/K or 0, so the kernel is either maximal or M. In either case, the kernel contains $\operatorname{rad}(M)$, so $f\operatorname{rad}(M) \subseteq K$. ∎

Lemma 4.2. *Let M be a module. Then $\operatorname{rad}(A)M \subseteq \operatorname{rad}(M)$, with equality if M is projective.*

Proof. For each $m \in M$, right multiplication by m defines a homomorphism $A \to M$, so $\operatorname{rad}(A)m \subseteq \operatorname{rad}(M)$ by Lemma 4.1. This proves the inclusion.

Next we prove that the inclusion is an equality for free modules. Let F be free with basis $(e_s)_{s \in S}$. Let $x = \sum_{s \in S} x_s e_s \in \operatorname{rad}(F)$ (with $x_s = 0$ for all but finitely many s). Applying Lemma 4.1 to the s-projection $F \to A$ gives $x_s \in \mathrm{rad}(A)$, for each $s \in S$. Hence $x \in \mathrm{rad}(A)F$, giving $\mathrm{rad}(F) \subseteq \mathrm{rad}(A)F$.

Now let P be any projective module. By Lemma 2.1(iv), there is an epimorphism $\pi\colon F \twoheadrightarrow P$ with a section $\iota\colon P \to F$, for some free F. So

$$\mathrm{rad}(P) = \pi\iota\,\mathrm{rad}(P) \subseteq \pi\,\mathrm{rad}(F) = \pi(\mathrm{rad}(A)F) \subseteq \mathrm{rad}(A)P,$$

using Lemma 4.1 twice. ■

Lemma 4.3. $\mathrm{rad}(A)^n = 0$ *for some* $n \geq 0$.

Proof. Since A is finite-dimensional, we can choose $n \geq 0$ minimizing the dimension of the A-module $\mathrm{rad}(A)^n$. Suppose that $\mathrm{rad}(A)^n \neq 0$. By finite-dimensionality again, $\mathrm{rad}(A)^n$ has a maximal submodule, so $\mathrm{rad}(\mathrm{rad}(A)^n)$ is a proper submodule of $\mathrm{rad}(A)^n$. But $\mathrm{rad}(A)^{n+1} \subseteq \mathrm{rad}(\mathrm{rad}(A)^n)$ by Lemma 4.2, so $\mathrm{rad}(A)^{n+1}$ is a proper submodule of $\mathrm{rad}(A)^n$, a contradiction. ■

Proposition 4.4. *Every nonzero projective module has a maximal submodule.*

Proof. Let P be a projective module with no maximal submodule. Then $P = \mathrm{rad}(P) = \mathrm{rad}(A)P$ by Lemma 4.2, so $P = \mathrm{rad}(A)^n P$ for all $n \geq 0$, so $P = 0$ by Lemma 4.3. ■

5 From simple modules to projective indecomposable modules

Here we show that for every simple module S, there is a unique projective indecomposable module P such that S is a quotient of P.

Roughly, our first lemma states that different projective indecomposable modules share no quotients.

Lemma 5.1. *Let P and P' be projective indecomposable modules, at least one of which is finitely generated. If some nonzero module is a quotient of both P and P' then $P \cong P'$.*

Proof. Suppose that P is finitely generated, and that there exist a nonzero module M and epimorphisms $\pi\colon P \twoheadrightarrow M$, $\pi'\colon P' \twoheadrightarrow M$. Since P and P' are projective, there exist homomorphisms

$$\begin{tikzcd} P \arrow[r, shift left, "\alpha'"] \arrow[dr, two heads, "\pi"'] & P' \arrow[l, shift left, "\alpha"] \arrow[d, two heads, "\pi'"] \\ & M \end{tikzcd}$$

such that $\pi'\alpha' = \pi$ and $\pi\alpha = \pi'$. Then $\alpha\alpha'$ is an endomorphism of P satisfying $\pi(\alpha\alpha') = \pi$. Since P is finitely generated and indecomposable, Corollary 3.2 implies that $\alpha\alpha'$ is nilpotent or invertible. If nilpotent then $(\alpha\alpha')^n = 0$ for some $n \geq 0$, so $\pi = \pi(\alpha\alpha')^n = \pi 0 = 0$, contradicting the fact that π is an epimorphism to a nonzero module. So $\alpha\alpha'$ is invertible, and in particular α is an epimorphism. By Lemma 2.1(iii), P is therefore a direct summand of P'. But P' is indecomposable and P is nonzero, so $P \cong P'$. ■

Lemma 5.2. *Every simple module is cyclic.*

Proof. Let S be a simple module. Since S is nonzero, we may choose a nonzero element $x \in S$. The submodule generated by x is nonzero, and is therefore S. ∎

Lemma 5.3. *Every simple module is a quotient of some cyclic projective indecomposable module.*

Proof. Let S be a simple module. By Lemma 5.2, S is a quotient of the A-module A. Thus, among all direct summands M of A with the property that S is a quotient of M, we may choose one of smallest dimension; call it P.

This module P is projective (by Lemma 2.1(ii)) and cyclic (being a quotient of A). To see that P is indecomposable, suppose that $P = M \oplus N$ for some submodules M and N. Take an epimorphism $\pi \colon P \twoheadrightarrow S$. Then $S = \pi M + \pi N$ and S is nonzero, so without loss of generality, πM is nonzero. Since S is simple, $\pi M = S$, so S is a quotient of M. But M is a direct summand of A, so minimality of P gives $M = P$, as required. ∎

The next result is a partner to Lemma 5.2. It implies, in particular, that projective indecomposable modules are finitely generated.

Lemma 5.4. *Every projective indecomposable module is cyclic.*

Proof. Let P be a projective indecomposable module. By Proposition 4.4, we may choose a maximal submodule of P, the quotient by which is a simple module: say $\pi \colon P \twoheadrightarrow S$. By Lemma 5.3, we may then choose $\pi' \colon P' \twoheadrightarrow S$ with P' cyclic and projective indecomposable. By Lemma 5.1, $P \cong P'$. Hence P is cyclic. ∎

Proposition 5.5. *For each simple module S, there is a projective indecomposable module, unique up to isomorphism, of which S is a quotient.*

Proof. Lemma 5.3 proves existence. Lemmas 5.1 and 5.4 prove uniqueness up to isomorphism. ∎

Given a simple module S, the unique projective indecomposable module of which S is a quotient is called its **projective cover**.

6 From projective indecomposable modules to simple modules

In the last section, we showed that for every simple module S, there is a unique projective indecomposable module of which S is a quotient. We now show that this process is bijective. In other words, we show that every projective indecomposable module has a unique simple quotient.

Lemma 6.1. *Every projective indecomposable module has exactly one maximal submodule.*

Proof. Let P be a projective indecomposable module. By Proposition 4.4, P has at least one maximal submodule. Now let M and M' be maximal submodules, and consider the inclusions and projections

$$\begin{tikzcd}[row sep=small] M \arrow[dr, "\iota"] & & P/M \\ & P \arrow[ur, two heads, "\pi"] \arrow[dr, two heads, "\pi'"'] & \\ M' \arrow[ur, "\iota'"'] & & P/M'. \end{tikzcd}$$

Since P/M' is simple, $\mathrm{im}(\pi'\iota)$ is either 0 or P/M'. If 0 then $M \subseteq \ker \pi' = M'$; but M and M' are maximal, so $M = M'$. It therefore suffices to prove that $\pi'\iota$ is not an epimorphism. Suppose that it is. Since P is projective, there exists a homomorphism ψ such that

$$\begin{tikzcd} & P \arrow[dl, dotted, "\psi"'] \arrow[dr, two heads, "\pi'"] & \\ M \arrow[r, "\iota"'] & P \arrow[r, two heads, "\pi'"'] & P/M' \end{tikzcd}$$

commutes. By Lemma 5.4, P is finitely generated, so by Corollary 3.2, the endomorphism $\iota\psi$ of P is nilpotent or invertible. If nilpotent then $(\iota\psi)^n = 0$ for some $n \geq 0$; but $\pi' = \pi'(\iota\psi)$, so $\pi' = \pi'(\iota\psi)^n = 0$, contradicting the fact that π' is an epimorphism to a nonzero module. If invertible then ι is an epimorphism, so $M = P$, also a contradiction. ∎

Proposition 6.2. *For each projective indecomposable module P, there is a simple module, unique up to isomorphism, that is a quotient of P.*

Proof. Immediate from Lemma 6.1. ∎

The unique simple quotient of a projective indecomposable module is called its **top** or **head**.

7 The bijection

Assembling the results of the last two sections, we obtain our main theorem.

Theorem 7.1. *There is a bijection between the set of isomorphism classes of projective indecomposable modules and the set of isomorphism classes of simple modules, given by matching a projective indecomposable module P with a simple module S if and only if there exists an epimorphism $P \twoheadrightarrow S$.*

Proof. Immediate from Propositions 5.5 and 6.2. ∎

Example 7.2. Let A be the algebra of 2×2 upper triangular matrices $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ over K. The A-module A has submodules

$$P_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in K \right\}, \qquad P_2 = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} : b, c \in K \right\}$$

satisfying $P_1 \oplus P_2 = A$. By Lemma 2.1(ii), P_1 and P_2 are projective.

Since P_1 is 1-dimensional, it is simple, and in particular indecomposable. The existence of the identity homomorphism $P_1 \twoheadrightarrow P_1$ implies that the simple module corresponding to the projective indecomposable module P_1 is P_1 itself.

By an elementary calculation, P_2 has just one nontrivial submodule M, consisting of the matrices $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$ ($b \in K$). It follows that P_2 is indecomposable. It also follows that M is the unique maximal submodule of P_2. The simple module corresponding to the projective indecomposable module P_2 is, therefore, P_2/M. Explicitly, P_2/M is the vector space K made into an A-module by the action $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \cdot c = zc$.

This example shows that a simple module need not be projective indecomposable, or vice versa, as mentioned in the Introduction. For if P_2/M were projective then by Lemma 2.1(iii), it would be a 1-dimensional direct summand of the 2-dimensional module P_2, which is indecomposable. Conversely, the projective indecomposable module P_2 is not simple, having a nontrivial submodule M.

Lemma 9.5 will imply that P_1 and P_2 are the *only* projective indecomposable A-modules, and, therefore, that P_1 and P_2/M are the only simple A-modules.

8 The space of homomorphisms

When a projective indecomposable module P corresponds to a simple module S (that is, when there exists an epimorphism $P \twoheadrightarrow S$), we can try to describe the space $\mathrm{Hom}_A(P, S)$ of all homomorphisms $P \to S$. This is made easier by:

Lemma 8.1. *Every homomorphism into a simple module is either zero or an epimorphism.*

Proof. The image of such a homomorphism is a submodule of the codomain S, and is therefore 0 or S. ∎

In particular, this implies the following result, a partner to Corollary 3.2 for indecomposable modules.

Lemma 8.2. *Every endomorphism of a simple module is either zero or invertible.*

Proof. Follows from Lemma 8.1, since a surjective endomorphism of a finite-dimensional vector space is invertible. ∎

When P and S correspond as in Theorem 7.1, we can describe $\mathrm{Hom}_A(P, S)$ in terms of S alone:

Proposition 8.3. *Let P be a projective indecomposable module and S a simple module. Then $\mathrm{Hom}_A(P, S)$ is isomorphic as a vector space to either 0 or $\mathrm{End}_A(S)$.*

In the latter case, the isomorphism is *not* canonical.

Proof. If $\mathrm{Hom}_A(P, S) \neq 0$ then we can choose a nonzero homomorphism $\pi\colon P \to S$. By Lemma 8.1, π is an epimorphism, so by Lemma 6.1, $\ker \pi$ is the unique maximal submodule of P. Composition with π defines a linear map

$$- \circ \pi\colon \mathrm{End}_A(S) \to \mathrm{Hom}_A(P, S),$$

which we will prove is an isomorphism. It is injective, as π is an epimorphism. To show that is surjective, let $\phi \in \mathrm{Hom}_A(P, S)$. By Lemma 8.1, ϕ is either 0 or an epimorphism, so $\ker \phi$ is either P or a maximal submodule of P. In either case, $\ker \phi \supseteq \ker \pi$. Hence ϕ factors through π, as required. ■

Although $\mathrm{Hom}_A(P, S)$ does not carry the structure of a K-algebra in any obvious way, $\mathrm{End}_A(S)$ does. We now analyse that structure.

Lemma 8.4. *Let S be a simple module. Then:*

i. the K-algebra $\mathrm{End}_A(S)$ *is a skew field;*

ii. if K is algebraically closed then the K-algebra $\mathrm{End}_A(S)$ *is canonically isomorphic to K.*

Proof. Part (i) is immediate from Lemma 8.2. For (ii), we prove that the K-algebra homomorphism

$$\begin{array}{ccc} K & \to & \mathrm{End}_A(S) \\ \lambda & \mapsto & \lambda \cdot \mathrm{id}_S \end{array}$$

is an isomorphism. It is injective, as S is nonzero. To prove surjectivity, let $\theta \in \mathrm{End}_A(S)$. Then θ is a linear endomorphism of a nonzero finite-dimensional vector space over an algebraically closed field, and so has an eigenvalue λ. But $\theta - \lambda \cdot \mathrm{id}_S$ is then a non-invertible A-endomorphism of S, so by Lemma 8.2, it must be zero. ■

Proposition 8.5. *Let P be a projective indecomposable module and S a simple module. Suppose that K is algebraically closed. Then* $\mathrm{Hom}_A(P, S)$ *is isomorphic as a vector space to either* 0 *or K.*

Proof. Follows from Proposition 8.3 and Lemma 8.4(ii). ■

In the latter case, the isomorphism $\mathrm{Hom}_A(P, S) \cong K$ is not canonical.

9 Finitely many isomorphism classes

We have shown that the set of isomorphism classes of projective indecomposable modules is in bijection with the set of isomorphism classes of simple modules. Here we show that both sets are finite. We give two alternative proofs, each using a standard theorem whose proof we omit.

The first uses the Jordan–Hölder theorem (Theorem 3.11 of [2] or Theorem 1.1.4 of [1]) to show that there are only finitely many simple modules. A **composition series** of a module M is a chain

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M \tag{1}$$

of submodules in which each quotient M_j / M_{j-1} is simple.

Theorem 9.1 (Jordan–Hölder)**.** *Every finitely generated module M has a composition series* (1)*, and the modules* $M_1/M_0, \ldots, M_r/M_{r-1}$ *are independent of the composition series chosen, up to reordering and isomorphism.*

These quotients M_j/M_{j-1} are called the **composition factors** of M. Thus, every finitely generated module has a well-defined set-with-multiplicity of composition factors, which are simple modules. In particular, this is true of the A-module A; write $S_1, \ldots, S_r$ for its composition factors. (They need not all be distinct.)

Whenever N is a submodule of a finitely generated module M, the composition factors of M are the composition factors of N together with the composition factors of M/N, adding multiplicities. Hence:

Lemma 9.2. *Every simple module is isomorphic to S_j for some $j \in \{1, \ldots, r\}$.*

Proof. Let S be a simple module. By Lemma 5.2, S is a quotient of the A-module A. Hence every composition factor of S is a composition factor S_j of A. But S is simple, so its unique composition factor is itself. ∎

Together with Theorem 7.1, this gives our first proof of:

Proposition 9.3. *There are only finitely many isomorphism classes of projective indecomposable modules, and only finitely many isomorphism classes of simple modules.* ∎

For the second proof of Proposition 9.3, we use the Krull–Schmidt theorem (Theorem 6.12 of [2] or Theorem 1.4.6 of [1], for instance).

Theorem 9.4 (Krull–Schmidt)**.** *Every finitely generated module is isomorphic to a finite direct sum $M_1 \oplus \cdots \oplus M_n$ of indecomposable modules, and $M_1, \ldots, M_n$ are unique up to reordering and isomorphism.*

In particular, the A-module A is isomorphic to $P_1 \oplus \cdots \oplus P_n$ for some indecomposable A-modules P_i, which are determined uniquely up to order and isomorphism. (They need not all be distinct.) By Lemma 2.1(ii), each P_i is projective. Conversely:

Lemma 9.5. *Every projective indecomposable module is isomorphic to P_i for some $i \in \{1, \ldots, n\}$.*

Proof. Let P be a projective indecomposable module. By Lemma 5.4, there is an epimorphism $A \twoheadrightarrow P$, so by Lemma 2.1(iii), $A \cong P \oplus Q$ for some module Q. Now Q is a quotient of A and therefore finitely generated, so by the Krull–Schmidt theorem, $Q = Q_1 \oplus \cdots \oplus Q_m$ for some indecomposable modules Q_j. This gives $A \cong P \oplus Q_1 \oplus \cdots \oplus Q_m$. Each of the summands is indecomposable, so by the uniqueness part of Krull–Schmidt, P is isomorphic to some P_i. ∎

Together with Theorem 7.1, this provides a second proof of Proposition 9.3.

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