On the geometry of complete submanifolds immersed in the hyperbolic space

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Abstract

We deal with n-dimensional complete submanifolds immersed with parallel nonzero mean curvature vector $\mathbf{H}$ in the hyperbolic space $\mathbb{H}^{n+p}$. In this setting, we establish sufficient conditions to guarantee that such a submanifold M^n must be pseudo-umbilical, which means that $\mathbf{H}$ is an umbilical direction. In particular, we conclude that M^n is a minimal submanifold of a small hypersphere of $\mathbb{H}^{n+p}$.

1 Introduction and statement of the main result

The study of Bernstein-type properties concerning complete hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$ constitutes a classical and interesting theme into the scope of the isometric immersions. In this branch, do Carmo and Lawson [7] used the well known Alexandrov's reflexion method to show that a complete hypersurface properly embedded with constant mean curvature in $\mathbb{H}^{n+1}$ with a single point at the asymptotic boundary must be a horosphere. They also observed that the statement is no longer true if we replace embedded by immersed. Later on, Alías and Dajczer [1] proved that the horospheres are the only surfaces properly immersed in $\mathbb{H}^3$ with constant mean curvature $-1 \leq H \leq 1$ and which are contained in a slab (that is, the region between two horospheres that share the same point in the asymptotic boundary).

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More recently, the first author jointly with Aquino [3] used some generalized maximum principles in order to obtain another characterization result for the horospheres of $\mathbb{H}^{n+1}$. Meanwhile, these same authors jointly Barros showed that the only complete constant mean curvature hypersurfaces immersed in $\mathbb{H}^{n+1}$ with scalar curvature bounded from below and whose angle function with respect to some fixed vector a does not change sign, and with $a^\top$ having Lebesgue integrable norm along them, are the totally umbilical ones (see Theorem 1.2 of [4]).

Our purpose in this paper is to study the geometry of n-dimensional complete submanifolds immersed with parallel nonzero mean curvature vector (that is, the mean curvature vector field is parallel as a section of the normal bundle) in the $(n+p)$-dimensional hyperbolic space $\mathbb{H}^{n+p}$, which we are considering as being a quadric of the $(n+p+1)$-dimensional Lorentz-Minkowski space $\mathbb{R}_1^{n+p+1}$ (for more details, see Section 2). In this setting, we use a technique developed by Alías and Romero [2] jointly with the application of a suitable extension of a generalized maximum principle of Yau [11] due to Caminha in [5] (cf. Lemma 1) to prove the following result

Theorem 1. *Let M^n be a complete submanifold immersed in $\mathbb{H}^{n+p} \subset \mathbb{R}_1^{n+p+1}$ with nonzero parallel mean curvature vector* $\mathbf{H}$ *and normalized scalar curvature bounded from below. Suppose that there exists a fixed vector $a \in \mathbb{R}_1^{n+p+1}$ such that $|a^\top| \in \mathcal{L}^1(M)$, a^N does not vanish on M^n and a^N is collinear to* $\mathbf{H}$. *Then, M^n is pseudo-umbilical and, in particular, M^n is a minimal submanifold of a small hypersphere of $\mathbb{H}^{n+p}$.*

Here, $a^\top$ and a^N denote, respectively, the tangential and normal components of the vector a with respect to the immersion $M^n \hookrightarrow \mathbb{H}^{n+p} \subset \mathbb{R}_1^{n+p+1}$, and $\mathcal{L}^1(M)$ stands for the space of Lebesgue integrable functions on the submanifold M^n. Moreover, we recall that a submanifold M^n of $\mathbb{H}^{n+p}$ is called *pseudo-umbilical* when its mean curvature vector is an umbilical direction.

We note that, when $p = 1$, the notion of pseudo-umbilical coincides with that of totally umbilical. Moreover, we also observe that the hypothesis that a^N does not vanish on M^n amounts to the angle function $f_a = \langle a, \nu \rangle$ having strict sign on it, where ν stands for the Gauss mapping of $M^n \hookrightarrow \mathbb{H}^{n+1}$. Consequently, Theorem 1 can be regarded as an extension of Theorem 1.2 of [4]. Section 3 is devoted to present the proof of Theorem 1.

2 Preliminaries

Let $\mathbb{R}_1^{n+p+1}$ be the $(n+p+1)$-dimensional Lorentz-Minkowski space endowed with metric tensor $\langle , \rangle$ of index 1, given by

$$\langle v, w \rangle = \sum_{i=1}^{n+p} v_i w_i - v_{n+p+1} w_{n+p+1},$$

and let $\mathbb{H}^{n+p}$ be the $(n+p)$-dimensional unitary hyperbolic space, that is,

$$\mathbb{H}^{n+p} = \{x \in \mathbb{R}_1^{n+p+1} ; \langle x, x \rangle = -1\},$$

which has constant sectional curvature equal to -1.

Along this work, we will consider $x : M^n \to \mathbb{H}^{n+p} \subset \mathbb{R}_1^{n+p+1}$ a submanifold isometrically immersed in $\mathbb{H}^{n+p}$. In this setting, we will denote by ∇°, $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{R}_1^{n+p+1}$, $\mathbb{H}^{n+p}$ and M^n, respectively, and $\nabla^\perp$ will stand for the normal connection of M^n in $\mathbb{H}^{n+p}$.

We will denote by α the second fundamental form of M^n in $\mathbb{H}^{n+p}$ and by A_ξ the shape operator associated to a fixed vector field ξ normal to M^n in $\mathbb{H}^{n+p}$. We note that, for each $\xi \in \mathfrak{X}^\perp(M)$, A_ξ is a symmetric endomorphism of the tangent space T_xM at $x \in M^n$. Moreover, A_ξ and α are related by

$$\langle A_\xi X, Y\rangle = \langle \alpha(X,Y), \xi\rangle, \tag{2.1}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$.

We also recall that the Gauss and Weingarten formulas of M^n in $\mathbb{H}^{n+p}$ are given by

$$\nabla^\circ_X Y = \overline{\nabla}_X Y + \langle X, Y\rangle x = \nabla_X Y + \alpha(X,Y) + \langle X, Y\rangle x, \tag{2.2}$$

and

$$\nabla^\circ_X \xi = \overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$ and normal vector field $\xi \in \mathfrak{X}^\perp(M)$.

As in [10], the curvature tensor R of the submanifold M^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[,]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(M)$.

A well known fact is that the curvature tensor R of M^n can be described in terms of its second fundamental form α and the curvature tensor $\overline{R}$ of the ambient spacetime $\mathbb{H}^{n+p}$ by the so-called Gauss equation, which is given by

$$\begin{aligned}\langle R(X,Y)Z, W\rangle = &\langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle \\ &+ \langle \alpha(X,Z), \alpha(Y,W)\rangle - \langle \alpha(X,W), \alpha(Y,Z)\rangle,\end{aligned} \tag{2.3}$$

for all tangent vector fields $X, Y, Z, W \in \mathfrak{X}(M)$. Moreover, Codazzi equation asserts that

$$(\nabla_X A_\xi)Y = (\nabla_Y A_\xi)X, \tag{2.4}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^\perp(M)$.

The mean curvature vector $\mathbf{H}$ of $M^n \hookrightarrow \mathbb{H}^{n+p}$ is defined by

$$\mathbf{H} = \frac{1}{n}\mathrm{tr}(\alpha).$$

We recall that M^n has *parallel mean curvature vector* when $\nabla^\perp_X \mathbf{H} \equiv 0$, for every $X \in \mathfrak{X}(M)$. Furthermore, according to [6], a submanifold M^n of $\mathbb{H}^{n+p}$ with $\mathbf{H} \neq 0$ is called *pseudo-umbilical* when there exists a nonzero constant λ such that

$$\langle \alpha(X,Y), \mathbf{H}\rangle = \lambda\langle X, Y\rangle,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$.

At this point, we will describe the main analytical tool which is used along the proofs of our results in the next section. In [11] Yau, generalizing a previous result due to Gaffney [8], established the following version of Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold M^n: *if $\omega \in \Omega^{n-1}(M)$ is an integrable $(n-1)$-differential form on M^n, then there exists a sequence B_i of domains on M^n such that $B_i \subset B_{i+1}$, $M^n = \bigcup_{i\geq 1} B_i$ and*

$$\lim_{i\to+\infty}\int_{B_i} d\omega = 0.$$

Suppose that M^n is oriented by the volume element dM. If $\omega = \iota_X dM$ is the contraction of dM in the direction of a smooth vector field X on M^n, then Caminha obtained a suitable consequence of Yau's result, which can be regarded as an extension of Hopf's maximum principle for complete Riemannian manifolds (cf. Proposition 2.1 of [5]). In what follows, $\mathcal{L}^1(M)$ and div denote the space of Lebesgue integrable functions and the divergence on M^n, respectively.

Lemma 1. *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold M^n, such that* $\mathrm{div}X$ *does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then* $\mathrm{div}X = 0$.

Remark 1. *Lemma 1 can also be seen as a consequence of the version of Stokes' Theorem given by Karp in [9]. In fact, using Theorem in [9], condition $|X| \in \mathcal{L}^1(M)$ can be weakened to the following technical condition:*

$$\liminf_{r\to+\infty}\frac{1}{r}\int_{B(2r)\setminus B(r)}|X|dM = 0,$$

where $B(r)$ denotes the geodesic ball of radius r center at some fixed origin $o \in M^n$. See also Corollary 1 and Remark in [9] for some another geometric conditions guaranteeing this fact.

3 Proof of Theorem 1

Initially, taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n, from (2.3) we get that the squared norm of second form fundamental α of M^n satisfies

$$|\alpha|^2 = \sum_{i,j}|\alpha(e_i, e_j)|^2 = n^2\langle \mathbf{H}, \mathbf{H}\rangle - n(n-1)(R+1), \tag{3.1}$$

where R stands for the normalized scalar curvature of M^n.

On the other hand, since we are supposing that M^n has nonzero parallel mean curvature vector $\mathbf{H}$, a simple computation allows us to verify that $\langle \mathbf{H}, \mathbf{H}\rangle$ is a nonzero constant. Consequently, since we are also assuming that M^n has normalized scalar curvature R bounded from below, from (3.1) we conclude that α is bounded on M^n.

Let $a \in \mathbb{R}_1^{n+p+1}$ be a fixed nonzero vector and put

$$a = a^\top + a^N - \langle a, x\rangle x, \tag{3.2}$$

where $a^\top \in \mathfrak{X}(M)$ and $a^N \in \mathfrak{X}^\perp(M)$ denote, respectively, the tangential and normal components of a with respect to $M^n \hookrightarrow \mathbb{H}^{n+p}$. By taking covariant derivative in (3.2) and using (2.2), we get for all tangent vector field $X \in \mathfrak{X}(M)$ that

$$\nabla_X a^\top = A_{a^N} X + \langle a, x\rangle X. \tag{3.3}$$

Hence, from (2.1) and (3.3) we obtain

$$\operatorname{div}(a^\top) = \operatorname{tr}(A_{a^N}) + n\langle a, x\rangle = n\langle a, \mathbf{H}\rangle + n\langle a, x\rangle. \tag{3.4}$$

Moreover, we also have that

$$\begin{aligned}\operatorname{tr}(\nabla_{a^\top} A_\xi) = &\sum_i \langle \nabla_{a^\top} A_\xi e_i, e_i\rangle - \sum_i \langle \nabla_{a^\top} e_i, A_\xi e_i\rangle \\ &+ n\langle \nabla^\perp_{a^\top}\mathbf{H}, \xi\rangle - \sum_i a^\top \langle A_\xi e_i, e_i\rangle.\end{aligned}$$

So, considering a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $A_\xi e_i = \lambda_i^\xi e_i$, with a straightforward computation we can verify that

$$\operatorname{tr}(\nabla_{a^\top} A_\xi) = n\langle \nabla^\perp_{a^\top}\mathbf{H}, \xi\rangle. \tag{3.5}$$

From Codazzi equation (2.4) jointly with the equations (3.3) and (3.5) we obtain, for all $\xi \in \mathfrak{X}^\perp(M)$,

$$\begin{aligned}\operatorname{div}(A_\xi a^\top) = &\, n\langle \nabla^\perp_{a^\top}\mathbf{H}, \xi\rangle + \operatorname{tr}(A_{a^N}\circ A_\xi) + \langle a, x\rangle \operatorname{tr}(A_\xi) \\ &+ \sum_i \langle \alpha(a^\top, e_i), \nabla^\perp_{e_i}\xi\rangle.\end{aligned} \tag{3.6}$$

Now, let us suppose that a^N is collinear to $\mathbf{H}$. Taking $\xi = \mathbf{H}$ in (3.6), we get

$$\operatorname{div}(A_{\mathbf{H}} a^\top) = \operatorname{tr}(A_{a^N}\circ A_{\mathbf{H}}) + \langle a, x\rangle \operatorname{tr}(A_{\mathbf{H}}). \tag{3.7}$$

On the other hand, from (3.4) we have

$$\langle a, x\rangle = \frac{1}{n}\operatorname{div}(a^\top) - \langle a, \mathbf{H}\rangle. \tag{3.8}$$

Consequently, from (3.7) and (3.8) we obtain

$$\operatorname{div}(A_{\mathbf{H}} a^\top) = \operatorname{tr}(A_{a^N}\circ A_{\mathbf{H}}) + \operatorname{tr}(A_{\mathbf{H}})\frac{1}{n}\operatorname{div}(a^\top) - \frac{1}{n}\operatorname{tr}(A_{a^N})\operatorname{tr}(A_{\mathbf{H}}). \tag{3.9}$$

Moreover, taking into account once more that $\mathbf{H}$ is parallel, we also have

$$\operatorname{div}\left(\operatorname{tr}(A_{\mathbf{H}}) a^\top\right) = \operatorname{tr}(A_{\mathbf{H}})\operatorname{div}(a^\top). \tag{3.10}$$

Hence, from (3.9) and (3.10) we get

$$\operatorname{div} X = \operatorname{tr}(A_{a^N}\circ A_{\mathbf{H}}) - \frac{1}{n}\operatorname{tr}(A_{a^N})\operatorname{tr}(A_{\mathbf{H}}), \tag{3.11}$$

where X is a tangent vector field on M^n given by

$$X = \left(A_{\mathbf{H}} - \frac{1}{n}\mathrm{tr}(A_{\mathbf{H}})I \right) a^{\top}.$$

Since a^N does not vanish on M^n, there exists a smooth function λ having strict sign on M^n such that $a^N = \lambda \mathbf{H}$. So, from (3.11) we get

$$\mathrm{div}X = \lambda \left(\mathrm{tr}(A^2_{\mathbf{H}}) - \frac{1}{n}\mathrm{tr}(A_{\mathbf{H}})^2 \right). \tag{3.12}$$

But, we observe that the function $u = \mathrm{tr}(A^2_{\mathbf{H}}) - \frac{1}{n}\mathrm{tr}(A_{\mathbf{H}})^2$ is always nonnegative with $u = 0$ if, and only if, $\mathbf{H}$ is a umbilical direction. Consequently, from (3.12) we conclude that $\mathrm{div}X$ does not change sign on M^n.

Furthermore, since $|a^{\top}| \in \mathcal{L}^1(M)$, we also have that

$$|X| \leq (|A_{\mathbf{H}}| + |\langle \mathbf{H}, \mathbf{H} \rangle|)\, |a^{\top}| \in \mathcal{L}^1(M).$$

Hence, we can apply Lemma 1 to conclude that $\mathrm{div}X = 0$ on M^n.

Therefore, returning to (3.12) we obtain that

$$\lambda \left(\mathrm{tr}(A^2_{\mathbf{H}}) - \frac{1}{n}\mathrm{tr}(A_{\mathbf{H}})^2 \right) = 0,$$

which implies that $\mathbf{H}$ is an umbilical direction. Finally, from Proposition 4.2 of [6] we conclude that M^n must also be a minimal submanifold of a small hypersphere of $\mathbb{H}^{n+p}$.

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