Existence of solutions for p-Laplacian equation with electromagnetic fields and critical nonlinearity

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Abstract

In this paper, we consider the existence and multiplicity of solutions for perturbed p-Laplacian equation problems with critical nonlinearity in $\mathbb{R}^N$: $-\varepsilon^p\Big[g\Big(\int_{\mathbb{R}^N}|\nabla_A u|^p dx\Big)\Big]\Delta_{p,A}u+V(x)|u|^{p-2}u=|u|^{p^*-2}u+h(x,|u|^p)|u|^{p-2}u$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, where $V(x)$ is a nonnegative potential, $\Delta_{p,A}u(x):=\operatorname{div}(|\nabla u+iA(x)u|^{p-2}(\nabla u+iA(x)u)$ and $\nabla_A u:=(\nabla+iA)u$. By using Lions' second concentration compactness principle and concentration compactness principle at infinity to prove that the $(PS)_c$ condition holds locally and by variational method, we show that this equation has at least one solution provided that $\varepsilon<\mathcal{E}$, for any $m\in\mathbb{N}$, it has m pairs of solutions if $\varepsilon<\mathcal{E}_m$, where $\mathcal{E}$ and $\mathcal{E}_m$ are sufficiently small positive numbers.

1 Introduction

The main purpose of this paper is to study the existence and multiplicity of solutions for the following perturbed p-Laplacian equation problems with critical

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nonlinearity of the form

$$-\varepsilon^p\Big[g\Big(\int_{\mathbb{R}^N}|\nabla_A u|^p dx\Big)\Big]\Delta_{p,A}u + V(x)|u|^{p-2}u = \\ |u|^{p^*-2}u + h(x,|u|^p)|u|^{p-2}u,\ x\in\mathbb{R}^N, \quad (1)$$

where $\Delta_{p,A}u(x) := \operatorname{div}(|\nabla u + iA(x)u|^{p-2}(\nabla u + iA(x)u)$, here i is the imaginary unit, $p^* := pN/(N-p)$ denotes the Sobolev critical exponent and $N \geq 3$.

We make the following assumptions on $V(x)$, $g(x)$ and $h(x)$ throughout this paper:

(V) $V(x) \in C(\mathbb{R}^N,\mathbb{R})$, $V(x_0) = \min V = 0$ and there is $\tau_0 > 0$ such that the set $V^{\tau_0} = \{x \in \mathbb{R}^N : V(x) < \tau_0\}$ has finite Lebesgue measure;

(G) (g_1) There exists $\alpha_0 > 0$ such that nondecreasing function $g(t) \geq \alpha_0$ for all $t \geq 0$;
(g_2) There exists θ satisfied $\frac{p}{\mu} < \theta < 1$ and $G(t) \geq \theta g(t)t$ for all $t \geq 0$, where $G(t) = \int_0^t g(s)ds$;

(A) $A_j(x) \in C(\mathbb{R}^N,\mathbb{R})(j = 1,2,\ldots,N)$ and $A(x_0) = 0$;

(H) (h_1) $h \in C(\mathbb{R}^N \times [0,+\infty),\mathbb{R})$ and $h(x,t) = o(|t|)$ uniformly in x as $t \to 0$;
(h_2) there are $C_0 > 0$ and $q \in (p,p^*)$ such that $|h(x,t)| \leq C_0(1 + t^{\frac{q-p}{p}})$;
(h_3) there $l_0 > 0$, $s > \frac{p}{\theta}$ and $p < \mu < p^*$ such that $H(x,t) \geq l_0|t|^{\frac{s}{p}}$ and $\mu H(x,t) \leq h(x,t)t$ for all (x,t), where $H(x,t) = \int_0^t h(x,s)ds$.

Mathematics is successfully applied in numerous technical problems, semi-subsistence agriculture systems, biology, farming systems research, agricultural production planning, control theory, etc [4, 5, 6, 21, 25, 26, 35, 41, 45]. There are many methods to delta with these problems, for example: variational method, Morse theory, mathematical programming and multi-variate analysis etc. We can use the methods of mathematical programming and multi-variate analysis to delta with the agricultural production planning. The agricultural production planning has been defined as a process for the spatial organization of agricultural and forestry products that allocates particular uses to preferential land areas in an attempt to attain sustainable development by optimizing the agricultural production systems according to environmental concerns and socioeconomic and structural conditions. The aim of this process is to determine a sustainable development path in the relationship between agriculture and its natural environment. Therefore, a profound knowledge of this complex system and its behaviour under different socio-economic conditions is necessary.

In this paper, we want to use variational method to delta with problem (1). We note that problem (1) with $p = 2$, $A(x) \equiv 0$ $g(t) = 1$, it reduces to the well-known Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \quad (2)$$

Different approaches have been taken to attack this problem under various hypotheses on the potential and the nonlinearity. See for examples [16, 17, 36, 37] and the references therein. Observe that in all these papers the nonlinearities are assumed to be subcritical

$$|f(x,u)| \leq c(1+|u|^{p-1}) \text{ with } p \in (2,2^*), \tag{3}$$

together with some other technical conditions of course. Under the condition $V(x) > 0$, there have been enormous investigations on problem (2). Much of the impetus for these studies seems to have originated from the pioneering paper [23] by Floer and Weinstein in which the one-dimensional case ($N = 1$) with a cubic nonlinearity was studied by assuming that $V(x)$ is a bounded potential having a single non-degenerate minimum point x_0 while $\inf_{\mathbb{R}} V > 0$. As a matter of fact, based on a Lyapounov-Schmidt reduction technique, it was shown there that (2) admits, for $\varepsilon > 0$ sufficiently small, a family of spike-like solutions which in the semiclassical limit (i.e. as $\varepsilon \to 0$) concentrate around x_0; see also [36, 37]. The extension of this important result to higher dimensions with condition (3) and $V(x)$ having a finite set of non-degenerate critical points was achieved in [38] while this last hypothesis was eventually removed in [18]; for complementary results obtained by perturbation or variational methods see [1, 40], as well as the recent monograph [2]. For more results, we refer the reader to [3, 11, 22]. If the nonlinearities are assumed to be critical, Clapp and Ding [15] studied problem (2) with $f(x,u) = \mu u + u^{2^*-1}$ and $V(x) \geq 0$ and has a potential well and is invariant under an orthogonal involution of $\mathbb{R}^N$, they established existence and multiplicity of solutions which change sign exactly once and these solutions localize near the potential well for μ small and λ large. Ding and Lin [19] showed that the existence and multiplicity of semiclassical solutions of perturbed nonlinear Schrödinger equations with critical nonlinearity. Ding and Wei [20] established the existence and multiplicity of semiclassical bound states of the nonlinear Schrödinger equations under the assumption of $V(x)$ changes sign and f is superlinear with critical or supercritical growth as $|u| \to \infty$.

In equation (1) with bounded domain, if we set $p = 2$, $A(x) \equiv 0$, $\varepsilon = 1$, $V(x) = 0$ and $g(t) = a + bt$, it reduces to the following Dirichlet problem of Kirchhoff type

$$\begin{cases} -\left(a + b\displaystyle\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

Problem (4) is a generalization of a model introduced by Kirchhoff [30]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{5}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. The equation (4) is related to the stationary analogue of problem (5). (4) received much attention only after Lions [33] proposed an abstract framework to the problem. Some important and interesting results can be found, see for example [27, 29, 32]. We note that the results dealing with the problem (4) with critical nonlinearity are relatively scarce.

In equation (1) with $p \neq 2$, $A(x) \equiv 0$, $\varepsilon = 1$, $V(x) = 0$, it reduces to the p-Kirchhoff type problem. p-Kirchhoff type problem began to attract the attention of several researchers mainly after the work of Lions [33], where a functional analysis approach was proposed to attack it. However, in this work, we use a different approach to those explored in [29], because here we are working with the p-Laplacian operator. Because p-Laplacian operator is nonlinear, some estimates for this type of operator can not be obtained using the same kind of ideas explored for the case $p = 2$. For example, We know that $W^{1,p}(\mathbb{R}^N)$ is not a Hilbert space for $1 < p < N$, except for $p = 2$. The space $W^{1,p}(\mathbb{R}^N)$ with $p \neq 2$ does not satisfy the Lieb lemma [42].

When $A(x) \not\equiv 0$, there are also many works dealing with the magnetic case. The first one seems to be [22] where the existence of standing waves was obtained for $\hbar > 0$ fixed and for special classes of magnetic fields. If A and W are periodic functions, the existence of various types of solutions for fixed $\hbar > 0$ was proved in [7] by applying minimax arguments. Concerning semiclassical bound states, it was proved in [31] that for $\hbar > 0$ small and admits a least energy solution which concentrates near the global minimum of W. A multiplicity result for solutions was obtained in [12] by using a topological argument. There it was also proved that the magnetic potential A only contributes to the phase factor of the solitary solutions for $\hbar > 0$ sufficiently small. In [13] single-bump bound states were obtained by using perturbation methods. These concentrate near a non-degenerate critical point of W as $\hbar \to 0$. Chabrowski and Szulkin [14] considered problems (4) under assumption that $V(x)$ changes sign, by using a min-max type argument based on a topological linking, they obtained a solution in the Sobolev space which defined in the paper. Assume $K(x) \equiv 1$, Han [28] studied the problem (7) and established the existence of nontrivial solutions in the critical case by means of variational method.

To the best of our knowledge, the existence and multiplicity of solutions to problem (1) on $\mathbb{R}^N$ has not ever been studied by variational methods. As we shall see in the present paper, problem (1) can be viewed as a Schrödinger equation coupled with a non-local term. The competing effect of the non-local term with the critical nonlinearity and the lack of compactness of the embedding of $W^{1,p}(\mathbb{R}^N)$ into the space $L^p(\mathbb{R}^N)$, prevents us from using the variational methods in a standard way. Some new estimates for such a Kirchhoff equation involving Palais-Smale sequences, which are key points to apply this kinds of theory, are needed to be re-established. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all, since the appearance of non-local term, we must consider our problem for suitable space and so we need more delicate estimates.

Our main result is the following:

Theorem 1.1. *Let (V), (G), (A) and (H) be satisfied. Thus*
(1) For any $\sigma > 0$ there is $\mathcal{E}_\sigma > 0$ such that problem (1) *has at least one solution u_ε for each $\varepsilon \leq \mathcal{E}_\sigma$ satisfying*

$$\frac{\theta\mu - 1}{p} \int_{\mathbb{R}^N} H(x, |u_\varepsilon|^p)dx + \left(\frac{\theta}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |u_\varepsilon|^{p^*} dx \leq \sigma\varepsilon^N \tag{6}$$

and

$$\left(\frac{\theta}{p} - \frac{1}{\mu}\right) \alpha_0 \int_{\mathbb{R}^N} |\nabla_A u_\varepsilon|^p dx + \left(\frac{1}{p} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \lambda V(x)|u_\varepsilon|^p dx \leq \sigma\varepsilon^N. \tag{7}$$

(2) Assume additionally that $h(x,t)$ is odd in t, for any $m \in N$ and $\sigma > 0$ there is $\mathcal{E}_{m\sigma} > 0$ such that if problem (1) *has at least m pairs of solutions u_ε which satisfy the estimates* (6) *and* (7) *whenever $\varepsilon \geq \mathcal{E}_{m\sigma}$.*

Remark 1.1. We should point out that Theorem 1.1 is different from the previous results of [19, 20, 44] in four main directions:

(1) $A(x) \not\equiv 0$ and $p \not\equiv 2$. There exist many functions $h(x,t)$ satisfying condition (H), for example, $h(x,t) = P(x)|t|^{p-2}t$, where $P(x)$ is a positive and bounded function.

(2) $g(t) \not\equiv C$. There exist many functions $g(t)$ satisfying condition (g_1)-(g_2), for example, $g(t) = a + bt$, $a, b > 0$ and $\theta = \frac{1}{2}$.

(3) The potential function $V(x)$ can also be assumed other cases such that the embedding from $E_\lambda \hookrightarrow L^p(\mathbb{R}^N)$ is compact holds. For example: (i) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\liminf_{|x|\to\infty} V(x) > V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$; (ii) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ with periodic function (or bounded function) and $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$.

(4) We use Lions' second concentration compactness principle and concentration compactness principle at infinity to prove that the $(PS)_c$ condition holds which is different from methods used in [19, 20, 44].

2 Main result

We set $\lambda = \varepsilon^{-p}$ and rewrite (1) in the following form

$$\begin{aligned} -\left[g\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big)\right]\Delta_{p,A} u + \lambda V(x)|u|^{p-2}u = \\ \lambda|u|^{p^*-2}u + \lambda h(x, |u|^p)|u|^{p-2}u,\ x \in \mathbb{R}^N. \end{aligned} \tag{8}$$

We are going to prove the following result:

Theorem 2.1. *Let (V), (G), (A) and (H) be satisfied. Thus*
(1) For any $\sigma > 0$ there is $\Lambda_\sigma > 0$ such that problem (8) *has at least one solution u_λ for each $\lambda \geq \Lambda_\sigma$ satisfying*

$$\frac{\theta\mu - 1}{p} \int_{\mathbb{R}^N} H(x, |u_\lambda|^p)dx + \left(\frac{\theta}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |u_\lambda|^{p^*} dx \leq \sigma\lambda^{-\frac{N}{p}} \tag{9}$$

and

$$\left(\frac{\theta}{p}-\frac{1}{\mu}\right)\alpha_0\int_{\mathbb{R}^N}|\nabla_A u_\lambda|^p dx+\left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\lambda V(x)|u_\lambda|^p dx\leq\sigma\lambda^{1-\frac{N}{p}}. \tag{10}$$

(2) Assume additionally that $h(x,t)$ is odd in t, for any $m\in N$ and $\sigma>0$ there is $\Lambda_{m\sigma}>0$ such that if problem (8) *has at least m pairs of solutions u_λ which satisfy the estimates* (9) *and* (10) *whenever $\lambda\geq\Lambda_{m\sigma}$.*

In order to prove these theorems, we introduce the space

$$E_\lambda:=\left\{u\in W^{1,p}(\mathbb{R}^N,\mathbb{C}):\int_{\mathbb{R}^N}\lambda V(x)|u|^p dx<\infty,\ \lambda>0\right\}$$

equipped with the norm

$$\|u\|_\lambda^p=\int_{\mathbb{R}^N}\left(|\nabla_A u|^p+\lambda V(x)|u|^p\right)dx,$$

where $\nabla_A u:=\nabla u+iAu$. It is known that E_λ is the closure of $C_0^\infty(\mathbb{R}^N,\mathbb{C})$. Similar to the diamagnetic inequality [22], we have the following inequality

$$|\nabla_A u(x)|\geq|\nabla|u(x)||,\quad \text{for } u\in W^{1,p}(\mathbb{R}^N,\mathbb{C}).$$

Indeed, since A is real-valued

$$|\nabla|u|(x)|=\left|\mathrm{Re}\left(\nabla u\frac{\overline{u}}{|u|}\right)\right|=\left|\mathrm{Re}\left(\nabla u+iAu\right)\frac{\overline{u}}{|u|}\right|\leq|\nabla u+iAu|,$$

(the bar denotes complex conjugation) this fact means that if $u\in E_\lambda$, then $|u|\in W^{1,p}(\mathbb{R}^N,\mathbb{C})$, and therefore $u\in L^s(\mathbb{R}^N)$ for any $s\in[p,p^*)$. Thus, for each $s\in[p,p^*]$, there is $c_s>0$ (independent of λ) such that if $\lambda>1$

$$\left(\int_{\mathbb{R}^N}|u|^s\right)^{\frac{1}{s}}\leq c_s\left(\int_{\mathbb{R}^N}|\nabla|u||^p\right)^{\frac{1}{p}}\leq c_s\left(\int_{\mathbb{R}^N}|\nabla_A u|^p\right)^{\frac{1}{p}}\leq c_s\|u\|_\lambda. \tag{11}$$

The energy functional $J_\lambda:E_\lambda\to\mathbb{R}$ associated with problem (8)

$$J_\lambda(u):=\frac{1}{p}G\Big(\int_{\mathbb{R}^N}|\nabla_A u|^p dx\Big)+\frac{1}{p}\int_{\mathbb{R}^N}\lambda V(x)|u|^p dx- \\ \frac{\lambda}{p^*}\int_{\mathbb{R}^N}|u|^{p^*}dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}H(x,|u|^p)dx$$

is well defined. Thus, it is easy to check that as arguments [39, 43] $J_\lambda\in C^1(E_\lambda,\mathbb{R})$ and its critical points are solutions of (8).

We call that $u\in E_\lambda$ is a weak solution of (8), if

$$\langle J_\lambda'(u),v\rangle=\mathrm{Re}\Big\{g\Big(\int_{\mathbb{R}^N}|\nabla_A u|^p dx\Big)\int_{\mathbb{R}^N}\Big(|\nabla_A u|^{p-2}\nabla_A u\cdot\overline{\nabla_A v}\Big)dx+ \\ \lambda\int_{\mathbb{R}^N}V(x)|u|^{p-2}u\overline{v}dx-\lambda\int_{\mathbb{R}^N}|u|^{p^*-2}u\overline{v}dx-\lambda\int_{\mathbb{R}^N}h(x,|u|^p)|u|^{p-2}u\overline{v}dx\Big\},$$

where $v\in E_\lambda$.

3 Behaviors of (PS) sequences

We recall the second concentration-compactness principle of Lions [34]

Lemma 3.1. *[34] Let $\{u_n\}$ be a weakly convergent sequence to u in $W^{1,p}(\mathbb{R}^N)$ such that $|u_n|^{p^*} \rightharpoonup \nu$ and $|\nabla u_n|^p \rightharpoonup \mu$ in the sense of measures. Then, for some at most countable index set I,*

(i) $\nu = |u|^{p^*} + \sum_{j\in I} \delta_{x_j}\nu_j,\ \nu_j > 0,$

(ii) $\mu \geq |\nabla u|^p + \sum_{j\in I} \delta_{x_j}\mu_j,\ \mu_j > 0,$

(iii) $\mu_j \geq S\nu_j^{p/p^*},$

where S is the best Sobolev constant, i.e. $S = \inf\Big\{\int_{\mathbb{R}^N} |\nabla u|^p dx : \int_{\mathbb{R}^N} |u|^{p^*} dx = 1\Big\}$, *$x_j \in \mathbb{R}^N$, δ_{x_j} are Dirac measures at x_j and μ_j, ν_j are constants.*

Lemma 3.2. *[10] Let $\{u_n\}$ be a weakly convergent sequence to u in $W^{1,p}(\mathbb{R}^N)$ and define*

(i) $\nu_\infty = \lim\limits_{R\to\infty} \limsup\limits_{n\to\infty} \int_{|x|>R} |u_n|^{p^*} dx,$

(ii) $\mu_\infty = \lim\limits_{R\to\infty} \limsup\limits_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p dx.$

The quantities ν_∞ and μ_∞ exist and satisfy

(iii) $\limsup\limits_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty,$

(iv) $\limsup\limits_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty,$

(v) $\mu_\infty \geq S\nu_\infty^{p/p^*}.$

We recall that a C^1 functional J_λ on Banach space E_λ is said to satisfy the Palais-Smale condition at level c ($(PS)_c$ in short) if every sequence $\{u_n\} \subset E_\lambda$ satisfying $\lim\limits_{n\to\infty} J_\lambda(u_n) = c$ and $\lim\limits_{n\to\infty} \|J_\lambda(u_n)\|_{E_\lambda^*} = 0$ has a convergent subsequence.

Lemma 3.3. *Suppose that (V), (A), (G) and (H) hold. Then any $(PS)_c$ sequence $\{u_n\}$ is bounded in E_λ and $c \geq 0$.*

Proof. Let $\{u_n\}$ be a sequence in E_λ such that

$$\begin{aligned} c + o(1) &= J_\lambda(u_n) = \frac{1}{p}G\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big) + \frac{1}{p}\int_{\mathbb{R}^N} \lambda V(x)|u_n|^p dx \\ &\quad -\frac{\lambda}{p^*}\int_{\mathbb{R}^N} |u_n|^{p^*} dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} H(x,|u_n|^p)dx, \end{aligned} \tag{12}$$

$$
\begin{aligned}
\langle J'_\lambda(u_n), v\rangle &= \mathrm{Re}\Big\{ g\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big)\int_{\mathbb{R}^N} |\nabla_A u_n|^{p-2}\nabla_A u_n\cdot\overline{\nabla_A v}dx \\
&\quad + \lambda\int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\overline{v}dx - \lambda\int_{\mathbb{R}^N}|u_n|^{p^*-2}u_n\overline{v}dx \\
&\quad -\lambda\int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^{p-2}u_n\overline{v}dx\Big\} = o(1)\|u_n\|. \qquad (13)
\end{aligned}
$$

By (12), (13) and together with conditions (G), (h_3), we have

$$
\begin{aligned}
c + o(1)\|u_n\| &= J_\lambda(u_n) - \frac{1}{\mu}\langle J'_\lambda(u_n), u_n\rangle \\
&= \frac{1}{p}G\Big(\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\Big) - \frac{1}{\mu}g\Big(\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\Big)\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx \\
&\quad + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\lambda V(x)|u_n|^p dx + \left(\frac{1}{\mu}-\frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p^*}dx \\
&\quad + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{\mu}h(x,|u_n|^p)|u_n|^p - \frac{1}{p}H(x,|u_n|^p)\right]dx \\
&\geq \left(\frac{\theta}{p}-\frac{1}{\mu}\right)\alpha_0\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\lambda V(x)|u_n|^p dx \qquad (14)
\end{aligned}
$$

Therefore, the inequality (14) imply that $\{u_n\}$ is bounded in E_λ. Taking the limit in (14) shows that $c \geq 0$. This completes the proof of Lemma (3.3). ∎

The main result in this section is the following compactness result.

Lemma 3.4. *Suppose that (V), (A), (G) and (H) hold. For any $\lambda \geq 1$, J_λ satisfies $(PS)_c$ condition, for all $c \in \left(0, \sigma_0\lambda^{1-\frac{N}{p}}\right)$, where $\sigma_0 := \left(\frac{1}{\mu}-\frac{1}{p^*}\right)(\alpha_0 S)^{\frac{N}{p}}$, that is any $(PS)_c$-sequence $(u_n) \subset E_\lambda$ has a strongly convergent subsequence in E_λ.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence, by Lemma 3.3, $\{u_n\}$ is bounded in E_λ. Hence, by diamagnetic inequality, $\{|u_n|\}$ is bounded in $W^{1,p}(\mathbb{R}^N,\mathbb{C})$. Then, for some subsequence, there is $u \in W^{1,p}(\mathbb{R}^N,\mathbb{C})$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N,\mathbb{C})$. We claim that

$$\int_{\mathbb{R}^N}|u_n|^{p^*}dx \to \int_{\mathbb{R}^N}|u|^{p^*}dx. \qquad (15)$$

In order to prove this claim, we suppose that

$$|\nabla|u_n||^p \rightharpoonup |\nabla|u||^p + \mu \quad \text{and} \quad |u_n|^{p^*} \rightharpoonup |u|^{p^*} + \nu \quad (\text{weak}^* \text{ sense of measures}).$$

Using the concentration compactness-principle due to Lions (cf. [[34], Lemma 1.2]), we obtain a countable index set I, sequences $\{x_j\} \subset \mathbb{R}^N$, $\{\mu_j\}, \{\nu_j\} \subset (0,\infty)$ such that

$$\nu = \sum_{j\in I}\delta_{x_j}\nu_j, \quad \mu \geq \sum_{j\in I}\delta_{x_j}\mu_j \quad \text{and} \quad \mu_j \geq S\nu_j^{p/p^*} \qquad (16)$$

for all $j \in I$, where δ_{x_j} are Dirac measures at x_j and μ_j, ν_j are constants.

Now, let x_j be a singular point of the measures μ and ν. We define a function

$\phi(x) \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\phi(x) = 1$ in $B(x_j, \varepsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N \setminus B(x_j, 2\varepsilon)$ and $|\nabla\phi| \le 2/\varepsilon$ in $\mathbb{R}^N$. Since $\{u_n\phi\}$ is bounded in $W^{1,p}(\mathbb{R}^N, \mathbb{C})$ and ϕ takes values in $\mathbb{R}$, a direct calculation shows that

$$\langle J'_\lambda(u_n), u_n\phi\rangle \to 0$$

and

$$\overline{\nabla_A(u_n\phi)} = i\overline{u_n}\nabla\phi + \phi\overline{\nabla_A u_n}.$$

Therefore,

$$\begin{aligned}
&g\Big(\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\Big)\int_{\mathbb{R}^N}|\nabla_A u_n|^p\phi dx + \int_{\mathbb{R}^N}\lambda V(x)|u_n|^p\phi dx \\
&= -g\Big(\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\Big)\mathrm{Re}\left(\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla_A\phi}dx\right) \\
&\qquad + \lambda\int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^p\phi dx + \lambda\int_{\mathbb{R}^N}|u_n|^{p^*}\phi dx + o_n(1). \qquad (17)
\end{aligned}$$

On the other hand, by Hölder's inequality we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\Big|\mathrm{Re}\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla\phi}dx\Big| \\
&\le \limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\right)^{(p-1)/p}\Big(\int_{\mathbb{R}^N}|\overline{u_n\nabla_A\phi}|^p dx\Big)^{1/p} \\
&\le C_1\Big(\int_{B(x_j,2\varepsilon)}|u|^p|\nabla_A\phi|^p dx\Big)^{1/p} \qquad (18) \\
&\le C_1\Big(\int_{B(x_j,2\varepsilon)}|\nabla_A\phi|^N dx\Big)^{1/N}\Big(\int_{B(x_j,2\varepsilon)}|u|^{p^*}dx\Big)^{1/p^*} \\
&\le C_2\Big(\int_{B(x_j,2\varepsilon)}|u|^{p^*}dx\Big)^{1/p^*} \to 0 \quad \text{as } \varepsilon \to 0\,.
\end{aligned}$$

Similarly, it follows from the definition of ϕ and condition (H) that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^p\phi dx = 0. \qquad (19)$$

Since ϕ has compact support, letting $n \to \infty$ in (17) we deduce from (18) and (20) that

$$\alpha_0\int_{\mathbb{R}^N}\phi d\mu \le -\int_{\mathbb{R}^N}\lambda V(x)|u|^p\phi dx + \lambda\int_{\mathbb{R}^N}\phi d\nu.$$

Letting $\varepsilon \to 0$, we obtain $\alpha_0\mu_j \le \lambda\nu_j$. Combing this with Lemma 3.1, we obtain $\nu_j \ge \alpha_0\lambda^{-1}S\nu_j^{\frac{p}{p^*}}$. This result implies that

$$\text{(I)} \quad \nu_j = 0 \qquad \text{or} \qquad \text{(II)} \quad \nu_j \ge \left(\alpha_0\lambda^{-1}S\right)^{\frac{N}{p}}.$$

To obtain the possible concentration of mass at infinity, similarly, we define a cut off function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x) = 0$ on $|x| < R$ and $\phi_R(x) = 1$ on $|x| > R+1$. Note that $\langle J'(u_n), u_n\phi_R\rangle \to 0$, this fact imply that

$$\begin{aligned} g\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big) \int_{\mathbb{R}^N} |\nabla_A u_n|^p \phi_R dx + \int_{\mathbb{R}^N} \lambda V(x)|u_n|^p \phi_R dx \\ = -g\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big) \mathrm{Re}\left(\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n \overline{\nabla_A \phi_R} dx\right) \\ + \lambda \int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^p \phi_R dx + \lambda \int_{\mathbb{R}^N} |u_n|^{p^*}\phi_R dx + o_n(1). \end{aligned} \tag{20}$$

It is easy to prove that

$$-\lim_{R\to\infty}\lim_{n\to\infty} \mathrm{Re}\left(\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n \overline{\nabla_A \phi_R} dx\right) = 0$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^p \phi_R dx = 0.$$

Letting $R \to \infty$, we obtain $\alpha_0\mu_\infty \le \lambda\nu_\infty$. By Lemma 3.2, we obtain $\nu_\infty \ge \alpha_0\lambda^{-1}S\nu_\infty^{\frac{p}{p^*}}$. This result implies that

$$\text{(III)} \quad \nu_\infty = 0 \qquad \text{or} \qquad \text{(IV)} \quad \nu_\infty \ge \left(\alpha_0\lambda^{-1}S\right)^{\frac{N}{p}}.$$

Next, we claim that (II) and (IV) cannot occur. If the case (IV) holds, for some $j \in I$, then by using Lemma 3.2 and condition (h_3), we have that

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle\right) \\ &\ge \left(\frac{\theta}{p} - \frac{1}{\mu}\right) g\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big) \int_{\mathbb{R}^N} |\nabla_A u_n|^p dx + \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \lambda V(x)|u_n|^p dx \\ &\quad + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{\mu}h(x,|u_n|^p)|u_n|^p - \frac{1}{p}H(x,|u_n|^p)\right]dx + \left(\frac{1}{\mu} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p^*}dx \\ &\ge \left(\frac{1}{\mu} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p^*}dx \ge \left(\frac{1}{\mu} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p^*}\phi_R dx \\ &= \left(\frac{1}{\mu} - \frac{1}{p^*}\right)\lambda\nu_\infty \ge \sigma_0\lambda^{1-\frac{N}{p}}, \end{aligned}$$

where $\sigma_0 = \left(\frac{1}{\mu} - \frac{1}{p^*}\right)(\alpha_0 S)^{\frac{N}{p}}$. This is impossible. Consequently, $\nu_j = 0$ for all $j \in I$. Similarly, if the case (II) holds, for some $j \in I$, then by condition (H), we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle\right) \\ &\ge \left(\frac{1}{\mu} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p^*}dx \ge \left(\frac{1}{\mu} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p^*}\phi dx \\ &= \left(\frac{1}{\mu} - \frac{1}{p^*}\right)\lambda\nu \ge \alpha_0\lambda^{1-\frac{N}{p}} \quad \text{as} \quad \varepsilon \to 0, \end{aligned}$$

which leads to a contradiction. Thus, we must have (II) cannot occur for each j. Thus

$$\int_{\mathbb{R}^N} |u_n|^{p^*} dx \to \int_{\mathbb{R}^N} |u|^{p^*} dx. \tag{21}$$

In order to prove $u_n \to u$ in E_λ, we adapt some arguments in [9] and [24]. Define functions

$$\tau_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

Fix a compact set $K \subset \mathbb{R}^N$ and take a cut-off function $\phi_K : \mathbb{R}^N \to \mathbb{R}$ satisfying:

$$\phi_K \in C_0^\infty(\mathbb{R}^N), \quad 0 \le \phi_K \le 1 \quad \text{and} \quad \phi_K(x) = 1 \quad \text{on} \quad K.$$

Then $\phi_K \tau_k(u_n - u) \in W^{1,p}(\mathbb{R}^N)$. Since $\{u_n\}$ is a $(PS)_c$ sequence, from the weak lower semicontinuity of the norm and (G), we have

$$\begin{aligned}
o(1) &= \langle J'_\lambda(u_n) - J'_\lambda(u), \phi_K \tau_k(u_n - u) \rangle \\
&= g\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big) \int_{\mathbb{R}^N} |\nabla_A u_n|^{p-2} \nabla u_n \overline{\nabla_A \left[\phi_K \tau_k(u_n - u)\right]} dx \\
&\quad - g\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big) \int_{\mathbb{R}^N} |\nabla_A u|^{p-2} \nabla_A u \overline{\nabla_A \left[\phi_K \tau_k(u_n - u)\right]} dx \\
&\quad + \int_{\mathbb{R}^N} \lambda V(x) \left(|u_n|^{p-2} u_n - |u|^{p-2} u\right) \overline{\phi_K \tau_k(u_n - u)} dx - I_1 - I_2 \\
&\ge \alpha_0 \int_{\mathbb{R}^N} \left(|\nabla_A u_n|^{p-2} \nabla_A u_n - |\nabla_A u|^{p-2} \nabla_A u\right) \overline{\nabla_A \left[\phi_K \tau_k(u_n - u)\right]} dx \\
&\quad + \int_{\mathbb{R}^N} \lambda V(x) \left(|u_n|^{p-2} u_n - |u|^{p-2} u\right) \overline{\phi_K \tau_k(u_n - u)} dx - I_1 - I_2,
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
I_1 &= \lambda \int_{\mathbb{R}^N} \left(h(x, |u_n|^p) |u_n|^{p-2} u_n - h(x, |u|^p) |u|^{p-2} u\right) \overline{\phi_K \tau_k(u_n - u)} dx, \\
I_2 &= \lambda \int_{\mathbb{R}^N} \left(|u_n|^{p^*-2} u_n - |u|^{p^*-2} u\right) \overline{\phi_K \tau_k(u_n - u)} dx.
\end{aligned}$$

By (21), we have $|I_1| \le ck$ and $|I_2| \le ck$. Thus, by (22), we conclude that $|e_n| \le ck$, where

$$e_n := \langle |\nabla_A u_n|^{p-2} \nabla_A u_n - |\nabla_A u|^{p-2} \nabla_A u, \phi_K \tau_k(u_n - u) \rangle.$$

It follows that $e_n \ge 0$ by the following well-known inequality. (See also Ghoussoub and Yuan [[24], Lemma 4.1])

$$\Big\langle |s|^{p-2} s - |t|^{p-2} t, s - t \Big\rangle \ge \begin{cases} C_p |s - t|^p, & \forall\ p \ge 2, \\ C_p \frac{|s-t|^2}{(|s|+|t|)^{2-p}}, & \forall\, p \le 2, \end{cases} \quad s, t \in \mathbb{R}^N, \tag{23}$$

where $\langle \cdot\,, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^N$.
Take $0 < \theta < 1$ and split K into

$$S_n^k = \{x \in K : \ |u_n - u| \le k\}, \quad G_n^k = \{x \in K : \ |u_n - u| \ge k\}.$$

Then

$$\int_{\mathbb{R}^N} e_n^\theta dx = \int_{S_n^k} e_n^\theta dx + \int_{G_n^k} e_n^\theta dx$$
$$\leq \left(\int_{S_n^k} e_n^\theta dx\right)^\theta |S_n^\theta|^{1-\theta} + \left(\int_{G_n^k} e_n^\theta dx\right)^\theta |G_n^\theta|^{1-\theta}.$$

Now, fixed k, then $|G_n^k| \to 0$ as $n \to \infty$ and from the uniform boundedness in L^1 we get

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} e_n^\theta dx \leq (Ck)^\theta |K|^{1-\theta}.$$

Let $k \to 0$ we get that $e_n^\theta \to 0$ in L^1. Finally, from the well-known inequality (23), we have $\nabla_A u_n \to \nabla_A u$ in $L^\tau(\mathbb{R}^N)$ for $1 < \tau \leq p$. By passing to a subsequence, we have $\nabla_A u_n \to \nabla_A u$ in $\mathbb{R}^N$. This fact together with (22) and (23) imply that $\nabla_A u_n \to \nabla_A u$ in E_λ. This completes the proof of Lemma 3.4. ∎

4 Proofs of Theorem 2.1

In the following, we always consider $\lambda \geq 1$. By the assumptions (V), (A), (G) and (H), one can see that $J_\lambda(u)$ has mountain pass geometry.

Lemma 4.1. *Assume* (V), (A), (G) *and* (H) *hold. There exist* $\alpha_\lambda, \rho_\lambda > 0$ *such that* $J_\lambda(u) > 0$ *if* $u \in B_{\rho_\lambda} \setminus \{0\}$ *and* $J_\lambda(u) \geq \alpha_\lambda$ *if* $u \in \partial B_{\rho_\lambda}$, *where* $B_{\rho_\lambda} = \{u \in E_\lambda : \|u\|_\lambda \leq \rho_\lambda\}$.

Proof. By (h_1) - (h_3), for $\delta \leq \left(2\min\left\{\frac{\theta\alpha_0}{p}, \frac{1}{p}\right\}\lambda c_p^p\right)^{-1}$ there is $C_\delta > 0$ such that

$$\frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx + \frac{1}{p}\int_{\mathbb{R}^N} H(x, |u|^p) dx \leq \delta |u|_p^p + C_\delta |u|_{p^*}^{p^*}.$$

Therefore, from condition (G) it follows that

$$\begin{aligned} J_\lambda(u) &:= \frac{1}{p} G\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big) + \frac{1}{p}\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx - \frac{\lambda}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx \\ &\qquad - \frac{\lambda}{p}\int_{\mathbb{R}^N} H(x, |u|^p) dx \\ &\geq \min\left\{\frac{\theta\alpha_0}{p}, \frac{1}{p}\right\} \|u\|_\lambda^p - \lambda\delta |u|_p^p - \lambda C_\delta |u|_{p^*}^{p^*} \\ &\geq \frac{1}{2}\min\left\{\frac{\theta\alpha_0}{p}, \frac{1}{p}\right\} \|u\|_\lambda^p - \lambda C_\delta c_{p^*}^{p^*} \|u\|_\lambda^{p^*}. \end{aligned}$$

Since $p^* > p$, we know that the conclusion of Lemma 4.1 holds. This completes the proof of Lemma 4.1. ∎

Lemma 4.2. *Under the assumption of Lemma 4.1, for any finite dimensional subspace* $F \subset E_\lambda$,

$$J_\lambda(u) \to -\infty \quad \textit{as} \quad u \in F,\ \|u\|_\lambda \to \infty.$$

Proof. On the one hand, by integrating (g_2), we obtain

$$G(t) \leq \frac{G(t_0)}{t_0^{1/\theta}} t^{1/\theta} = C_0 t^{1/\theta} \quad \text{for all } t \geq t_0 > 0. \tag{24}$$

Using conditions (V) and (H), we can get

$$J_\lambda(u) \leq \frac{C_0}{p} \|u\|_\lambda^{\frac{p}{\theta}} + \frac{1}{p}\|u\|_\lambda^p - \frac{\lambda}{p^*}|u|_{p^*}^{p^*} - \lambda l_0 |u|_s^s$$

for all $u \in F$. Since all norms in a finite-dimensional space are equivalent and $\frac{p}{\theta} < p^*$, $p < p^*$. This completes the proof of Lemma 4.2. ∎

Since $J_\lambda(u)$ does not satisfy the $(PS)_c$ condition for all $c > 0$. Thus, in the following we will find a special finite-dimensional subspaces by which we construct sufficiently small minimax levels.

Recall that assumption (V) implies that there is $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x \in \mathbb{R}^N} V(x) = 0$. Without loss of generality we assume from now on that $x_0 = 0$.

Observe that, by (h_3) we have

$$\frac{\lambda}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx + \lambda \int_{\mathbb{R}^N} H(x, |u|^p) dx \geq l_0 \lambda \int_{\mathbb{R}^N} |u|^s dx.$$

Definite the function $I_\lambda \in C^1(E_\lambda, \mathbb{R})$ by

$$I_\lambda(u) := \frac{1}{p} G\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big) + \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx - l_0 \lambda \int_{\mathbb{R}^N} |u|^s dx.$$

Then $J_\lambda(u) \leq I_\lambda(u)$ for all $u \in E_\lambda$ and it suffices to construct small minimax levels for I_λ.

Note that

$$\inf\left\{\int_{\mathbb{R}^N} |\nabla\phi|^p dx : \phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}), |\phi|_p = 1\right\} = 0.$$

For any $1 > \delta > 0$ one can choose $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ with $|\phi_\delta|_p = 1$ and $\operatorname{supp} \phi_\delta \subset B_{r_\delta}(0)$ so that $|\nabla \phi_\delta|_p^p < \delta$. Set

$$f_\lambda = \phi_\delta(\lambda^{\frac{1}{p}} x), \tag{25}$$

then

$$\operatorname{supp} f_\lambda \subset B_{\lambda^{-\frac{1}{p}} r_\delta}(0).$$

Thus, for $t \geq 0$,

$$
\begin{aligned}
I_\lambda(tf_\lambda) &\leq \frac{C_0}{p} t^{\frac{p}{\theta}} \left(\int_{\mathbb{R}^N} |\nabla_A f_\lambda|^p dx \right)^{1/\theta} + \frac{t^p}{p} \int_{\mathbb{R}^N} \lambda V(x) |f_\lambda|^p dx - t^s l_0 \lambda \int_{\mathbb{R}^N} |f_\lambda|^s dx \\
&= \lambda^{1-\frac{N}{p}} \left[\frac{C_0}{p} t^{\frac{p}{\theta}} \left(\lambda^{1-\frac{N}{p}} \right)^{\frac{1}{\theta}-1} \left(\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx \right)^{1/\theta} \right. \\
&\quad \left. + \frac{t^p}{p} \int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{p}} x\right) |\phi_\delta|^p dx - t^s l_0 \int_{\mathbb{R}^N} |\phi_\delta|^s dx \right] \\
&\leq \lambda^{1-\frac{N}{p}} \left[\frac{C_0}{p} t^{\frac{p}{\theta}} \left(\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx \right)^{1/\theta} + \frac{t^p}{p} \int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{p}} x\right) |\phi_\delta|^p dx \right. \\
&\quad \left. - t^s l_0 \int_{\mathbb{R}^N} |\phi_\delta|^s dx \right] \\
&= \lambda^{1-\frac{N}{p}} \Psi_\lambda(t\phi_\delta),
\end{aligned}
$$

where $\Psi_\lambda \in C^1(E_\lambda, \mathbb{R})$ defined by

$$
\Psi_\lambda(u) := \frac{C_0}{p} \left(\int_{\mathbb{R}^N} |\nabla_A u|^p dx \right)^{1/\theta} + \frac{1}{p} \int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{p}} x\right) |u|^p dx - l_0 \int_{\mathbb{R}^N} |u|^s dx.
$$

Since $s > \frac{p}{\theta}$, thus there exists finite number $t_0 \in [0, +\infty)$ such that

$$
\begin{aligned}
\max_{t \geq 0} \Psi_\lambda(t\phi_\delta) &= \frac{C_0}{p} t_0^{\frac{p}{\theta}} \left(\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx \right)^{1/\theta} + \frac{t_0^p}{p} \int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{p}} x\right) |\phi_\delta|^p dx \\
&\quad - t_0^s l_0 \int_{\mathbb{R}^N} |\phi_\delta|^s dx \\
&\leq \frac{C_0}{p} t_0^{\frac{p}{\theta}} \left(\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx \right)^{1/\theta} + \frac{t_0^p}{p} \int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{p}} x\right) |\phi_\delta|^p dx.
\end{aligned}
$$

On the one hand, since $V(0) = 0$ and note that $\sup \phi_\delta \subset B_{r_\delta}(0)$, there is $\Lambda_\delta > 0$ such that

$$
V\left(\lambda^{-\frac{1}{p}} x\right) \leq \frac{\delta}{|\phi_\delta|_p^p} \text{ for all } |x| \leq r_\delta \text{ and } \lambda \geq \Lambda_\delta.
$$

This implies that

$$
\max_{t \geq 0} \Psi_\lambda(t\phi_\delta) \leq \frac{C_0}{p} t_0^{\frac{p}{\sigma}} \delta^{1/\sigma} + \frac{t_0^p}{p} \delta \leq T^* \delta. \tag{26}
$$

where $T^* := \left(\frac{C_0}{p} t_0^{\frac{p}{\sigma}} + \frac{t_0^p}{p} \right)$. Therefore, for all $\lambda \geq \Lambda_\delta$,

$$
\max_{t \geq 0} J_\lambda(t\phi_\delta) \leq T^* \delta \lambda^{1-\frac{N}{p}}. \tag{27}
$$

Thus we have the following lemma.

Lemma 4.3. *Under the assumption of Lemma 4.1, for any $\kappa > 0$ there exists $\Lambda_\kappa > 0$ such that for each $\lambda \geq \Lambda_\kappa$, there is $\widehat{f_\lambda} \in E_\lambda$ with $\|\widehat{f_\lambda}\| > \rho_\lambda$, $J_\lambda(\widehat{f_\lambda}) \leq 0$ and*

$$\max_{t\in[0,1]} J_\lambda(t\widehat{f_\lambda}) \leq \kappa\lambda^{1-\frac{N}{p}}. \tag{28}$$

Proof. Choose $\delta > 0$ so small that $T^*\delta \leq \kappa$. Let $f_\lambda \in E_\lambda$ be the function defined by (25). Taking $\Lambda_\kappa = \Lambda_\delta$. Let $\widehat{t_\lambda} > 0$ be such that $\widehat{t_\lambda}\|f_\lambda\|_\lambda > \rho_\lambda$ and $J_\lambda(tf_\lambda) \leq 0$ for all $t \geq \widehat{t_\lambda}$. By (27), let $\widehat{f_\lambda} = \widehat{t_\lambda} f_\lambda$ we know that the conclusion of Lemma 4.3 holds. ■

For any $m^* \in \mathbb{N}$, one can choose m^* functions $\phi_\delta^i \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\delta^i \cap \operatorname{supp}\phi_\delta^k = \emptyset$, $i \neq k$, $|\phi_\delta^i|_s = 1$ and $|\nabla\phi_\delta^i|_p^p < \delta$. Let $r_\delta^{m^*} > 0$ be such that $\operatorname{supp}\phi_\delta^i \subset B_{r_\delta}^i(0)$ for $i = 1, 2, \cdots, m^*$. Set

$$f_\lambda^i(x) = \phi_\delta^i(\lambda^{\frac{1}{p}}x), \ \text{ for } i = 1, 2, \cdots, m^* \tag{29}$$

and

$$H_{\lambda\delta}^{m^*} = \operatorname{span}\{f_\lambda^1, f_\lambda^2, \cdots, f_\lambda^{m^*}\}.$$

Observe that for each $u = \sum_{i=1}^{m^*} c_i f_\lambda^i \in H_{\lambda\delta}^{m^*}$,

$$\int_{\mathbb{R}^N} |\nabla_A u|^p dx = \sum_{i=1}^{m^*} |c_i|^p \int_{\mathbb{R}^N} |\nabla_A f_\lambda^i|^p dx,$$

$$\int_{\mathbb{R}^N} V(x)|u|^p dx = \sum_{i=1}^{m^*} |c_i|^p \int_{\mathbb{R}^N} V(x)|f_\lambda^i|^p dx,$$

$$\frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx = \frac{1}{p^*}\sum_{i=1}^{m^*} |c_i|^{p^*} \int_{\mathbb{R}^N} |f_\lambda^i|^{p^*} dx$$

and

$$\int_{\mathbb{R}^N} H(x, |u|^p) dx = \sum_{i=1}^{m^*} \int_{\mathbb{R}^N} H(x, c_i f_\lambda^i) dx.$$

Therefore

$$J_\lambda(u) = \sum_{i=1}^{m^*} J_\lambda(c_i f_\lambda^i)$$

and as before

$$J_\lambda(c_i f_\lambda^i) \leq \lambda^{1-\frac{N}{p}}\Psi(|c_i| f_\lambda^i).$$

Set

$$\beta_\delta := \max\{|\phi_\delta^i|_p^p : j = 1, 2, \cdots, m^*\}$$

and choose $\Lambda_{m^*\delta} > 0$ so that

$$V(\lambda^{-\frac{1}{p}}x) \leq \frac{\delta}{\beta_\delta} \text{ for all } |x| \leq r_\delta^{m^*} \text{ and } \lambda \geq \Lambda_{m^*\delta}.$$

As before, we can obtain the following

$$\max_{u \in H^{m^*}_{\lambda\delta}} J_\lambda(u) \le m^* T^* \delta \lambda^{1-\frac{N}{p}} \tag{30}$$

for all $\lambda \ge \Lambda_{m^*\delta}$.

Using this estimate we have the following.

Lemma 4.4. *Under the assumptions of Lemma 4.1, for any $m^* \in \mathbb{N}$ and $\kappa > 0$ there exists $\Lambda_{m^*\kappa} > 0$ such that for each $\lambda \ge \Lambda_{m^*\kappa}$, there exists an m^*-dimensional subspace $F_{\lambda m^*}$ satisfying*

$$\max_{u \in F_{\lambda m^*}} J_\lambda(u) \le \kappa \lambda^{1-\frac{N}{p}}.$$

Proof. Choose $\delta > 0$ so small that $m^* T^* \delta \le \kappa$. Taking $F_{\lambda m^*} = H^{m^*}_{\lambda\delta} = \operatorname{span}\{f^1_\lambda, f^2_\lambda, \cdots, f^{m^*}_\lambda\}$, where $f^i_\lambda(x) = \phi^i_\delta(\lambda^{\frac{1}{p}} x)$, for $i = 1, 2, \cdots, m^*$ are given by (29). From (30), we know that the conclusion of Lemma 4.4 holds. ∎

We now establish the existence and multiplicity results.

Proof of Theorem 2.1. Using Lemma 4.3, we choose $\Lambda_\sigma > 0$ and define for $\lambda \ge \Lambda_\sigma$, the minimax value

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(t\widehat{f}_\lambda)$$

where

$$\Gamma_\lambda := \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0 \text{ and } \gamma(1) = \widehat{f}_\lambda\}.$$

By Lemma 4.1, we have $\alpha_\lambda \le c_\lambda \le \sigma_0 \lambda^{1-\frac{N}{p}}$. In virtue of Lemma 3.4, we know that J_λ satisfies the $(PS)_{c_\lambda}$ condition, there is $u_\lambda \in E_\lambda$ such that $J'_\lambda(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda$. Then u_λ is a solution of (8). Moreover, it is well known that such a Mountain-Pass solution is a least energy solution of (8).

Such u_λ is a critical point of J_λ, for $\tau \in \left[\frac{p}{\theta}, p^*\right]$,

$$\begin{aligned}
\sigma\lambda^{1-\frac{N}{p}} \ &\ge\ J_\lambda(u_\lambda) = J_\lambda(u_\lambda) - \frac{1}{\tau} J'_\lambda(u_\lambda) u_\lambda \\
&=\ \frac{1}{p} G\Big(\int_{\mathbb{R}^N} |\nabla_A u_\lambda|^p dx\Big) - \frac{1}{\tau} g\Big(\int_{\mathbb{R}^N} |\nabla_A u_\lambda|^p dx\Big) \int_{\mathbb{R}^N} |\nabla_A u_\lambda|^p dx \\
&\quad + \left(\frac{1}{p} - \frac{1}{\tau}\right) \int_{\mathbb{R}^N} \lambda V(x) |u_\lambda|^p dx + \left(\frac{1}{\tau} - \frac{1}{p^*}\right) \lambda \int_{\mathbb{R}^N} |u_\lambda|^{p^*} dx \\
&\quad + \lambda \int_{\mathbb{R}^N} \left[\frac{1}{\tau} h(x, |u_\lambda|^p) |u_\lambda|^p - \frac{1}{p} H(x, |u_\lambda|^p)\right] dx \\
&\ge\ \left(\frac{\theta}{p} - \frac{1}{\tau}\right) \alpha_0 \int_{\mathbb{R}^N} |\nabla_A u_\lambda|^p dx + \left(\frac{1}{p} - \frac{1}{\tau}\right) \int_{\mathbb{R}^N} \lambda V(x) |u_\lambda|^p dx \\
&\quad + \left(\frac{1}{\tau} - \frac{1}{p^*}\right) \lambda \int_{\mathbb{R}^N} |u_\lambda|^{p^*} dx + \left(\frac{\mu}{\tau} - \frac{1}{p}\right) \lambda \int_{\mathbb{R}^N} H(x, |u_\lambda|^p) dx,
\end{aligned} \tag{31}$$

where μ is the constant in (H). Taking $\tau = \frac{p}{\theta}$ yields the estimate (9), and taking $\tau = \mu$ gives the estimate (10) hence the existence is proved.

Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ, for each $Z \in \Sigma$. Let $\mathrm{gen}(Z)$ be the Krasnoselski genus and

$$i(Z) := \min_{h \in \Gamma_{m^*}} \mathrm{gen}(h(Z) \cap \partial B_{\rho_\lambda}),$$

where Γ_{m^*} is the set of all odd homeomorphisms $h \in C(E_\lambda, E_\lambda)$ and ρ_λ is the number from Lemma 4.1. Then i is a version of Benci's pseudoindex [8]. Let

$$c_{\lambda i} := \inf_{i(Z) \geq i} \sup_{u \in Z} J_\lambda(u), \quad 1 \leq i \leq m^*.$$

Since $J_\lambda(u) \geq \alpha_\lambda$ for all $u \in \partial B^+_{\rho\lambda}$ and since $i(F_{\lambda m^*}) = \dim F_{\lambda m^*} = m^*$,

$$\alpha_\lambda \leq c_{\lambda 1} \leq \cdots \leq c_{\lambda m^*} \leq \sup_{u \in H_{\lambda m^*}} J_\lambda(u) \leq \sigma \lambda^{1-\frac{N}{p}}.$$

It follows from Lemma 3.4 that J_λ satisfies the $(PS)_{c_\lambda}$ condition at all levels c_i. By the usual critical point theory, all c_i are critical levels and J_λ has at least m^* pairs of nontrivial critical points. ∎

5 A special case of problem (1)

We consider the following the special case of problem (1):

$$\begin{cases} -\varepsilon^p \left(a + b \int_{\mathbb{R}^N} |\nabla_A u|^p dx\right) \Delta_p u + V(x)|u|^{p-2}u = |u|^{p^*-2}u + h(x,u), \ x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{32}$$

where $1 < p < \frac{\mu}{2}$, a and b are positive constants.

Set $g(t) = a + bt$. Then, $g(t) \geq a$ and

$$G(t) = \int_0^1 g(s)ds = at + \frac{1}{2}bt^2 \geq \frac{1}{2}(a+bt)t = \sigma g(t)t,$$

where $\sigma = 1/2$. Hence the conditions (g_1) and (g_2) are satisfied. In view of Theorem 1.1, we have the following corollary.

Corollary 5.1. *Let* (V), (A) *and* (H) *be satisfied. Thus*

(I) *For any* $\kappa > 0$ *there is* $\mathcal{E}_\kappa > 0$ *such that if* $\varepsilon \leq \mathcal{E}_\kappa$ *problem* (32) *has at least one solution* u_ε *satisfying*

$$\frac{\theta\mu - 1}{p} \int_{\mathbb{R}^N} H(x, |u_\varepsilon|^p)dx + \left(\frac{\theta}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |u_\varepsilon|^{p^*} dx \leq \kappa \varepsilon^N, \tag{33}$$

$$\left(\frac{\theta}{p} - \frac{1}{\mu}\right) \alpha_0 \int_{\mathbb{R}^N} |\nabla_A u_\varepsilon|^p dx + \left(\frac{1}{p} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \lambda V(x)|u_\varepsilon|^p dx \leq \kappa \varepsilon^N. \tag{34}$$

Moreover, $u_\varepsilon \to 0$ *in* $W^{1,p}(\mathbb{R}^N)$ *as* $\varepsilon \to 0$.

(II) *Assume additionally that* $h(x,t)$ *is odd in* t, *for any* $m \in N$ *and* $\kappa > 0$ *there is* $\mathcal{E}_{m\kappa} > 0$ *such that if* $\varepsilon \leq \mathcal{E}_{m\kappa}$, *problem* (1) *has at least* m *pairs of solutions* $u_{\varepsilon,i}$, $u_{\varepsilon,-i}$, $i = 1, 2, \cdots, m$ *which satisfy the estimates* (33) *and* (34). *Moreover,* $u_{\varepsilon,i} \to 0$ *in* $W^{1,p}(\mathbb{R}^N)$ *as* $\varepsilon \to 0$, $i = 1, 2, \cdots, m$.

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