Optimal extension of the Cesàro operator in $L^p([0,1])$

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Abstract

The Cesàro operator $C_p : L^p([0,1]) \to L^p([0,1])$, a classical kernel operator, induces the vector measure $m_p : A \mapsto C_p(\chi_A)$ which generates a factorization of C_p through $L^1(m_p)$ via the integration map $I_{m_p} : f \mapsto \int_0^1 f \, dm_p$, for $f \in L^1(m_p)$. This provides a technique to investigate various operator theoretic properties of C_p. Even though the variation measure $|m_p|$ of m_p is finite it turns out, atypically for a kernel operator, that the restriction of I_{m_p} to $L^1(|m_p|) \subseteq L^1(m_p)$ is not an extension of C_p, that is, C_p fails to factorize through the more traditional space $L^1(|m_p|)$.

1 Introduction and main results.

Let L be a Banach function space (briefly, B.f.s.) over some positive measure space (Ω, Σ, μ), X be a Banach space and $T : L \to X$ be a continuous linear operator. Under favorable circumstances the set function $m_T : A \mapsto T(\chi_A)$, provided that each $\chi_A \in L$, is a σ-additive, X-valued measure (i.e. a vector measure) and so the associated B.f.s. $L^1(m_T)$ consisting of all the m_T-integrable functions is available together with the X-valued integration map $I_{m_T} : f \mapsto \int_\Omega f \, dm_T$, for $f \in L^1(m_T)$. For a simple function s it is clear that $I_{m_T}(s) = T(s)$ and so, if $L \subseteq L^1(m_T)$ continuously, then I_{m_T} is a continuous X-valued extension of T. That is, one can factorize T as $T = I_{m_T} \circ J$, where $J : L \to L^1(m_T)$ is the natural inclusion. This particular factorization has the desirable feature that I_{m_T} is the *optimal*

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extension of T within the class of all B.f.s.' on (Ω, Σ, μ) which have order continuous norm and contain L continuously; see [13]. Typical operators T which are susceptible to this approach are those generated by a measurable kernel (usually with certain properties); see, for instance, [2], [3], [4], [5], [6], [13], [14] and the references therein. For the variation measure $|m_T| : \Sigma \to [0, \infty]$, [7, p.2], of m_T we always have that $L^1(|m_T|) \subseteq L^1(m_T)$ continuously, [10, Theorem 4.1]. If $|m_T|$ is a finite measure, then it is usually the case that also $L \subseteq L^1(|m_T|)$. Then the restriction of I_{m_T} to the intermediatory space $L^1(|m_T|)$ provides another factorization of T. An advantage of this factorization, if available, is that $L^1(|m_T|)$ is a classical B.f.s., which may be preferable to the less wieldy space $L^1(m_T)$.

The aim of this note is to invoke this approach to investigate a classical kernel operator of a different type; it is generated by a kernel that does not satisfy the usual properties required in the above cited literature. More precisely, let λ be Lebesgue measure on the σ-algebra $\mathcal{B}$ of all Borel subsets of $\Omega := [0, 1]$. For each $1 \le p < \infty$ let L^p denote the Banach space of all p-th power integrable functions f on Ω equipped with the norm $\|f\|_p := (\int_\Omega |f|^p \, d\lambda)^{1/p}$. Of course, L^∞ has its essential sup-norm $\|\cdot\|_\infty$. A well known result of G.H. Hardy ensures that the kernel operator

$$C_p f : x \mapsto \frac{1}{x} \int_0^x f(y) \, dy, \qquad x \in (0, 1], \tag{1.1}$$

is continuous from L^p into itself, [8, p.240]. The operator $C_p : L^p \to L^p$ is traditionally called the *Cesàro operator*. In order to formulate the main results concerning C_p we require a few preliminaries.

Since L^p has order continuous norm (i.e., whenever a sequence $\{f_n\}_{n=1}^\infty \subseteq L^p$ decreases to 0 a.e. in $[0, 1]$, then $\|f_n\|_p \to 0$ for $n \to \infty$), the finitely additive set function $m_p : \mathcal{B} \to L^p$ defined by

$$m_p(A) := C_p \chi_A, \qquad A \in \mathcal{B}, \tag{1.2}$$

is actually σ-additive, for each $1 < p < \infty$. Clearly $m_p(A) \in L^\infty$, for each $A \in \mathcal{B}$; see (1.1) and (1.2). Concerning the B.f.s. $L^1(m_p)$, [13, Ch. 3], recall that a measurable function $f : \Omega \to \mathbb{C}$ belongs to $L^1(m_p)$ precisely when:

(I-1) $\int_\Omega |f| \, d|\langle m_p, \xi \rangle| < \infty$ for every $\xi \in L^{p'}$, and

(I-2) for each $A \in \mathcal{B}$, there exists an element of L^p, denoted by $\int_A f \, dm_p$, satisfying $\langle \int_A f \, dm_p, \xi \rangle = \int_A f \, d\langle m_p, \xi \rangle$ for all $\xi \in L^{p'}$.

Here $\frac{1}{p} + \frac{1}{p'} = 1$ and $\langle m_p, \xi \rangle$ is the $\mathbb{C}$-valued measure defined on $\mathcal{B}$ via $A \mapsto \langle m_p(A), \xi \rangle$. Since L^p is reflexive, condition (I-2) actually follows from (I-1), [10, Theorem 5.1]. The (Banach lattice) norm in $L^1(m_p)$ is given by

$$\|f\|_{L^1(m_p)} := \sup_{\|\xi\|_{p'} \le 1} \int_\Omega |f| \, d|\langle m_p, \xi \rangle|, \qquad f \in L^1(m_p). \tag{1.3}$$

Theorem 1.1. *Let* $1 < p < \infty$ *and* $m_p : \mathcal{B} \to L^p$ *be given by* (1.2).

(i) *The vector measure* m_p *has the same null sets as* λ. *Furthermore,* m_p *has finite variation given by*

$$|m_p|(A) = (p-1)^{-1/p} \int_A (y^{1-p} - 1)^{1/p} \, dy, \qquad A \in \mathcal{B}, \tag{1.4}$$

and the natural inclusion

$$L^1(|m_p|) \subsetneqq L^1(m_p) \tag{1.5}$$

is continuous and proper. In addition, the range $m_p(\mathcal{B})$ of m_p is a relatively compact subset of L^p.

(ii) *The Banach space $L^p \not\subseteq L^1(|m_p|)$ and so also $L^1 \not\subseteq L^1(|m_p|)$. In particular, the restricted integration map $I_{|m_p|} : L^1(|m_p|) \to L^p$, necessarily continuous, is **not** an extension of $C_p : L^p \to L^p$.*

(iii) *The Banach space $L^1(|m_p|) \not\subseteq L^1$. Hence, L^1 and $L^1(|m_p|)$ are not comparable.*

(iv) *The inclusion $L^p \subsetneqq L^1(m_p)$ is proper and $I_{m_p} f \in L^p$ is the function*

$$I_{m_p} f : x \mapsto \frac{1}{x} \int_0^x f(y)\, dy, \qquad x \in (0,1], \tag{1.6}$$

for each $f \in L^1(m_p)$.

(v) *Neither of the two integration operators $I_{|m_p|} : L^1(|m_p|) \to L^p$ or $I_{m_p} : L^1(m_p) \to L^p$ is compact.*

As mentioned earlier, that $L^1(|m_p|)$ is *not* intermediatory between L^p and $L^1(m_p)$ is unexpected. In this regard we note that $L^1(|m_p|)$ is equipped with its classical L^1-norm $\int_\Omega |f|\, d|m_p|$ and not the norm inherited from $L^1(m_p)$.

Let L^0 denote the space of (equivalence classes λ-a.e. of) all measurable functions on Ω. The kernel $K : (0,1] \times [0,1] \to [0,\infty)$ which induces the Cesàro operator (1.1), namely

$$K(x,y) := \frac{1}{x}\, \chi_{[0,x]}(y) = \frac{1}{x}\, \chi_{[y,1]}(x), \tag{1.7}$$

satisfies both of the conditions (K1) and (K2) in Section 2 of [5]. Define the *optimal lattice domain* of C_p as in [5] by

$$[C_p, L^p] := \{f \in L^0 : C_p|f| \in L^p\}, \tag{1.8}$$

where $C_p|f| \in L^p$ means that $\int_\Omega K(x,y)|f(y)|\, dy < \infty$ for a.e. $x \in \Omega$ and the resulting function $C_p|f| : x \mapsto \int_\Omega K(x,y)|f(y)|\, dy$ belongs to L^p. Then the linear space $[C_p, L^p]$, equipped with the norm

$$\|f\|_{[C_p,L^p]} := \|C_p|f|\,\|_p, \qquad f \in [C_p, L^p], \tag{1.9}$$

is a B.f.s. (over $(\Omega, \mathcal{B}, \lambda)$) containing L^∞ and the so defined linear map $C_p : [C_p, L^p] \to L^p$ is a *positive operator* between Banach lattices with operator norm 1, [5, Proposition 2.1]. Clearly, $L^p \subseteq [C_p, L^p]$ continuously. What is the connection between $[C_p, L^p]$ and $L^1(m_p)$?

Theorem 1.2. *Let $1 < p < \infty$. Then $[C_p, L^p] = L^1(m_p)$ with*

$$\|f\|_{L^1(m_p)} = \|f\|_{[C_p,L^p]} = \|C_p|f|\,\|_p, \qquad f \in L^1(m_p). \tag{1.10}$$

Moreover,

$$\int_\Omega f\, dm_p = I_{m_p} f = C_p f, \qquad f \in L^1(m_p). \tag{1.11}$$

It is important to note that Theorem 1.2 does *not* follow from Proposition 5.2 of [2]. Indeed, in order to be able to apply that result it is required that the range $m_p(\mathcal{B})$ of m_p lies in the Banach space $C(\Omega)$. Unfortunately, this is *not* the case; see Proposition 1.3 below. The results of Section 3 in [5] are also *not* applicable to m_p. Indeed, for each $0 < b < 1$, direct calculation shows that $m_p((0, b])$ is the function

$$x \mapsto \chi_{(0,b]}(x) + \frac{b}{x}\chi_{[b,1]}(x), \qquad x \in (0,1], \tag{1.12}$$

and hence, $\|m_p((0,b])\|_\infty = 1$. Accordingly, $m_p : \mathcal{B} \to L^\infty$ is *not* σ-additive, thereby violating a necessary condition assumed throughout Section 3 of [5].

Proposition 1.3. *Let* $1 < p < \infty$.

(i) *For each* $f \in L^1$, *the function* $C_p f : x \mapsto \frac{1}{x}\int_0^x f(y)\,dy$ *is continuous on* $(0,1]$.

(ii) *There exists a set* $A \in \mathcal{B}$ *such that the continuous function* $C_p\chi_A = m_p(A) : (0,1] \to [0,\infty)$ *fails to be right-continuous at* 0, *i.e.,* $m_p(A) \notin C(\Omega)$.

2 Proofs and further results

Throughout this section we assume that $1 < p < \infty$. Define the function $F_p : [0,1] \to L^p$ by $F_p(y) := K_y$, where K_y is the partial function $x \mapsto K(x,y)$, for $x \in (0,1]$, and K is the kernel (1.7). That F_p really is L^p-valued is clear from

$$\|F_p(y)\|_p = \Big(\int_0^1 x^{-p}\chi_{[y,1]}(x)\,dx\Big)^{1/p} = (p-1)^{-1/p}(y^{1-p}-1)^{1/p}, \tag{2.1}$$

for $y \in (0,1]$. Whenever $0 < t < s \le 1$ we have $F_p(t) - F_p(s)$ is the function $x \mapsto \frac{1}{x}\chi_{[t,s]}(x)$, for $x \in (0,1]$, from which it follows that

$$\|F_p(t) - F_p(s)\|_p = (p-1)^{-1/p}(t^{1-p} - s^{1-p})^{1/p}. \tag{2.2}$$

Since the function $y \mapsto y^{1-p}$ is continuous on $(0,1]$, we can deduce from (2.2) that $F_p : (0,1] \to L^p$ in continuous. Then the separability of L^p and the Pettis measurability theorem, [7, p.42], ensure that F_p is strongly λ-measurable. According to [7, p.45, Theorem 2] the function F_p is then *Bochner λ-integrable* because (2.1) implies that

$$\begin{aligned}\int_0^1 \|F_p(y)\|_p\,dy &= (p-1)^{-1/p}\int_0^1 (y^{1-p}-1)^{1/p}\,dy \\ &\le (p-1)^{-1/p}\int_0^1 (y^{1-p})^{1/p}\,dy = (p-1)^{-1/p}\int_0^1 y^{-1/p'}\,dy < \infty.\end{aligned} \tag{2.3}$$

Hence, the L^p-valued vector measure $A \mapsto \int_A F_p\,d\lambda$, for $A \in \mathcal{B}$, has *finite variation* given by

$$A \mapsto \int_A \|F_p(y)\|_p\,dy = (p-1)^{-1/p}\int_A (y^{1-p}-1)^{1/p}\,dy, \qquad A \in \mathcal{B}, \tag{2.4}$$

[7, p.46, Theorem 4(iv)]. For each $0 \leq \xi \in L^{p'}$, an application of Fubini's theorem reveals that

$$\left\langle \int_A F_p \, d\lambda, \xi \right\rangle = \int_0^1 \int_0^1 \chi_A(y)\chi_{[0,x]}(y)\xi(x)/x \, dx \, dy = \langle C_p \chi_A, \xi \rangle,$$

for each $A \in \mathcal{B}$, which implies that

$$m_p(A) := C_p \chi_A = \int_A F_p \, d\lambda, \qquad A \in \mathcal{B}. \tag{2.5}$$

This establishes that m_p has finite variation with $|m_p|$ given by (1.4); see (2.4) and (2.5). Since F_p is Bochner λ-integrable, it also a consequence of (2.5) that $m_p(\mathcal{B})$ is a relatively compact subset of L^p, [7, p.56, Corollary 9(c)].

Recall that a set $A \in \mathcal{B}$ is *m_p-null* if $m_p(B) = 0$ for every $B \in \mathcal{B}$ with $B \subseteq A$, [13, pp.106-107]. It is then clear from (1.1) and (1.2) that m_p and λ have the same null sets. In this case, one also says that the operator C_p is *λ-determined*, [13, p.187].

As already noted in Section 1, the inclusion (1.5) always holds and is continuous, [10, Theorem 4.1]. Moreover, with $X(\mu) = E := L^p$ and $T := C_p$, it follows from Theorem 4.14 of [13] that

$$L^p \subseteq L^1(m_p), \tag{2.6}$$

with a continuous inclusion. In view of (2.6) and $L^1(|m_p|) \subseteq L^1(m_p)$, to show that the inclusion (1.5) is *proper* it suffices to establish the following

Lemma 2.1. *For each $1 < p < \infty$, it is the case that $L^p \not\subseteq L^1(|m_p|)$.*

Proof. Fix $1 < p < \infty$. Suppose, on the contrary, that $L^p \subseteq L^1(|m_p|)$. Then (1.4) implies that $\int_0^1 |f(y)|\varphi(y) \, dy < \infty$, for all $f \in L^p$, where $\varphi(y) := (y^{1-p}(1-y^{p-1}))^{1/p}$ on $(0,1]$ satisfies $0 \leq \varphi \in L^{p'}$. Note that $\delta := (1-2^{-p})^{1/(p-1)}$ belongs to $(0,1)$ and that $(1-y^{p-1})^{1/p} \geq \frac{1}{2}$ for all $y \in [0,\delta]$. It follows that $\varphi(y) \geq \frac{1}{2}y^{-1/p'}$, for $y \in [0,\delta]$. Since $\varphi\chi_{[0,\delta]} \in L^{p'}$, we can conclude that $y \mapsto y^{-1/p'}\chi_{[0,\delta]}(y)$ belongs to $L^{p'}$, which is however, *not* the case. So, $L^p \not\subseteq L^1(|m_p|)$. ∎

Up to this stage we observe that parts (i), (ii) of Theorem 1.1 have been completely verified.

To establish Theorem 1.1(iii), define h by $h(y) := (1-y^{p-1})^{-1}$ for $y \in [\frac{1}{2}, 1]$ and 0 otherwise on $[0,1]$. It follows from (1.4) that

$$\int_0^1 h \, d|m_p| = \int_{1/2}^1 (1-y^{p-1})^{-1/p'} y^{-1/p'} \, dy \leq 2^{1/p'} \int_{1/2}^1 (1-y^{p-1})^{-1/p'} \, dy. \tag{2.7}$$

Set $u := 2^{1-p}$, so that $0 < u < 1$, and substitute $y = t^{1/(p-1)}$ yields

$$\int_{1/2}^1 (1-y^{p-1})^{-1/p'} \, dy = (p-1)^{-1} \int_u^1 (1-t)^{-1/p'} t^{(2-p)/(p-1)} \, dt. \tag{2.8}$$

Since $t \mapsto t^{(2-p)/(p-1)}$ is bounded on $[u,1]$, it follows that (2.8) is finite and hence, via (2.7), that $h \in L^1(|m_p|)$.

On the other hand, for the same substitution $y = t^{1/(p-1)}$, we have

$$\int_0^1 h(y)\, dy = \int_{1/2}^1 (1 - y^{p-1})^{-1}\, dy = \int_u^1 (1-t)^{-1} t^{(2-p)/(p-1)}\, dt$$
$$\geq K_p \int_u^1 (1-t)^{-1}\, dt$$

with $K_p := \inf\{t^{(2-p)/(p-1)} : t \in [u, 1]\} > 0$. It it then clear that $h \notin L^1$. This completes the proof of Theorem 1.1(iii).

If it were the case that $L^1(|m_p|) = L^1(m_p)$, then (2.6) yields $L^p \subseteq L^1(|m_p|)$. This is a contradiction to Lemma 2.1. So, the inclusion $L^1(|m_p|) \subsetneq L^1(m_p)$ must be *proper*. Granted the validity of Theorem 1.2 for the moment, it is then clear that (1.6) is valid. This completes the proof of part (iv) of Theorem 1.1.

In order to avoid any "circular logic" let us immediately provide the

Proof of Theorem 1.2. Let $f \geq 0$ belong to $L^1(m_p)$. Choose $\mathcal{B}$-simple functions $0 \leq f_n \uparrow f$ a.e. on Ω. It is clear from (1.2) that $C_p f_n = \int_\Omega f_n\, dm_p \in L^p$, for each $n \in \mathbb{N}$, i.e., $\{f_n\}_{n=1}^\infty \subseteq [C_p, L^p]$. For $n \geq m$ we have, via (1.9), that $\|f_n - f_m\|_{[C_p, L^p]} = \|C_p|f_n - f_m|\,\|_p$. Since $m_p(A) \geq 0$ on Ω for all $A \in \mathcal{B}$, we also have that

$$\|C_p|f_n - f_m|\,\|_p = \left\| \int_\Omega |f_n - f_m|\, dm_p \right\|_p = \|f_n - f_m\|_{L^1(m_p)},$$

[13, Lemma 3.13]. Accordingly, $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $[C_p, L^p]$ and so there exists $g \in [C_p, L^p]$ with $f_n \to g$ in $[C_p, L^p]$ for $n \to \infty$. Since $[C_p, L^p]$ is a B.f.s. (over $(\Omega, \mathcal{B}, \lambda)$) which contains all $\mathcal{B}$-simple functions, there exists a subsequence $f_{n(k)} \to g$ (λ-a.e.) for $k \to \infty$, [13, Proposition 2.2(ii)]. But, $f_n \uparrow f$ and so $f = g$, i.e., $f \in [C_p, L^p]$. Since every function in $L^1(m_p)$ is a linear combination of at most four non-negative functions from $L^1(m_p)$, it follows that $L^1(m_p) \subseteq [C_p, L^p]$.

Conversely, suppose that $f \in [C_p, L^p]$, i.e., $C_p|f| \in L^p$. Let $\{h_n\}_{n=1}^\infty$ be a sequence of non-negative $\mathcal{B}$-simple functions increasing to $|f|$ pointwise on Ω. Fix any $A \in \mathcal{B}$. Since the kernel $K \geq 0$ (cf. (1.7)), by the monotone convergence theorem the sequence $\{C_p h_n \chi_A\}_{n=1}^\infty$ increases pointwise on Ω to the function $C_p|f|\chi_A \leq C_p|f| \in L^p$. So, via order continuity of the norm in L^p, the sequence $\{\int_A h_n\, dm_p\}_{n=1}^\infty = \{C_p h_n \chi_A\}_{n=1}^\infty$ is convergent in L^p. By the arbitrariness of $A \in \mathcal{B}$ it follows that $|f|$ hence, also f, belongs to $L^1(m_p)$, [13, Theorem 3.5]. This implies that $[C_p, L^p] \subseteq L^1(m_p)$. Accordingly, $[C_p, L^p] = L^1(m_p)$.

The identities in (1.10) now follow from the equality $[C_p, L^p] = L^1(m_p)$, definition (1.9) and Lemma 3.13 of [13]. Clearly (1.11) is satisfied for every $\mathcal{B}$-simple function f. Since such functions are dense in $L^1(m_p)$, [13, Theorem 3.7(ii)], it follows from (1.10) that (1.11) actually holds for all $f \in L^1(m_p)$. This completes the proof of Theorem 1.2. ∎

To establish part (v) of Theorem 1.1 it suffices to show that $I_{|m_p|} : L^1(|m_p|) \to L^p$ is not compact, because then $I_{|m_p|} = I_{m_p} \circ \widetilde{J}$, with $\widetilde{J} : L^1(|m_p|) \to L^1(m_p)$ denoting the (continuous) identity inclusion, implies that also I_{m_p} fails to be compact.

To this effect, observe if we replace the function g_p in Lemma 5.1 of [14] with F_p, then the proof of the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) in that lemma also applies to the vector measure (1.2) induced by the Cesàro operator C_p. Accordingly, if $H_p : (0,1] \to L^p$ is the function given by

$$H_p(y) := (p-1)^{1/p}(y^{1-p}-1)^{-1/p}F_p(y) = (p-1)^{1/p}y^{1/p'}(1-y^{p-1})^{-1/p}F_p(y),$$

for $y \in (0,1]$, then for each $f \in L^1(|m_p|)$ we have

$$I_{|m_p|}f = \int_\Omega f\, dm_p = \int_\Omega f \|F_p(\cdot)\|_p^{-1} F_p\, d|m_p| = \int_\Omega f H_p\, d|m_p|.$$

Note that $\|H_p(y)\|_p = 1$, for $y \in (0,1]$, and that H_p is continuous on $(0,1]$ because both F_p and $y \mapsto y^{1/p'}(1-y^{p-1})^{-1/p}$ are continuous on $(0,1]$. Arguing as in the proof of Proposition 5.2(ii) in [14] it suffices to show that the closure $\overline{H_p((0,1])}$ is *not* compact in L^p.

Assume, on the contrary, that $\overline{H_p((0,1])}$ is compact in L^p. Let $y_n \to 0^+$ in $(0,1]$. By compactness there exists $\varphi \in L^p$ and a subsequence $\{y_{n(k)}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty} H_p(y_{n(k)}) = \varphi, \qquad \text{in } L^p. \tag{2.9}$$

Passing to a further subsequence, if necessary, we may also suppose that $H_p(y_{n(k)}) \to \varphi$ pointwise a.e. as $k \to \infty$. Since the function $H_p(y_{n(k)}) \geq 0$ on $(0,1]$, for each $k \in \mathbb{N}$, it is clear that $\varphi \geq 0$ a.e. Define $\xi \in L^{p'}$ by $\xi(x) := x$, for $x \in \Omega$. Then

$$\lim_{k\to\infty} \langle H_p(y_{n(k)}), \xi\rangle = \langle \varphi, \xi\rangle = \|\varphi\xi\|_1. \tag{2.10}$$

On the other hand, for each $y \in (0,1]$, we have

$$\begin{aligned}\langle H_p(y), \xi\rangle &= (p-1)^{1/p}y^{1/p'}(1-y^{p-1})^{-1/p}\langle F_p(y), \xi\rangle \\ &= (p-1)^{1/p}y^{1/p'}(1-y^{p-1})^{-1/p}(1-y)\end{aligned}$$

from which it follows that $\lim_{k\to\infty}\langle H_p(y_{n(k)}), \xi\rangle = 0$. Then (2.10) yields that $\varphi = 0$. But, (2.9) together with $\|H_p(y_{n(k)})\|_p = 1$, for each $k \in \mathbb{N}$, implies that $\|\varphi\|_p = 1$; contradiction! This ends the proof of part (v) of Theorem 1.1 and thereby the entire proof of Theorem 1.1. ∎

It is known, [9], that the spectrum of C_p is the closed disc

$$\sigma(C_p) = \{z \in \mathbb{C} : |z - \frac{p'}{2}| \leq \frac{p'}{2}\}, \qquad 1 < p < \infty,$$

and hence, C_p cannot be a compact operator. Since $C_p = I_{m_p} \circ J$, with $J : L^p \to L^1(m_p)$ the natural inclusion, this gives an alternate proof of the fact that I_{m_p} cannot be compact.

Proof of Proposition 1.3. (i) Fix $f \in L^1$. Let $x \in (0,1]$ and suppose that the sequence $x_n \to x$ in $(0,1]$. Then the dominated convergence theorem yields that

$\lim_{n\to\infty}(C_p f)(x_n) = (C_p f)(x)$. This establishes the continuity of $C_p f$ at the point x.

(ii) For $k \in \mathbb{N}$, choose $0 < a_{k+1} < b_{k+1} < a_k/2^{k+1} < a_k < b_k \leq 1$ such that $a_k = \frac{b_k}{2}$ for k even and $a_k = \frac{2b_k}{3}$ for k odd. Consider the disjoint union $A := \bigcup_{k=1}^{\infty}(a_k, b_k)$. For each $k \in \mathbb{N}$ we have

$$\frac{b_k - a_k}{b_k} = \frac{1}{b_k}\int_{a_k}^{b_k} \chi_A \, d\lambda \leq \frac{1}{b_k}\int_0^{b_k} \chi_A \, d\lambda = (C_p\chi_A)(b_k)$$

and also that

$$(C_p\chi_A)(b_k) = \frac{1}{b_k}\int_{a_k}^{b_k} \chi_A \, d\lambda + \frac{1}{b_k}\int_0^{b_{k+1}} \chi_A \, d\lambda \leq \frac{b_k - a_k}{b_k} + \frac{b_{k+1}}{b_k}.$$

Since $\frac{b_{k+1}}{b_k} \leq \frac{b_{k+1}}{a_k} \leq \frac{a_k/2^{k+1}}{a_k} = \frac{1}{2^{k+1}}$, it follows that

$$\frac{b_k - a_k}{b_k} \leq (C_p\chi_A)(b_k) \leq \frac{b_k - a_k}{b_k} + \frac{1}{2^{k+1}}, \qquad k \in \mathbb{N},$$

with $\frac{b_k - a_k}{b_k} = \frac{1}{2}$ for k even and $\frac{b_k - a_k}{b_k} = \frac{1}{3}$ for k odd. Accordingly, when $k \to \infty$ with k even we have $(C_p\chi_A)(b_k) \to \frac{1}{2}$ whereas $(C_p\chi_A)(b_k) \to \frac{1}{3}$ when $k \to \infty$ with k odd. Since $b_k \downarrow 0$, we can conclude that $C_p\chi_A$ is right discontinuous at 0. ∎

Remark 2.2. There surely exist *some* sets $A \in \mathcal{B}$ such that $C_p\chi_A = m_p(A)$ is right continuous at 0; see (1.12), for example. ∎

We conclude with two further results concerning $L^1(m_p)$ and I_{m_p}. A continuous linear operator between Banach spaces is *completely continuous* if it maps every weakly convergent sequence to a norm convergent sequence. Compact operators are always completely continuous. The converse is not valid, in general, unless the domain space of the operator is reflexive. Clearly the domain $L^1(|m_p|)$ of $I_{|m_p|}$ is not reflexive; what about the domain $L^1(m_p)$ of I_{m_p}?

Proposition 2.3. *For each $1 < p < \infty$, the Banach space $L^1(m_p)$ is not reflexive.*

Proof. Fix $p \in (1, \infty)$. According to Theorem 1.1(i) above the vector measure $m_p : \mathcal{B} \to L^p$ is non-atomic. So, if $L^1(m_p)$ was reflexive, then necessarily $L^1(|m_p|) = \{0\}$, [13, Corollary 3.23(ii)] (for real Banach spaces, see [1, Remark, pp. 3804-3805]), which is clearly *not* the case; see (1.4). ∎

Proposition 2.4. *Let $1 < p < \infty$.*

(i) *The map $I_{|m_p|} : L^1(|m_p|) \to L^p$ is completely continuous.*

(ii) *The map $I_{m_p} : L^1(m_p) \to L^p$ is not completely continuous.*

Proof. (i) Since m_p has finite variation and its range $m_p(\mathcal{B})$ is a relatively compact subset of L^p (cf. Theorem 1.1(i)), the desired conclusion follows from [13, Proposition 3.56(II)(v)].

(ii) According to [11, Theorem 2.c.5] the Banach space L^p has an unconditional basis. Moreover, being reflexive, L^p cannot contain an isomorphic copy of the Banach sequence space ℓ^1. So, if I_{m_p} was completely continuous, then $L^1(m_p) = L^1(|m_p|)$ would follow, [12, Theorem 1.2], which contradicts (1.5) above. ∎

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