Cyclic Convolution Operators on the Hardy Spaces

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Abstract

Using Banach algebra structure of the Hardy space, we describe all finite codimensional invariant subspaces of a cyclic convolution operator on the Hardy space H^p of the unit disc for $1 \leq p \leq \infty$. We also observe that every operator in the commutant of such operators is not weakly supercyclic.

1 Introduction

Let X be a Banach space and T be a bounded linear operator on X. A subspace M of X is an invariant subspace for T, if $TM \subseteq M$; if further, $\dim X/M < \infty$, it is called a finite codimensional invariant subspace for T. The famous "invariant subspace problem" asks whether every bounded linear operator on a Hilbert space, or generally, on a Banach space, admits a non-trivial invariant subspace. Of course, the problem has been answered, negatively, by Read [13] for the Banach space setting. A more specific problem, in this connection, is the characterization of all invariant subspaces of an operator. Among bounded operators, the classes of shift operators, multiplication operators, and integral operators have a special position. The invariant subspaces of Volterra integral operator on the Hilbert Hardy space is studied by Donoghue [8], and a complete characterization of such subspaces in a Banach spaces of analytic functions in the open unit disc, containing the Hardy, Bergman and Dirichlet spaces is obtained in 2008 by Aleman and Korenblum in [4]. Furthermore, characterizing finite codimensional

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invariant subspaces of operators, specially on spaces of analytic functions, is considered by several authors. In 1969, Gellar determined the finite codimensional invariant subspaces of weighted shifts [10]. Axler and Bourdon [6] and also Aleman [2] described these subspaces for the operator multiplication by z on the Bergman spaces. Later, Aleman [3] generalized their results to a Hilbert space of analytic functions. Abdollahi and Seddighi, in 1999, investigated this problem on a Banach space of analytic functions [1]. Moreover, Carisson, in 2011, obtained a characterization for a finite codimensional invariant subspaces of multiplication by a polynomial in Banach spaces of vector-valued analytic functions in several variables [7].

Supercyclicity of operators is also a concept closely related to the invariant subspace problem. It was first introduced by Hilden and Wallen in [11]. Recall that an operator T is (weakly) supercyclic if the set of all scalar multiples of the vectors in $\{x, Tx, T^2x, \cdots\}$ is (weakly) dense in X. In this case, the vector x is called a (weakly) supercyclic vector for T. Sanders proved that there exists a weakly supercyclic operator that fail to be norm supercyclic [14]. Sufficient conditions for non-weak supercyclicity of a weighted composition operator on a Hilbert space of analytic functions are given in [12]. Non-supercyclicity of the Volterra operator on $L^p[0,1]$, $1 \leq p < \infty$ is proved in [9]. Later, Shkarin [16] generalized this result, and proved that operators commuting with the Volterra operator on such spaces are not weakly supercyclic. In this paper, we are going to prove that operators in the commutant of a cyclic convolution operator on the Hardy space are not weakly supercyclic.

This paper is organized as follows. After presenting some preliminaries in Section 2, we are going to characterize all finite codimensional invariant subspaces of a cyclic convolution operator, in Section 3. The paper proceeds, in Section 4, with the study of weak supercyclicity of the operators commuting with a cyclic convolution operator; indeed, we prove that they are not weakly supercyclic.

2 Preliminary Results

Throughout this paper, the open unit disc is denoted by $\mathbb{D}$ and $\partial\mathbb{D}$ is its boundary. Also, $H(\mathbb{D})$ is the set of all analytic functions on $\mathbb{D}$. To a function f in $H(\mathbb{D})$, we associate a family of functions $\{f_r\}_r$ on $\partial\mathbb{D}$ defined by

$$f_r(e^{i\theta}) = f(re^{i\theta}) \quad (0 \leq r < 1).$$

Let m denote the normalized Lebesgue measure on $\partial\mathbb{D}(i.e., dm = \frac{d\theta}{2\pi})$. By the p-norm of f_r we mean

$$\begin{aligned} ||f_r||_p &= (\int_{\partial\mathbb{D}} |f_r|^p dm)^{\frac{1}{p}} \quad (1 \leq p < \infty); \\ ||f_r||_\infty &= \sup_{0 \leq \theta < 2\pi} |f(re^{i\theta})|. \end{aligned}$$

Also, we define the p-norm of f by

$$||f||_p = sup\{||f_r||_p : 0 \leq r < 1\} \quad (1 \leq p \leq \infty).$$

The Hardy space $H^p(1 \leq p \leq \infty)$ is the class of all analytic functions f on $\mathbb{D}$, for which $||f||_p < \infty$. The norm $||.||_p$ turns H^p into a Banach space. It shall be easily verified that if $1 \leq q \leq p \leq \infty$, then

$$H^\infty \subseteq H^p \subseteq H^q.$$

Furthermore, the functional $f \mapsto f^{(n)}(z)$ is continuous on H^p, for every $z \in \mathbb{D}$ and $n \geq 0$.

Recall that the space H^p, $1 \leq p \leq \infty$ with the product

$$\begin{aligned}(f * g)(z) &= \frac{d}{dz}\int_0^z f(z-t)g(t)dt \\ &= \int_0^z f'(z-t)g(t)dt + f(0)g(z)\end{aligned}$$

becomes a commutative Banach algebra with the constant function 1 as the identity. Indeed, there exists a constant C_p depending only on p such that $||f * g||_p \leq C_p||f||_p||g||_p$. Moreover, $f \in H^p$ is invertible if and only if $f(0) \neq 0$ (see [17]).

For $\varphi \in H^\infty$ define the multiplication operator M_φ on H^p by $M_\varphi f = \varphi f$. It is known that $||M_\varphi|| \leq ||\varphi||_\infty$. Moreover, for $f \in H^p$ the multiplication operator with $*$-product on H^p is denoted by D_f which is given by $D_f g = f * g$. The operator D_f is bounded; indeed,

$$||D_f|| \leq C_p||f||_p.$$

Let $1 \leq p \leq \infty$ and $w \in H^p$. The convolution operator K_w on H^p is defined by

$$(K_w g)(z) = \int_0^z w(z-t)g(t)dt.$$

When w is the constant function 1, the operator K_w is the Volterra integral operator which is denoted by V.

3 Finite Codimensional Invariant Subspaces

It is known that the Volterra integral operator is a compact, quasinilpotent operator [5]. The following result generalizes this fact.

Proposition 1. *Let $1 \leq p \leq \infty$ and $w \in H^p$. Then the convolution operator K_w is a bounded, compact (for $p \neq \infty$) and quasinilpotent operator on H^p.*

Proof. Suppose that $g \in H^p$ and $z \in \mathbb{D}$. Then

$$\begin{aligned}(D_{V(w)}g)(z) = (V(w) * g)(z) = \int_0^z (V(w))'(z-t)g(t)dt \\ = \int_0^z w(z-t)g(t)dt = (K_w g)(z).\end{aligned}$$

Consequently, $K_w = D_{V(w)}$ is a bounded operator. It is shown in [5] that for any polynomial q the operator

$$T_q(f)(z) = \frac{1}{z}\int_0^z q(t)f(t)dt$$

is a compact operator on H^p, $(1 \leq p < \infty)$. Thus, the operator

$$S_q(f) = \frac{1}{z}\int_0^z q(z-t)f(t)dt$$

is compact. Hence for $\varphi(z) = z$, the operator $K_q = M_\varphi S_q$ is also compact. Now, let $(q_n)_n$ be a sequence of polynomials converging to w in H^p. Since

$$||K_{q_n} - K_w|| = ||D_{V(q_n)-V(w)}||$$

$$\leq C_p||V(q_n - w)|| \leq C_p||V|| \, ||q_n - w||,$$

we conclude that K_w is a compact operator.

Note that $f \in H^p$ is invertible (as an element of a Banach algebra) if and only if $f(0) \neq 0$. Thus, the spectrum of f is $\sigma(f) = \{f(0)\}$. This, in turn, implies that

$$\sigma(K_w) = \sigma(D_{V(w)}) = \sigma(V(w))$$

$$= \{(V(w))(0)\} = \{0\}.$$

Hence K_w is a quasinilpotent operator. ■

In the sequel, we assume that $1 \leq p \leq \infty$. Recall that if X is a Banach space and T is a bounded linear operator on X then $x \in X$ is called a cyclic vector for T whenever $\bigvee\{T^n x : n \geq 0\} = X$; here $\bigvee\{.\}$ denotes the closed linear span of the set $\{.\}$. Also, if T has a cyclic vector then it is called a cyclic operator.

Theorem 1. *Suppose that the convolution operator K_w is cyclic on H^p. Then a vector $f \in H^p$ is a cyclic vector for K_w if and only if $f(0) \neq 0$.*

Proof. Take a cyclic vector $f \in H^p$ for K_w. So there is a sequence of polynomials $(P_n)_n$ such that $P_n(K_w)f \to 1$ in H^p. Consequently,

$$(P_n(K_w)f)(0) = P_n(0)f(0) \to 1,$$

which implies that $f(0) \neq 0$. Thus, the operator D_f is invertible. On the other hand, since H^p is a commutative Banach algebra

$$K_w D_f = D_f K_w, \tag{1}$$

and so

$$D_f^{-1} K_w = K_w D_f^{-1} .$$

Since f is a cyclic vector for K_w, the above equality shows that the constant function 1 is a cyclic vector for K_w.

Now, let f be an arbitrary element in H^p such that $f(0) \neq 0$. Since the constant function 1 is a cyclic vector for K_w we conclude from (1) that f is a cyclic vector of K_w. ■

Corollary 1. *A vector $f \in H^p$ is a cyclic vector for the Volterra operator V if and only if $f(0) \neq 0$.*

Proof. Since $V^n 1 = z^n/n!$ for every $n \geq 0$, the operator V is cyclic. So the result follows from Theorem 1. ∎

In what follows, we use the notation $f^{*n} = \underbrace{f * f * \cdots * f}_{n\text{-times}}$ for every $n \geq 1$ and every $f \in H^p$.

Proposition 2. *If K_w is cyclic then for every $n \geq 0$,*

$$\overline{z^{*n} * H^p} = \bigvee\{z^n, K_w z^n, K_w^2 z^n, \cdots\} = z^n H^p.$$

Proof. By Theorem 1, the constant function 1 is a cyclic vector for K_w; so the above equality holds for $n = 0$. Suppose that $n \geq 1$ and let Q_n be the set of all polynomials q so that

$$q(0) = q'(0) = \cdots = q^{(n-1)}(0) = 0.$$

Since $z^{*n} * q^{(n)} = q$ for every polynomial q and Q_n is dense in $z^n H^p$ we conclude that $z^n H^p \subseteq \overline{z^{*n} * H^p}$. Moreover, for $f \in H^p$ there is a sequence of polynomials $(q_i)_i$ such that

$$q_i(K_w)z^n = D_{z^n} q_i(K_w)1 \to D_{z^n} f$$

as $i \to \infty$. Therefore,

$$z^n H^p \subseteq \overline{z^{*n} * H^p} \subseteq \bigvee\{z^n, K_w z^n, K_w^2 z^n, \cdots\}.$$

On the other hand, it is easy to see that

$$(f * g)^{(k)}(z) = \int_0^z f^{(k)}(z-t)g'(t)dt + \sum_{i=0}^{k-1} f^{(i)}(0)g^{(k-i)}(z) + g(0)f^{(k)}(z)$$

for all f and g in H^p. Put $f(z) = z^n$. So $(z^n * g)^{(k)}(0) = 0$ for all $g \in H^p$ and $0 \leq k \leq n-1$. Therefore, for every $i \geq 0$ and $0 \leq k \leq n-1$, we have

$$(K_w^i z^n)^{(k)}(0) = ((Vw)^{*i} * z^n)^{(k)}(0) = 0.$$

Consequently,

$$\bigvee\{z^n, K_w z^n, K_w^2 z^n, \cdots\} \subseteq z^n H^p.$$

Hence

$$\overline{z^{*n} * H^p} = \bigvee\{z^n, K_w z^n, K_w^2 z^n, \cdots\} = z^n H^p.$$ ∎

It is known [4] that for every nonzero invariant subspace M of the Volterra operator V on the space H^p, $p > 1$ there is a nonnegative integer n such that $M = z^n H^p$. In the next theorem, we characterize all finite codimensional invariant subspaces of a cyclic convolution operator on H^p.

Theorem 2. *Suppose that K_w is cyclic on H^p. Then the following results hold.*

(a) Every proper invariant subspace of K_w is a subset of zH^p.

(b) Every invariant subspace of K_w of finite codimension, has the form $\overline{K_w^n H^p}$ for some $n \geq 0$.

Proof. (a) Suppose that M is a proper invariant subspace of K_w and $f \in M$. If $f(0) \neq 0$ then by Theorem 1, the vector f is a cyclic vector of K_w, and so $M = H^p$; which is a contradiction. Hence $M \subseteq zH^p$.

(b) Let M be an invariant subspace of K_w such that $\dim\ H^p/M = m \geq 0$. Define the linear transformation $T : H^p/M \to H^p/M$ by $T(f + M) = K_w f + M$, which is well defined thanks to the fact that M is an invariant subspace for K_w. Let $q(z)$ be the characteristic polynomial of T. Then $q(z)$ is a monic polynomial of degree m such that, by the Cayley-Hamilton theorem, $q(T) = 0$.

Suppose $n \geq 0$ is the multiplicity of the zero of $q(z)$ at the origin. Thus $q(z) = z^n t(z)$ where $t(z)$ is a polynomial whose roots are nonzero. Now, if $\lambda \neq 0$ is a root of $t(z)$, then $K_w - \lambda$ is invertible. Indeed, $\sigma(K_w - \lambda) = \{\lambda\}$ therefore, $t(K_w)$ is invertible and so

$$K_w^n H^p = K_w^n t(K_w) H^p = q(K_w) H^p \subseteq M. \tag{2}$$

The last inclusion occurs because $q(T) = 0$. On the other hand, since K_w is cyclic

$$\dim\ H^p/M \leq \dim\ H^p/\overline{K_w^n H^p} \leq n \leq m = \dim\ H^p/M.$$

Hence $M = \overline{K_w^n H^p}$. ∎

Remark. In the above proof, although $q(T) = 0$, but $q(K_w)$ is never zero. Because otherwise, (2) shows that K_w is a nilpotent operator. But this is impossible, thanks to the fact that the constant function 1 is a cyclic vector for K_w.

4 Non-weak Supercyclicity of the Commutant

Recall that for an operator T on a Banach space X the commutant of T, which is denoted by $\{T\}'$, is the set of all bounded operators S on X such that $TS = ST$. First, we obtain the commutant of a cyclic convolution operator on the Hardy space.

Proposition 3. *Suppose that K_w is cyclic on the space H^p. Then $A \in \{K_w\}'$ if and only if $A = D_f$ where $f = A1$.*

Proof. Suppose that A commutes with K_w. Since $K_w = D_{V(w)}$, $AD^n_{V(w)} = D^n_{V(w)}A$ for all $n \geq 0$. Hence

$$A(V(w)^{*n}) = AD^n_{V(w)}1 = D^n_{V(w)}A1 = D_f(V(w)^{*n})$$

where $f = A1$.

Since the constant function 1 is a cyclic vector for K_w, $\bigvee\{1, V(w), V^{*2}(w), \cdots\} = H^p$. Consequently, $A = D_f$. The reverse inclusion is obvious. ∎

Theorem 3. *Every operator in the commutant of a cyclic convolution operator K_w on H^p is not weakly supercyclic.*

Proof. Suppose that $T \in \{K_w\}'$. By the previous proposition $T = D_h$ for some $h \in H^p$. If $h(0) = 0$ then $(D_h^n g)(0) = 0$ for every $n \geq 0$ and every $g \in H^p$. Hence T cannot be a weakly supercyclic operator. Thus, we may assume that $h(0) \neq 0$. Now consider the bounded linear functional Λ on H^p defined by $\Lambda(f) = f(0)$. Observe that $\ker\Lambda$ is a closed invariant subspace of T of codimension 1. Define the bounded linear operator S on $\ker\Lambda \oplus \mathbb{C}$ by

$$S(f \oplus \lambda) = \frac{1}{h(0)}(T|_{\ker\Lambda} f + \lambda(h - h(0))) \oplus \lambda$$

and consider the bounded linear bijective transformation $A : \ker\Lambda \oplus \mathbb{C} \to H^p$ by $A(f \oplus \lambda) = f + \lambda$.

Now for every $f \in H^p$, the fact that $D_h(f(0)) = f(0)h$ implies that

$$\begin{aligned}(ASA^{-1})(f) &= (AS)((f - f(0)) \oplus f(0)) \\ &= A(\frac{1}{h(0)}(D_h(f - f(0)) + f(0)(h - h(0))) \oplus f(0)) \\ &= \frac{1}{h(0)}(D_h(f - f(0)) + D_h(f(0))) \\ &= \frac{1}{h(0)} Tf.\end{aligned}$$

Assume, on the contrary that, the operator T is weakly supercyclic. Since similarity preserves weak supercyclicity, we conclude that the operator S is also weakly supercyclic. Thus, Corollary 7.11 and Lemma 7.12 of [15] imply that $B := T|_{\ker\Lambda}$ is weakly hypercyclic; i.e., there is a vector $g \in \ker\Lambda$ such that the set $\{g,\ Bg\ B^2 g, \cdots\}$ is weakly dense in $\ker\Lambda$. Then, for every $f \in \ker\Lambda$ there is a net $(n_\alpha)_\alpha$ such that $B^{n_\alpha} g = D_h^{n_\alpha} g = h^{*n_\alpha} * g$ converges weakly to f. Furthermore, in view of the continuity of the linear functional $f \mapsto f'(0)$, we see that

$$(h^{*n_\alpha} * g)'(0) = h^{*n_\alpha}(0) g'(0) = h^{n_\alpha}(0) g'(0) \to f'(0).$$

Taking $f(z) = z$ in the above formula, we see that $g'(0) \neq 0$ and $|h(0)| = 1$. But when $f(z) = z^2$ we get a contradiction. ∎

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