L_p-approximation via Abel convergence

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Abstract

In the present paper, using the concept of the Abel convergence method, we give a Korovkin type approximation theorem for a sequence of positive linear operators acting from $L_p\,[a,b]$ into itself. We also study some quantitative estimates for L_p approximation via Abel convergence.

1 Introduction

The classical Korovkin theorem [1] yields the uniform convergence in $C[a,b]$, the space of continuous functions on a compact interval, of a sequence of positive linear operators by checking the convergence only on three test functions $\{1, x, x^2\}$. Some results concerning the Korovkin type approximation theorem in the space $L_p[a,b]$ of the Lebesgue integrable functions on a compact interval may be found in [2]. If the sequence of positive linear operators does not converge to the identity operator then it might be useful to use some matrix summability methods (see e.g. [3]). Recently the Abel method, a nonmatrix summability method, has been used in the Korovkin type approximation of functions in the weighted space (see [4], [5]).

In this paper, we develop the main aspects of the Korovkin type L_p approximation theory with the use of the Abel method which is a sequence-to-function transformation. Recall that the main point of using the Abel method has always been to make a non-convergent sequence to converge. On the other hand the Abel method is much more effective in approximating functions as we get a function after applying the Abel method to a sequence of positive linear operators. Using

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modulus of smoothness and the K-functional of Peetre we also give a quantitative estimate for L_p approximation via Abel convergence in $L_p[a,b]$.

First of all, we recall some basic definitions and notations used in the paper.

Let $L_p[a,b]$, $1 \le p < \infty$, denote the space of measurable real valued pth power Lebesgue integrable functions f on $[a,b]$ with $\|f\|_p := \|f\|_{L_p[a,b]} := \left(\int_a^b |f|^p \, d\mu \right)^{1/p}$.

Let $T : L_p \to L_p$ be a linear operator. If $Tf \ge 0$ whenever $f \ge 0$, then T is called positive. If T is a positive linear operator then $f \le g$ implies that $Tf \le Tg$. The operator norm $\|T\|_{L_p \to L_p}$ is given by $\|T\|_{L_p \to L_p} = \sup\limits_{\|f\|_p = 1} \|Tf\|_p$.

The following approximation theorem for a sequence of positive linear operators acting from $L_p[a,b]$ into itself may be found in [2].

Theorem A. Let $\{T_n\}$ be a uniformly bounded sequence of positive linear operators from $L_p[a,b]$ into itself, $1 \le p < \infty$. Then the sequence $\{T_n f\}$ converges to f in L_p norm for any function $f \in L_p[a,b]$ if and only if

$$\lim_n \|T_n(f_i) - f_i\|_p = 0 \; for \; i = 0,1,2$$

where $f_i(t) = t^i$ for $i = 0,1,2$.

Some extensions of this theorem may be found in [6], [7] and [8].

In the present paper, using the Abel convergence method we will give another variation of Theorem A. We also present an example of a sequence of positive linear operators acting from L_p into itself so that Theorem A does not hold but our result does hold. This shows that our result is stronger than Theorem A.

Let us recall the Abel convergence:

If the series

$$\sum_k a_k y^k$$

converges for all $y \in (0,1)$ and

$$\lim_{y \to 1^-} (1-y) \sum_k a_k y^k = L \tag{1.1}$$

then we say that the sequence $a = (a_k)$ is Abel convergent to L.

Since $\frac{1}{1-y} = \sum\limits_k y^k$, $0 < y < 1$, (1.1) is equivalent to the fact that

$$\lim_{y \to 1^-} (1-y) \sum_k (a_k - L) y^k = 0.$$

Note that any convergent sequence is Abel convergent to the same value but not conversely ([9], [10]).

Let $\{T_n\}$ be a sequence of positive linear operators from $L_p[a,b]$, $1 \le p < \infty$, into itself such that

$$H := \sup_{y \in (0,1)} (1-y) \sum_n \|T_n\|_{L_p \to L_p} y^n < \infty. \tag{1.2}$$

Then for all $f \in L_p[a,b]$ and $y \in (0,1)$ the operator U_y defined by

$$U_y(f) := U_y(f;x) := (1-y)\sum_n T_n(f;x)y^n$$

is a positive linear operator from $L_p[a,b]$ into itself, since it follows from (1.2) that

$$\begin{aligned}\left\|U_y\right\|_{L_p \to L_p} &= \sup_{\|f\|_p=1}(1-y)\left\|\sum_n T_n(f)y^n\right\|_p \\ &\leq \sup_{y\in(0,1)}(1-y)\sum_n \|T_n\|_{L_p\to L_p} y^n < \infty.\end{aligned}$$

2 Abel convergence of the sequence of positive linear operators in $L_p\,[a,b]$

In this section using Abel convergence instead of ordinary convergence we give a Korovkin type L_p approximation theorem for the sequence of positive linear operators.

Theorem 1. *Let $\{T_n\}$ be a sequence of positive linear operators from $L_p[a,b]$, $1 \leq p < \infty$ into itself such that (1.2) holds. Then for any function $f \in L_p[a,b]$*

$$\lim_{y\to 1^-}\left\|U_y(f)-f\right\|_p = 0 \tag{2.1}$$

if and only if

$$\lim_{y\to 1^-}\left\|U_y(f_i)-f_i\right\|_p = 0 \; for \; i = 0,1,2 \tag{2.2}$$

where $f_i(t) = t^i$ for $i = 0,1,2$.

Proof. It is obvious that (2.1) implies (2.2). Now assume that (2.2) holds. Let $f \in L_p[a,b]$. Following the Lusin theorem, for each given $\varepsilon > 0$ there exists $\varphi \in C[a,b]$ such that

$$\|f-\varphi\|_p < \varepsilon. \tag{2.3}$$

By the continuity of φ on $[a,b]$, for each given $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|\varphi(t)-\varphi(x)| < \varepsilon$ for all $x,t \in [a,b]$ satisfying $|t-x| < \delta$. As it is proved in [[8], (2.7)-(2.9)] we have for any $x,t \in [a,b]$

$$|\varphi(t)-\varphi(x)| < \varepsilon + \frac{2M}{\delta^2}\psi_x(t), \tag{2.4}$$

where $\psi_x(t) = (t-x)^2$ and $M := \|\varphi\|_{C[a,b]}$. Using the positivity and linearity of operators T_n and inequality (2.3), we obtain

$$\begin{aligned}\left\|U_y(f)-f\right\|_p &\leq \left\|U_y(f-\varphi)\right\|_p + \left\|U_y(\varphi)-\varphi\right\|_p + \|f-\varphi\|_p \\ &< \varepsilon\left(1+(1-y)\sum_n \|T_n\|_{L_p\to L_p} y^n\right) + \left\|U_y(\varphi)-\varphi\right\|_p. \end{aligned} \tag{2.5}$$

Now considering the monotonicity of operators T_n and (2.4), the second term on the right hand side of above inequality may be written as follows:

$$\begin{aligned}\left\|U_y(\varphi)-\varphi\right\|_p &\leq \left\|U_y(|\varphi(t)-\varphi(x)|\,;x)\right\|_p + M\left\|U_y(f_0)-f_0\right\|_p \\ &< \left\|U_y(\varepsilon+\frac{2M}{\delta^2}\psi_x(t);x)\right\|_p + M\left\|U_y(f_0)-f_0\right\|_p \\ &\leq \varepsilon + (\varepsilon + M + \frac{2M\beta^2}{\delta^2})\left\|U_y(f_0)-f_0\right\|_p \\ &+ \frac{4M\beta}{\delta^2}\left\|U_y(f_1)-f_1\right\|_p + \frac{2M}{\delta^2}\left\|U_y(f_2)-f_2\right\|_p,\end{aligned} \tag{2.6}$$

where $\beta := \max\{|a|,|b|\}$. It follows from (2.5) and (2.6), for all $y \in (0,1)$, that

$$\begin{aligned}\left\|U_y(f)-f\right\|_p &< \varepsilon(2+H) \\ &+ (\varepsilon + M + \frac{2M\beta^2}{\delta^2})\left\|U_y(f_0)-f_0\right\|_p \\ &+ \frac{4M\beta}{\delta^2}\left\|U_y(f_1)-f_1\right\|_p + \frac{2M}{\delta^2}\left\|U_y(f_2)-f_2\right\|_p.\end{aligned} \tag{2.7}$$

Since $\varepsilon > 0$ is arbitrary, letting $y \to 1^-$ in both sides of (2.7) we get

$$\lim_{y\to 1^-}\left\|U_y(f)-f\right\|_p = 0$$

which concludes the proof. ■

3 Quantitative estimate for L_p approximation via Abel convergence

In this section using Abel convergence we give a quantitative estimate for L_p approximation of positive linear operators considered in Theorem 1. Furthermore we obtain the rate of Abel convergence of these operators. We will need the following Lemma to prove our main Theorem 2.

First of all, we recall some basic definitions and notations used in this section. Let

$$L_p^{(2)}[a,b] = \left\{f \in L_p[a,b] : f' \text{ absolutely continuous and } f'' \in L_p[a,b]\right\},$$

where f' and f'' denote the first and second derivatives of f, respectively.

For $f \in L_p[a,b]$, $1 \leq p < \infty$, and $t > 0$, the K-functional of Peetre (see [11]) is defined by

$$K_{2,p}(f;t) = \inf_{g\in L_p^{(2)}[a,b]}\{\|f-g\|_p + t(\|g\|_p + \left\|g''\right\|_p)\}. \tag{3.1}$$

Following [12] and [13], for $f \in L_p[a,b]$, $1 \le p < \infty$, the second-order modulus of smoothness $w_{2,p}(f)$ is defined by

$$w_{2,p}(f,h) = \sup_{0<t\le h} \|f(x+t) - 2f(x) + f(x-t)\|_{L_p[a+t,b-t]},$$

where $[a+t, b-t] \subset [a,b]$.

By a well known relation between modulus of smoothness and the K-functional of Peetre, [14], we have

$$\begin{aligned} C_1 \left\{ \min(1,t^2) \|f\|_p + w_{2,p}(f;t) \right\} &\le K_{2,p}(f;t^2) \\ &\le C_1^{-1} \left\{ \min(1,t^2) \|f\|_p + w_{2,p}(f;t) \right\}. \end{aligned} \tag{3.2}$$

where $C_1 > 0$ is independent of f and p.

Let $\mu_{yp} := \left(\max\limits_{i=0,1,2} \left\{ \left\| U_y(f_i) - f_i \right\|_p \right\} \right)^{p/2p+1}$ where $f_i(t) = t^i$ for $i = 0,1,2$ and $\{T_n\}$ is a sequence of positive linear operators from $L_p[a,b]$ into itself.

Lemma 1. *Let $\{T_n\}$ be a sequence of positive linear operators from $L_p[a,b]$, $1 \le p < \infty$, into itself and assume that (1.2) holds. Then for all $x,t \in [a,b]$, $g \in L_p^{(2)}[a,b]$ and for all y sufficiently close to 1 from the left hand side*

$$\left\| U_y(g) - g \right\|_p \le \begin{cases} C_p'(\|g\|_p + \left\| g'' \right\|_p)\mu_{yp}^2, & \text{if } \mu_{yp} < 1, \\ C_p'(\|g\|_p + \left\| g'' \right\|_p)\mu_{yp}^4, & \text{if } \mu_{yp} \ge 1 \end{cases} \tag{3.3}$$

where $C_p' > 0$ is independent of g and y.

Proof. Let $g \in L_p^{(2)}[a,b]$ and assume, that g is extended outside of $[a,b]$ so that $g''(x) = 0$ if $x \notin [a,b]$. For $x,t \in [a,b]$ we know that

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t (t-u)g''(u)du. \tag{3.4}$$

Using (3.4) we get

$$\begin{aligned} \left\| U_y(g(t) - g(x);x) \right\|_p \le{}& \left\| g' \right\|_\infty \{ \left\| U_y(f_1) - f_1 \right\|_p + \beta \left\| U_y(f_0) - f_0 \right\|_p \} \\ &+ \left\| U_y(\int_x^t (t-u)g''(u)du; x) \right\|_p, \end{aligned} \tag{3.5}$$

where $\beta := \max\{|a|, |b|\}$.

Fix $\delta > 0$. For $x,t \in [a,b]$ we know that (see [15])

$$\left| \int_x^t (t-u)g''(u)du \right| \le \delta \int_0^\delta \left| g''(x+u) \right| du + \frac{(t-x)^2}{\delta^{1/p}} \left\| g'' \right\|_p.$$

Therefore by the monotonicity of the operators T_n we obtain

$$\begin{aligned}&\left\| U_y(\int_x^t (t-u)g''(u)du;x)\right\|_p \\ &\leq \left\| \delta(\int_0^\delta \left|g''(x+u)\right| du)U_y(f_0;x)\right\|_p + \frac{\left\|g''\right\|_p}{\delta^{1/p}}\left\|U_y((t-x)^2;x)\right\|_p .\end{aligned} \tag{3.6}$$

By the Hölder inequality and the generalized Minkowski inequality, the first term on the right hand side of (3.6) may be written as

$$\begin{aligned}&\left\| \delta\int_0^\delta \left|g''(x+u)\right| du\left(U_y(f_0;x)-f_0(x)\right)+\delta\int_0^\delta\left|g''(x+u)\right|du\right\|_p \\ &\leq \delta^{2-1/p}\left\|g''\right\|_p\left\|U_y(f_0)-f_0\right\|_p+\delta\int_0^\delta\left\|g''(x+u)\right\|_p du \\ &= \left\|g''\right\|_p\{\delta^{2-1/p}\left\|U_y(f_0)-f_0\right\|_p+\delta^2\}.\end{aligned}$$

Hence

$$\begin{aligned}\left\| U_y(\int_x^t (t-u)g''(u)du;x)\right\|_p \leq \left\|g''\right\|_p\{\delta^2+\frac{1}{\delta^{1/p}}[&\left\|U_y(f_2)-f_2\right\|_p \\ &+2\beta\left\|U_y(f_1)-f_1\right\|_p \\ &+(\delta^2+\beta^2)\left\|U_y(f_0)-f_0\right\|_p]\}.\end{aligned}$$

For y sufficiently close to 1 from the left hand side, we can choose $\delta := \mu_{yp}$ to obtain

$$\left\| U_y(\int_x^t (t-u)g''(u)du;x)\right\|_p \leq C\left\|g''\right\|_p\mu_{yp}^2, \tag{3.7}$$

provided that $\mu_{yp} < 1$ where $C > 0$ is an absolute constant.
Using (3.5) and (3.7) and considering $\mu_{yp} < 1$, we have

$$\begin{aligned}\left\|U_y(g)-g\right\|_p &\leq \left\|U_y(g(t)-g(x);x)\right\|_p+\|g\|_\infty\left\|U_y(f_0)-f_0\right\|_p \\ &\leq \left\|g'\right\|_\infty\{\left\|U_y(f_1)-f_1\right\|_p+\beta\left\|U_y(f_0)-f_0\right\|_p\} \\ &\quad +C\left\|g''\right\|_p\mu_{yp}^2+\|g\|_\infty\left\|U_y(f_0)-f_0\right\|_p .\end{aligned}$$

By Theorem 3.1 in [16], we obtain from the above inequality that

$$\left\|U_y(g) - g\right\|_p \leq C_p'(\|g\|_p + \left\|g''\right\|_p)\mu_{yp}^2.$$

If $\mu_{yp} \geq 1$ we get (3.3) similarly. This completes the proof. ∎

Now we present the following quantitative estimate for L_p approximation via Abel convergence.

Theorem 2. *Let $\{T_n\}$ be a sequence of positive linear operators from $L_p[a,b]$, $1 \leq p < \infty$, into itself, and assume that (1.2) holds. Then for all $x, t \in [a,b]$, $f \in L_p[a,b]$ and for all y sufficiently close to 1 from the left hand side*

$$\left\|U_y(f) - f\right\|_p \leq \begin{cases} C_p\left\{\min(1,\mu_{yp}^2)\left\|f\right\|_p + w_{2,p}(f;\mu_{yp})\right\}, & \text{if } \mu_{yp} < 1, \\ C_p\left\{\min(1,\mu_{yp}^4)\left\|f\right\|_p + w_{2,p}(f;\mu_{yp}^2)\right\}, & \text{if } \mu_{yp} \geq 1 \end{cases} \tag{3.8}$$

where $C_p > 0$ is independent of f and y.

Proof. Let $f \in L_p[a,b]$ and $g \in L_p^{(2)}[a,b]$ and for all y sufficiently close to 1 from the left hand side, if $\mu_{yp} < 1$ Lemma 1 yields that

$$\begin{aligned} &\left\|U_y(f) - f\right\|_p \\ &\leq \|f-g\|_p \sup_{y\in(0,1)} (1-y)\sum_n \|T_n\|_{L_p\to L_p} y^n + \left\|U_y(g) - g\right\|_p + \|f-g\|_p \\ &\leq (1+H)\|f-g\|_p + C_p'(\|g\|_p + \left\|g''\right\|_p)\mu_{yp}^2. \end{aligned}$$

Taking infimum over all $g \in L_p^{(2)}[a,b]$ and using (3.1) and (3.2), we have

$$\left\|U_y(f) - f\right\|_p \leq C_p\left\{\min(1,\mu_{yp}^2)\|f\|_p + w_{2,p}(f;\mu_{yp})\right\}.$$

If $\mu_{yp} \geq 1$ we obtain similarly (3.8) which concludes the proof. ∎

The following rate of Abel convergence in $L_p[a,b]$ follows from Theorem 2 immediately.

Corollary 1. *Let $\{T_n\}$ be a sequence of positive linear operators from $L_p[a,b]$, $1 \leq p < \infty$, into itself. Assume that (1.2) holds and $\mu_{yp} \to 0$ (as $y \to 1^-$). Then for all $f \in L_p[a,b]$ we have*

$$\lim_{y\to 1^-} \left\|U_y(f) - f\right\|_p = 0.$$

4 Remarks

Let $\{T_n\}$ be a sequence of positive linear operators from $L_p[a,b]$ into itself satisfying the hypotheses of Theorem A. Consider the sequence $\alpha = (\alpha_n)$ given by $\alpha_n = 1$ if n is a perfect square and $\alpha_n = 0$ otherwise. Note that α is Abel convergent to zero but not convergent. Let $\{P_n\}$ be a sequence of positive linear operators acting from $L_p[a,b]$ into itself defined by

$$P_n(f;x) = (1+\alpha_n)T_n(f;x)$$

for $f \in L_p[a,b]$. Observe that the sequence $\{P_n\}$ satisfies the hypotheses of Theorem 1, but it does not satisfy Theorem A.

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