Poincaré's problem in the class of almost periodic type functions

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Abstract

We consider the Poincaré's classical problem of approximation for second order linear differential equations in the class of almost periodic type functions. We obtain an explicit form for solutions of these equations by studying a Riccati equation associated with the logarithmic derivative of a solution. The fixed point Banach argument allows us to find almost periodic and asymptotically almost periodic solutions of the Riccati equation. A decomposition property of the perturbations induces a decomposition on the Riccati equation and its solutions. In particular, by using this decomposition we obtain asymptotically almost periodic and also p-almost periodic solutions to the Riccati equation.

1 Introduction

The theory of almost periodic functions has been developed in connection with problems of differential equations, stability theory, dynamical systems and many others. The applications include not only ordinary differential equations, but also difference equations, partial differential equations or differential equations in Banach spaces. Since there are plenty of results in literature, let us just quote [8, 9, 10, 29, 30], for their applications in engineering [15, 21, 22, 28] and life sciences [1, 11, 19].

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Received by the editors in May 2014.
Communicated by P. De Maesschalck.
2010 *Mathematics Subject Classification :* 34A05, 34A30, 34E10, 34E05.
Key words and phrases : Perturbed second order linear differential equation, Almost periodic solutions, Riccati equation.

Poincaré [32] in 1885 established the existence of a solution y of the equation

$$y^{(n)} + \sum_{i=0}^{n-1} (a_i + r_i(t)) y^{(i)} = 0 \tag{1}$$

such that $y'(t)/y(t)$ converges as $t \to \infty$, under the hypothesis: the roots $\lambda_1, \ldots, \lambda_n$ of equation $\lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i = 0$ have distinct real parts and $r_i \in C_0$, *i.e.*, $r_i(t) \to 0$ as $t \to +\infty$, for $i = 0, \ldots, n-1$. Perron [25], at the beginning of the twentieth century, improved this result assuring the existence of n linearly independent solutions $y_1, \ldots, y_n$, such that $y_i'(t)/y_i(t)$ converges to λ_i as $t \to +\infty$, under the same hypothesis. In case $r_i \in L^p$ for some $p \geq 1$, there are results for systems of linear differential equations, which can be applied to this equation. These results are due to Levinson [2, 6, 12, Th. 1.3.1] for $p = 1$ and to Hartman-Wintner [20, 12, Th.1.5.1] for $p \in (1, 2]$. In [16]-[18], by using a nonlinear transformation and a general Riccati equation we have studied second and third order equations (1). In the class of C_0 and L^p functions we have obtained precise formulae and estimated their errors. In almost diagonal linear systems, perhaps the first results in this sense were obtained in Pinto *et al.* [31, 30, 29].

In this paper, by using a certain Riccati equation, [16, 26, 2, chap. 6], we generalize Poincaré's and Perron's classical problem of approximation to the class of almost periodic type functions, namely, we consider equation

$$y'' + (a_1 + r_1(t))y' + (a_0 + r_0(t))y = 0, \tag{2}$$

where a_i are constants and r_i are almost periodic complex-valued functions for $i = 0, 1$. Under sufficient conditions, we have obtained an almost explicit formula for the solutions. In particular, if $\lambda_\pm$ are the roots of the polynomial $P(\lambda) = \lambda^2 + a_1\lambda + a_0$, $\gamma = \lambda_+ - \lambda_-$, with $\operatorname{Re}\gamma \neq 0$, r_0 and r_1 are sufficiently small in a L^∞-sense and almost periodic then there exists a fundamental system of solutions $y_\pm : \mathbb{R} \to \mathbb{C}$ satisfying

$$\begin{aligned} y_\pm(t) &= e^{\lambda_\pm t} \exp\left(\mp \frac{1}{\gamma} \int_0^t [r_0(s) + \lambda_\pm r_1(s) + r_1(s) z_\pm(s) \right. \\ &\qquad\qquad \left. + z_\pm^2(s)]\, ds \mp \frac{1}{\gamma}[z_\pm(t) - z_\pm(0)] \right), \\ y_\pm'(t) &= (\lambda_\pm + z_\pm(t)) y_\pm(t), \end{aligned}$$

where $z_\pm$ satisfy a Riccati equation, is an almost periodic complex-valued function and

$$z_\pm(t) = O\left(\mp \int_t^{\mp\infty} e^{\mp\alpha(t-s)} |r_0(s) + \lambda_\pm r_1(s)|\, ds \right)$$

for some $0 < \alpha < |\operatorname{Re}\gamma|$ (see Theorem 1 and (21)). Let us point out that without loss of generality, we always can assume that $\operatorname{Re}\gamma > 0$. These formulae remember us the Floquet's Theorem for the periodic case, which is not longer true for almost periodic type coefficients.

On the other hand, this result is also true for r_i, $i = 0, 1$ in either C_0 or L^p, see [2, 6, 12, 16]. In general, $z_\pm$ are related with the logarithmic derivative of $y_\pm$

and represent the error function belonging to spaces under consideration C_0, L^p or almost periodic type functions, and are small with respect to the norm in the space; L^∞ in our case. Our procedure allows us to address a new class of perturbations, introducing a new class of functions in the context of almost periodicity: the p-almost periodic functions. See Definition 2 and Remark 7. In other words, there exist several classes of almost periodic type functions verifying these results. This is the case of those with a summand C_0 or L^p (asymptotically almost periodic functions or p-almost periodic functions), namely, $r_i = \mu_i + \nu_i$, $i = 0, 1$ with $\mu_i \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ and either $\nu_i \in C_0$ or $\nu_i \in L^p$ for $i = 0, 1$, see Theorems 2-4. An important consequence is that this decomposition of the coefficients r_i, $i = 0, 1$ induces the decomposition $z_\pm = \theta_\pm + \psi_\pm$, where $\theta_\pm$ is the almost periodic part and $\psi_\pm$ is C_0 or L^p, respectively. These functions $\theta_\pm$ and $\psi_\pm$ satisfy their own equations, which can be treated and solved separately, implying the existence of new solutions. See equations (22)-(23) and Theorem 4. In section 4, we will discuss the results through an illustrative example. Finally, let us emphasize that the used method is scalar [16, 17, 18, 26], reducing the order of the equation and avoiding the usual diagonalization process [6, 12, 13, 14].

2 Almost periodic functions and Riccati equations

In this section we shall present a certain class of almost periodic type functions, review some facts to Green's type operators and show some existence results for a nonlinear equation, which we will apply to a Riccati equation related with (2). In order to make a better exposition of these topics, we consider three subsections.

2.1 Almost periodic type functions

First, let us introduce the notion of almost periodic function used throughout this paper. See [8, 23, 33].

Definition 1. *a*) A continuous function $f : \mathbb{R} \to \mathbb{C}$ is *almost periodic* (in the Bochner sense [3]) if for every sequence $\{\alpha'_n\} \subset \mathbb{R}$ there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that $\lim_{n\to+\infty} f(t + \alpha_n)$ exists uniformly on the real line. The set of almost periodic functions from $\mathbb{R}$ to $\mathbb{C}$ will be denoted by $\mathcal{AP}(\mathbb{R}, \mathbb{C})$.

b) A continuous function $f : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ satisfies $f \in \mathcal{AP}(\mathbb{R} \times \mathbb{C}, \mathbb{C})$ if for every sequence $\{\alpha'_n\} \subset \mathbb{R}$ there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that $\lim_{n\to+\infty} f(t + \alpha_n, z)$ exists uniformly for $t \in \mathbb{R}$ and for z on compact subsets of $\mathbb{C}$.

In order to perturb almost periodic functions we present the following result.

Lemma 1. *Let $f \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ and $p \geq 1$. If $f(t) \to 0$ as $t \to +\infty$ then $f \equiv 0$. If $f \in L^p[0, +\infty)$ then $f \equiv 0$.*

More generally, if $f \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ and $f(t) \to c$ as $t \to +\infty$ then $f \equiv c$. This Lemma is straightforward and, for the sake of self-containment, we shall present a proof here.

Proof: First, assume that $f(t) \to 0$ as $t \to +\infty$. By assumptions, there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ and a function g such that $f(t+n_k) \to g(t)$ as $k \to +\infty$ uniformly for $t \in \mathbb{R}$. Since, for every $t \in \mathbb{R}$, $t+n_k \to +\infty$, we have that $f(t+n_k) \to 0 = g(t)$ as $k \to +\infty$. Hence, $g(t) = 0$ for all $t \in \mathbb{R}$ and due to uniform convergence, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(t+n_k)| < \varepsilon$ for all $k \geq N$ and $t \in \mathbb{R}$, namely, for all $\varepsilon > 0$, $|f(t)| < \varepsilon$ for all $t \in \mathbb{R}$. Therefore, $f \equiv 0$.

Now, assume that $f \in L^p[0,+\infty)$, *i.e.*, $\int_0^\infty |f|^p < +\infty$. Then, we get that

$$\lim_{n\to+\infty} \int_0^{+\infty} |f(t+n)|^p \, dt = \lim_{n\to+\infty} \int_n^{+\infty} |f(t)|^p \, dt = 0.$$

Thus, $f(\cdot+n)$ converges to 0 in $L^p[0,+\infty)$. Hence, there exists a subsequence $n_k \in \mathbb{N}$ such that $f(\cdot+n_k)$ converges a.e. to 0 as $k \to +\infty$. On the other hand, there exists a subsequence, that we will still denote by $\{n_k\}_k$, with $n_k \to +\infty$ as $k \to +\infty$ and a function f^*, such that $f(\cdot+n_k) \to f^*$ uniformly in $\mathbb{R}$ as $k \to +\infty$, in view of $f \in \mathcal{AP}(\mathbb{R},\mathbb{C})$. Therefore, by the uniform convergence of $\{f(\cdot+n_k)\}_k$, we conclude that $f^* \equiv 0$ in $[0,+\infty)$. Finally, we find that $f \equiv 0$, since $f^* \equiv 0$ in $[0,+\infty)$ implies $f(t) \to 0$ as $t \to +\infty$. ■

Let us introduce the following function spaces: $BC(\mathbb{R},\mathbb{C})$ is the set of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{C}$. In addition, we define $BC_0(\mathbb{R},\mathbb{C}) = \{f \in BC(\mathbb{R},\mathbb{C}) \mid f(t) \to 0 \text{ as } t \to +\infty\}$, $C_{00}(\mathbb{R},\mathbb{C}) = \{f \in BC_0(\mathbb{R},\mathbb{C}) \mid f(t) \to 0 \text{ as } t \to -\infty\}$, and $L_0^p = \{f : \mathbb{R} \to \mathbb{C} \mid \int_0^\infty |f|^p < \infty\}$. In other words, from Lemma 1 we have that $\mathcal{AP}(\mathbb{R},\mathbb{C}) \cap BC_0(\mathbb{R},\mathbb{C}) = \{0\}$ and $\mathcal{AP}(\mathbb{R},\mathbb{C}) \cap L_0^p = \{0\}$. Thus, we set

Definition 2. *c*) A bounded continuous function $f : \mathbb{R} \to \mathbb{C}$ is *asymptotically almost periodic* if $f = \phi + g$ with $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $\lim_{t\to+\infty} g(t) = 0$. The set of asymptotically almost periodic functions from $\mathbb{R}$ to $\mathbb{C}$ will be denoted by $\mathcal{AAP}(\mathbb{R},\mathbb{C})$. In addition, if $\lim_{t\to-\infty} g(t) = 0$, we will say $f \in \mathcal{AAP}_0(\mathbb{R},\mathbb{C})$. See [5, 10].

d) A bounded continuous function $f : \mathbb{R} \to \mathbb{C}$ is *p-almost periodic* with $p \geq 1$, if $f = \phi + g$ with $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $g \in L_0^p$. The set of all p-almost periodic functions from $\mathbb{R}$ to $\mathbb{C}$ will be denoted by $\mathcal{AP}(\mathbb{R},\mathbb{C},p)$. In addition, if $g \in L^p(\mathbb{R})$, we will say $f \in \mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$. See [10, section 4.3; page 46].

e) A bounded continuous function $f : \mathbb{R}\times\mathbb{C} \to \mathbb{C}$ satisfied $f \in \mathcal{AAP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ if $f = \phi + g$ with $\phi \in \mathcal{AP}(\mathbb{R}\times\mathbb{C})$ and $\lim_{t\to+\infty} g(t,x) = 0$ uniformly for x on compact subsets of $\mathbb{C}$. In addition, if $\lim_{t\to-\infty} g(t,x) = 0$ uniformly for x on compact subsets of $\mathbb{C}$, we will say $f \in \mathcal{AAP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C})$.

f) A bounded continuous function $f : \mathbb{R}\times\mathbb{C} \to \mathbb{C}$ satisfied $f \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C},p)$ if $f = \phi + g$ with $\phi \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ and $g(\cdot,x) \in L_0^p$ uniformly for x on compact subsets of $\mathbb{C}$. In addition, if $g(\cdot,x) \in L^p(\mathbb{R})$ uniformly for x on compact subsets of $\mathbb{C}$, we will say $f \in \mathcal{AP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C},p)$.

Remark 1. It follows that $\mathcal{AP}(\mathbb{R},\mathbb{C}) \subset \mathcal{AAP}_0(\mathbb{R},\mathbb{C}) \subset \mathcal{AAP}(\mathbb{R},\mathbb{C}) \subset BC(\mathbb{R},\mathbb{C})$ and the direct sums $\mathcal{AAP}(\mathbb{R},\mathbb{C}) = \mathcal{AP}(\mathbb{R},\mathbb{C}) \oplus BC_0(\mathbb{R},\mathbb{C})$ and $\mathcal{AAP}_0(\mathbb{R},\mathbb{C}) = \mathcal{AP}(\mathbb{R},\mathbb{C}) \oplus C_{00}(\mathbb{R},\mathbb{C})$. Furthermore, $\mathcal{AP}(\mathbb{R},\mathbb{C})$, $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$ and $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ are closed subspaces of the Banach space $BC(\mathbb{R},\mathbb{C})$ endowed with supremum norm $\|\cdot\|_\infty$. Also, $\mathcal{AP}(\mathbb{R},\mathbb{C},p) = \mathcal{AP}(\mathbb{R},\mathbb{C}) \oplus [L_0^p \cap BC(\mathbb{R},\mathbb{C})]$ and $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p) = \mathcal{AP}(\mathbb{R},\mathbb{C}) \oplus [L^p(\mathbb{R}) \cap BC(\mathbb{R},\mathbb{C})]$. Notice that $f = \phi + g$ with $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $g \in L_0^p$ then $g|_{[t_0,+\infty)} \in L^p[t_0,+\infty)$ for any $t_0 \in \mathbb{R}$.

2.2 Green's type operators

Given $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$ and $\alpha > 0$, we define the operators acting on a function $r : \mathbb{R} \to \mathbb{C}$ as follows

$$G_{\pm}^{\gamma}[r](t) = -\int_t^{\mp\infty} e^{\mp\gamma(t-s)}\, r(s)\, ds \qquad \text{and} \qquad \mathcal{L}_{\pm}^{\alpha}[r](t) = \mp\int_t^{\mp\infty} e^{\mp\alpha(t-s)}\, |r(s)|\, ds.$$

The Green's operator $G_{\pm}^{\gamma}$ and $\mathcal{L}_{\pm}^{\alpha}$ allow us to state our results and do satisfy the following useful inequality.

Lemma 2 ([17]). *If $\beta < \alpha$ then*

$$\left| G_{\pm}^{\alpha}\left[b\, G_{\pm}^{\beta}[a]\right](t) \right| \le \mathcal{L}_{\pm}^{\alpha}\left[b\mathcal{L}_{\pm}^{\beta}[a]\right](t) \le \mathcal{L}_{\pm}^{\alpha-\beta}[b](t)\mathcal{L}_{\pm}^{\beta}[a](t). \tag{3}$$

The following inequality implies that the map $(0,+\infty) \ni \alpha \mapsto G_{\pm}^{\alpha}[r] \in BC(\mathbb{R},\mathbb{C})$ is a continuous function for $r \in BC(\mathbb{R},\mathbb{C})$ fixed.

Lemma 3. *Let $r \in BC(\mathbb{R},\mathbb{C})$ fixed. For all $\alpha, \beta > 0$ it holds*

$$\left\| G_{\pm}^{\alpha}[r] - G_{\pm}^{\beta}[r] \right\|_\infty \le \|r\|_\infty \left| \frac{1}{\alpha} - \frac{1}{\beta} \right|.$$

Notice that the same inequality still holds true for the operators $\mathcal{L}_{\pm}^{\alpha}$ and $\mathcal{L}_{\pm}^{\beta}$ with any function $r \in BC(\mathbb{R},\mathbb{C})$, since $\mathcal{L}_{\pm}^{\alpha}[r] = \pm G_{\pm}^{\alpha}\left[|r|\right]$.

Proof: Without loss of generality, assume that $\alpha < \beta$. Hence, it follows that

$$\left| G_{\pm}^{\alpha}[r](t) - G_{\pm}^{\beta}[r](t) \right| \le \|r\|_\infty \left| \int_t^{\mp\infty} \left| e^{\mp\alpha(t-s)} - e^{\mp\beta(t-s)} \right| ds \right| = \|r\|_\infty \left| \frac{1}{\alpha} - \frac{1}{\beta} \right|$$

for all $t \in \mathbb{R}$, in view of

$$\int_t^{\mp\infty} \left| e^{\mp\alpha(t-s)} - e^{\mp\beta(t-s)} \right| ds = \int_t^{\mp\infty} \left[e^{\mp\alpha(t-s)} - e^{\mp\beta(t-s)} \right] ds = \mp\left(\frac{1}{\alpha} - \frac{1}{\beta} \right).$$

Therefore, the conclusion follows. ∎

Let us recall the following result concerning the Green's operators which has been useful in asymptotic integration, see [2, 6, 7, 12, 20].

Lemma 4. *Let $\gamma \in \mathbb{C}$, $\alpha = \operatorname{Re}\gamma > 0$ and let $r \in BC(\mathbb{R},\mathbb{C})$ be a bounded continuous function. If $r \in BC_0(\mathbb{R},\mathbb{C})$ (resp. $C_{00}(\mathbb{R},\mathbb{C})$) then $\mathcal{L}^{\alpha}_{\pm}[r] \in BC_0(\mathbb{R},\mathbb{C})$ (resp. $C_{00}(\mathbb{R},\mathbb{C})$). If $r \in L^p_0$ (resp. $L^p(\mathbb{R})$) for some $p \geq 1$ then $\mathcal{L}^{\alpha}_{\pm}[r] \in BC_0(\mathbb{R},\mathbb{C}) \cap L^p_0$ (resp. $C_{00}(\mathbb{R},\mathbb{C}) \cap L^p(\mathbb{R})$). Similarly, if $r \in L^p(-\infty,0]$ for some $p \geq 1$ then $\mathcal{L}^{\alpha}_{\pm}[r](t) \to 0$ as $t \to -\infty$ and $\mathcal{L}^{\alpha}_{\pm}[r] \in L^p(-\infty,0]$.*

Notice that $G^{\gamma}_{\pm}[r]$ is a solution of the linear equation $y' = \mp\gamma y + r$. In order to study linear equations with $r \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and nonlinear perturbations of them we have the following invariance property of the Green's operator.

Lemma 5. *Let $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 0$. It holds $G^{\gamma}_{\pm} : E \to E$, where either $E = \mathcal{AP}(\mathbb{R},\mathbb{C})$, $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$. Similarly, if $r \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $r \in \mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$) then $G^{\gamma}_{\pm}[r] \in \mathcal{AP}(\mathbb{R},\mathbb{C},p) \cap \mathcal{AAP}(\mathbb{R},\mathbb{C})$ (resp. $G^{\gamma}_{\pm}[r] \in \mathcal{AP}_0(\mathbb{R},\mathbb{C},p) \cap \mathcal{AAP}_0(\mathbb{R},\mathbb{C})$).*

Proof: First, note that for any $\tau \in \mathbb{R}$ we have that

$$G^{\gamma}_{\pm}[r](t+\tau) = -\int_t^{\mp\infty} e^{\mp\gamma(t-s)} r(s+\tau)\,ds = G^{\gamma}_{\pm}[r(\cdot+\tau)](t).$$

On the other hand, let $\{\alpha'_n\}_n \subset \mathbb{R}$ be a sequence. Then, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that $\lim_{n\to+\infty} r(t+\alpha_n)$ exists uniformly on the real line. Denote for every $t \in \mathbb{R}$, $r^*(t) = \lim_{n\to+\infty} r(t+\alpha_n)$. Hence, we find that

$$\begin{aligned}&\left|G^{\gamma}_{\pm}[r](t+\alpha_n) - G^{\gamma}_{\pm}[r^*](t)\right| \\ &\qquad \leq \pm\int_t^{\pm\infty} e^{\pm\alpha(t-s)} |r^*(s) - r(s+\alpha_n)|\,ds \leq \frac{1}{\alpha}\sup_{\mathbb{R}} |r^* - r(\cdot+\alpha_n)|,\end{aligned}$$

where $\alpha = \operatorname{Re}\gamma$. Therefore, $\lim_{n\to+\infty} G^{\gamma}_{\pm}[r](t+\alpha_n) = G^{\gamma}_{\pm}[r^*](t)$ uniformly for $t \in \mathbb{R}$.

Finally, it is enough to observe that if $f = \phi + g$ with $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ then we obtain that $G^{\gamma}_{\pm}[\phi] \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $G^{\gamma}_{\pm}[f] = G^{\gamma}_{\pm}[\phi] + G^{\gamma}_{\pm}[g]$. Conclusions follows from the Definition 2 and Lemma 4. ■

Remark 2. Notice that it is also possible to consider $E = \mathcal{PAP}(\mathbb{R},\mathbb{C})$, $\mathcal{AA}(\mathbb{R},\mathbb{C})$, $\mathcal{AAA}(\mathbb{R},\mathbb{C})$ or $\mathcal{PAA}(\mathbb{R},\mathbb{C})$, namely, the sets of pseudo-almost periodic functions, almost automorphic functions, asymptotically almost automorphic functions and pseudo-almost automorphic functions respectively, see for instance [4, 5, 9, 10, 24, 27, 30, 33].

2.3 Riccati equation

In order to study equation (2), we consider a new variable $z = (y'/y) - \lambda$, where $\lambda \in \mathbb{C}$ is a root of P. We will find such a function z with property $z \in \mathcal{AP}(\mathbb{R},\mathbb{C})$, when $r_i \in \mathcal{AP}(\mathbb{R},\mathbb{C})$, $i = 1,2$ are sufficiently small. We will need the following results, see [2, 12].

Lemma 6. *Suppose that the roots $\lambda_\pm$ of P are distinct. Then there are two solutions $y_\pm$ of* (2)*, such that*

$$y_\pm(t) = \exp\left(\int_0^t [\lambda_\pm + z_\pm(s)]\, ds \right), \qquad t \in \mathbb{R} \tag{4}$$

where $z_\pm$, satisfy

$$z'_\pm = -(r_0(t) + \lambda_\pm r_1(t)) \mp \gamma z_\pm - r_1(t) z_\pm - z_\pm^2, \quad \gamma = \lambda_+ - \lambda_-. \tag{5}$$

Let us stress that if we know a solution to Ricatti equation (5) then we find a solution to equation (2). This transformation reduces the order of the equation, avoiding the usual diagonalization process. Since Riccati equations are nonlinear, we present the results in a more general way. Consider the scalar differential equation

$$z' \pm \gamma z = a(t) + f(t,z), \tag{6}$$

where $\alpha = \operatorname{Re}\gamma > 0$. In order to study the existence of solution to (6) we will need the following fact.

Lemma 7. *Assume that $f \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$). If $z \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$) then $f(\cdot,z(\cdot)) \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$).*

This is also a straightforward fact and again for the sake of completeness we shall present a proof here.

Proof: Denote $u(t) = f(t,z(t))$. Given a sequence $\{\alpha'_n\} \subset \mathbb{R}$, let $\{\alpha_n\} \subset \{\alpha'_n\}$ a subsequence such that $\{z(\cdot+\alpha_n)\}_n$ converges to z^* uniformly on $\mathbb{R}$ and $f(t+\alpha_n,x)$ converges uniformly for t in $\mathbb{R}$ and x on compact subsets of $\mathbb{C}$. Denote $g(t,x) = \lim_{n\to+\infty} f(t+\alpha_n,x)$, so that for every $M > 0$

$$\lim_{n\to\infty} \sup_{|x|\le M} |f(t+\alpha_n,x) - g(t,x)| = 0, \qquad \text{uniformly for } t \in \mathbb{R}.$$

Then, by choosing $M > 0$ such that $\|z\|_\infty \le M$, it is clear that

$$|u(t+\alpha_n) - g(t,z(t+\alpha_n))| \le \sup_{|x|\le M} |f(t+\alpha_n,x) - g(t,x)|$$

implies that $u(\cdot+\alpha_n)$ converges uniformly to $g(\cdot,z^*)$ and we conclude that $u \in \mathcal{AP}(\mathbb{R},\mathbb{C})$.

Now, assume that $f = \psi + h \in \mathcal{AAP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ (resp. $\mathcal{AAP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C})$) with $\psi \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ and $\lim_{t\to\infty} h(t,x) = 0$ (resp. as $t \to -\infty$) uniformly for x on compact subsets of $\mathbb{C}$. If $z = \phi + g \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ (resp. $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$) with $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $g \in BC_0(\mathbb{R},\mathbb{C})$ (resp. $C_{00}(\mathbb{R},\mathbb{C})$) then $\psi(\cdot,\phi(\cdot)) \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and

$$f(t,z(t)) = \psi(t,\phi(t)) + \Big[\psi\big(t,\phi(t)+g(t)\big) - \psi(t,\phi(t))\Big] + h\big(t,\phi(t)+g(t)\big).$$

Hence, it is clear that $\lim_{t\to\infty} h\big(t,\phi(t)+g(t)\big)=0$, in view of $\lim_{t\to\infty}\sup_{|x|\le M}|h\big(t,x\big)|=0$, (resp. as $t\to-\infty$) with $M=\|\phi+g\|_\infty$, and also, it follows $\lim_{t\to\infty}\Big[\psi\big(t,\phi(t)+g(t)\big)-\psi(t,\phi(t))\Big]=0$, since there exists an increasing sequence $\{n_k\}_k$ and two functions ϕ^* and ψ^* such that $\phi(\cdot+n_k)$ converges uniformly in $\mathbb{R}$ to ϕ^* and $\psi(\cdot+n_k,x)$ converges uniformly in $\mathbb{R}$ for x on compact subsets of $\mathbb{C}$ to $\psi^*(\cdot,x)$, so that

$$\lim_{k\to\infty}\Big[\psi\big(t+n_k,\phi(t+n_k)+g(t+n_k)\big)-\psi\big(t+n_k,\phi(t+n_k)\big)\Big]=$$
$$\psi^*\big(t,\phi^*(t)\big)-\psi^*\big(t,\phi^*(t)\big)=0. \quad \blacksquare$$

Given a function $f=f(t,x)$, we shall assume that for some constant $M>0$ there exists a bounded function $\xi_M:\mathbb{R}\to[0,+\infty)$ such that for all $t\in\mathbb{R}$ and $|z_i|\le M$, $i=1,2$

$$|f(t,z_1)-f(t,z_2)|\le\xi_M(t)|z_1-z_2|. \tag{7}$$

Lemma 8. *Suppose that $a\in\mathcal{AP}(\mathbb{R},\mathbb{C})$ (resp. $a\in\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $a\in\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$), $f(\cdot,0)=0$ and $f\in\mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C})$). If f satisfies (7), $\|\mathcal{L}^\alpha_\pm[\xi_M]\|_\infty\le\varepsilon_0<1$ and $\|G^\gamma_\pm[a]\|_\infty\le(1-\varepsilon_0)M$ for some $M>0$, then there is a solution $z=z_\pm$ of (6) such that $z\in\mathcal{AP}(\mathbb{R},\mathbb{C})$ (resp. $z\in\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $z\in\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$), is a solution of the integral equation*

$$z=G^\gamma_\pm[a+f(\cdot,z)]. \tag{8}$$

Moreover, for some $0<\beta<\alpha$, $z_\pm$ satisfy the estimate

$$z_\pm=O\left(\mathcal{L}^\beta_\pm[a]\right). \tag{9}$$

Proof: Consider the space $\mathcal{AP}(\mathbb{R},\mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$) which is a Banach space with the norm $\|z\|_\infty=\sup\{|z(t)|\mid t\in\mathbb{R}\}$. Then define the operators $T=T_\pm$ as

$$Tz(t)=-\int_t^{\pm\infty}e^{\pm\gamma(t-s)}\big[a(s)+f(s,z(s))\big]\,ds=G^\gamma_\pm[a+f(\cdot,z)](t).$$

Note that if $z\in B=\{g\in\mathcal{AP}(\mathbb{R},\mathbb{C})\mid\|g\|_\infty\le M\}$ then $f(\cdot,z(\cdot))\in\mathcal{AP}(\mathbb{R},\mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$ in case $z\in\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $z\in\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$), by using Lemma 7. In addition, by Lemma 5 we have that $G^\gamma_\pm[a]\in\mathcal{AP}(\mathbb{R},\mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$ in case $a\in\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $a\in\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$). Therefore, $T:B\to\mathcal{AP}(\mathbb{R},\mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$ in case $B\subset\mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $B\subset\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$). Now, if $z\in B$ then $|f(t,z)|\le\xi_M(t)|z|$ and thus

$$|Tz(t)|\le|G^\gamma_\pm[a](t)|+\mathcal{L}^\alpha_\pm[\xi_M z](t)\le M,$$

since $|G^\gamma_\pm[a](t)|\le(1-\varepsilon_0)M$ and $\mathcal{L}^\alpha_\pm[\xi_M](t)\le\varepsilon_0$ for all $t\in\mathbb{R}$. Hence, $T:B\to B$ is a contractive operator in view of

$$|Tz_1(t)-Tz_2(t)|\le\mathcal{L}^\alpha_\pm[\xi_M](t)\|z_1-z_2\|_\infty\le\varepsilon_0\|z_1-z_2\|_\infty.$$

So, there exists a unique $z \in B$ such that $Tz = z$. Thus, there is a solution z of (6) such that $z \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ (resp. $\mathcal{AAP}(\mathbb{R}, \mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R}, \mathbb{C})$) and satisfies (8).

In order to prove (9), let us take the sequence $\{z_n\}_{n \geq 0}$ given by $z_0 = 0$ and $z_{n+1} = Tz_n$ for $n \geq 0$. Since T is a contractive operator then as $n \to \infty$, there holds $z_n \to z$. By Lemma 3 there are constants $0 < \beta < \alpha$ and $\varepsilon_0 \leq K < 1$ such that $\|\mathcal{L}_{\pm}^{\alpha-\beta}[\xi_M]\|_\infty \leq K$. We choose N such that $N \geq (1-K)^{-1}$, so that $1 + KN \leq N$. Now, we will prove that for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$

$$|z_n(t)| \leq N\mathcal{L}_{\pm}^{\beta}[a](t). \tag{10}$$

The induction in (10) is clear for $n = 0, 1$. Suppose that (10) is true for $n = k$. So, for $n = k+1$ we get that by using Lemma 2

$$\begin{aligned} |z_{k+1}(t)| &\leq \mathcal{L}_{\pm}^{\alpha}[a](t) + \mathcal{L}_{\pm}^{\alpha}[\xi_M |z_k|](t) \leq \mathcal{L}_{\pm}^{\alpha}[a](t) + N\mathcal{L}_{\pm}^{\alpha}\left[\xi_M \mathcal{L}_{\pm}^{\beta}[a]\right](t) \\ &\leq \mathcal{L}_{\pm}^{\alpha}[a](t) + N\mathcal{L}_{\pm}^{\alpha-\beta}[\xi_M](t)\mathcal{L}_{\pm}^{\beta}[a](t) \leq [1 + KN]\mathcal{L}_{\pm}^{\beta}[a](t). \end{aligned}$$

Therefore, (10) is true and the Lemma follows. ∎

Remark 3. Observe that similar results hold if $a \in V$, where V is an invariant closed subspace of $BC(\mathbb{R}, \mathbb{C})$ under the operator T. Thus, there is a solution $z = z_{\pm}$ of (6) and (8) such that $z \in B \cap V$, where $B = \{g \in BC(\mathbb{R}, \mathbb{C}) \mid \|g\|_\infty \leq M\}$. However, notice that $L_0^p \cap BC(\mathbb{R}, \mathbb{C})$ and $L^p(\mathbb{R}) \cap BC(\mathbb{R}, \mathbb{C})$ are not closed subspaces of $BC(\mathbb{R}, \mathbb{C})$, so that, we cannot obtain directly a version of Lemma 8 in the subspaces $\mathcal{AP}(\mathbb{R}, \mathbb{C}, p)$ or $\mathcal{AP}_0(\mathbb{R}, \mathbb{C}, p)$. Despite of this loss of completeness, we shall find such solutions in $\mathcal{AP}(\mathbb{R}, \mathbb{C}, p)$ or $\mathcal{AP}_0(\mathbb{R}, \mathbb{C}, p)$ by exploiting a decomposition property.

Under the same assumptions of the previous Lemma, assume that $a = \phi + g \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$ with $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, $g \in BC_0(\mathbb{R}, \mathbb{C})$ and $f = \varphi + h \in \mathcal{AAP}(\mathbb{R} \times \mathbb{C}, \mathbb{C})$ with $\varphi \in \mathcal{AP}(\mathbb{R} \times \mathbb{C}, \mathbb{C})$ and $h(\cdot, x) \in BC_0(\mathbb{R}, \mathbb{C})$ uniformly for x in compact subsets of $\mathbb{C}$. Then, the predicted solution by the previous Lemma $z = z_{\pm}$ to (6) and (8) satisfy $z \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$. Moreover, it holds that $z = \theta + \psi$ with $\theta \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ and $\psi \in BC_0(\mathbb{R}, \mathbb{C})$, where

$$\theta = G_{\pm}^{\gamma}[\phi + \varphi(\cdot, \theta)] \qquad \text{and} \qquad \psi = G_{\pm}^{\gamma}[g + h(\cdot, \theta) + f(\cdot, \theta + \psi) - f(\cdot, \theta)]. \tag{11}$$

In fact, we know that $z = \theta + \psi \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$ for some functions $\theta \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ and $\psi(t) \to 0$ as $t \to \infty$. Replacing in (8) we get that

$$\theta + \psi = G_{\pm}^{\gamma}[\phi + \varphi(\cdot, \theta)] + G_{\pm}^{\gamma}[g + f(\cdot, \theta + \psi) - \varphi(\cdot, \theta)]$$

and from the assumptions on ϕ, g, φ and h and Lemmata 4, 5 and 7 we find that

$$\mathcal{AP}(\mathbb{R}, \mathbb{C}) \ni \theta - G_{\pm}^{\gamma}[\phi + \varphi(\cdot, \theta)] = -\psi + G_{\pm}^{\gamma}[g + f(\cdot, \theta + \psi) - \varphi(\cdot, \theta)] \in BC_0(\mathbb{R}, \mathbb{C}).$$

Thus, we conclude (11), in view of $\mathcal{AAP}(\mathbb{R}, \mathbb{C}) = \mathcal{AP}(\mathbb{R}, \mathbb{C}) \oplus BC_0(\mathbb{R})$. More precisely, we have that the decomposition in a sum of a and f induces the direct sum of the solution z and the equation (8) in the direct sum (11). We have just proven the following

Corollary 1. *Suppose that $a \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ (resp. $a \in \mathcal{AAP}_0(\mathbb{R},\mathbb{C})$), $f(\cdot,0) = 0$ and $f \in \mathcal{AAP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ (resp. $\mathcal{AAP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C})$). If f satisfies* (7), $\|\mathcal{L}^{\alpha}_{\pm}[\xi_M]\|_\infty \le \varepsilon_0 < 1$ *and* $\|G^{\gamma}_{\pm}[a]\|_\infty \le (1-\varepsilon_0)M$ *for some $M > 0$, then there is a solution $z = z_\pm$ of* (6) *such that $z \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ (resp. $z \in \mathcal{AAP}_0(\mathbb{R},\mathbb{C})$), is a solution of the integral equation* (8), *satisfies the estimate* (9) *and $z = \theta + \psi$ with $\theta \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $\psi \in BC_0(\mathbb{R},\mathbb{C})$ (resp. $\psi \in C_{00}(\mathbb{R},\mathbb{C})$)), where θ and ψ satisfy* (11).

Now, we present a result which will be useful to study the problem (5) when a belongs to this new class of almost periodic functions $\mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$) given in Definition 2 and f is a suitable function that allows us to find solutions to (5) in this class of functions.

Lemma 9. *Assume that $f \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ and satisfies* (7). *If $z \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $z \in \mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$) then $f(\cdot,z(\cdot)) \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $f(\cdot,z(\cdot)) \in \mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$). Furthermore, if $f = \varphi + h \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C},p)$ (resp. $f \in \mathcal{AP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C},p)$) with $\varphi \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ satisfying* (7) *as above and $h(\cdot,x) \in L^p_0$ (resp. $L^p(\mathbb{R})$) uniformly for x in compact subsets of $\mathbb{C}$ and $z \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$), then $f(\cdot,z(\cdot)) \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$).*

Proof: Let $z = \phi + g \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$) with $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $g \in L^p_0$ (resp. $L^p(\mathbb{R})$) then for $M = \max\{\|z\|_\infty, \|\phi\|_\infty\}$, $f(\cdot,\phi(\cdot)) \in \mathcal{AP}(\mathbb{R},\mathbb{C})$, $\xi_M g \in L^p_0$, so that

$$f(t,z(t)) = f(t,\phi(t)+g(t)) - f(t,\phi(t)) + f(t,\phi(t)) = O(\xi_M(t)|g(t)|) + f(t,\phi(t))$$

and the conclusions follows. Now, if $f = \varphi + h \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C},p)$ (resp. $f \in \mathcal{AP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C},p)$) then $\varphi(\cdot,\phi(\cdot)+g(\cdot)) \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$) as above and $h(\cdot,\phi(\cdot)+g(\cdot)) \in L^p_0$ (resp. $L^p(\mathbb{R})$), in view of $\sup\limits_{|x|\le M} |h(\cdot,x)| \in L^p_0$ (resp. $L^p(\mathbb{R})$) with $M = \|z\|_\infty$. ■

In order to state the next result, let us denote for a suitable given function θ,

$$\widetilde{f}_\theta(t,w) = f(t,\theta+w) - f(t,\theta). \tag{12}$$

Moreover, assume that $\widetilde{f}_\theta$ satisfies (7) with $\zeta_M = \zeta_M(t)$ instead of ξ_M for some $M > 0$.

Lemma 10. *Suppose that $a = \phi + g \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ (resp. $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$) with $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $g \in L^p_0$ (resp. $L^p(\mathbb{R})$), $f(\cdot,0) = 0$ and $f = \varphi + h \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C},p)$ (resp. $\mathcal{AP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C},p)$) with $\varphi \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ and $h(\cdot,x) \in L^p_0$ (resp. $L^p(\mathbb{R})$) uniformly for x in compact subsets of $\mathbb{C}$, so that, there is a solution $\theta = \theta_\pm \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ of the integral equation*

$$\theta = G^{\gamma}_{\pm}[\phi + \varphi(\cdot,\theta)]. \tag{13}$$

If $\|\mathcal{L}^{\alpha}_{\pm}[\zeta_M]\|_\infty \le \varepsilon_0 < 1$ and $\|G^{\gamma}_{\pm}[g + h(\cdot,\theta)]\|_\infty \le (1-\varepsilon_0)M$, then there is a solution $w = w_\pm$ of the integral equation

$$w = G^{\gamma}_{\pm}[g + h(\cdot,\theta) + \widetilde{f}_\theta(\cdot,w)] \tag{14}$$

where $\widetilde{f}_\theta$ *is given by* (12) *and* θ *satisfies* (13), *satisfying for some* $0 < \beta < \alpha$ *the estimate*

$$w_\pm = O\left(\mathcal{L}^\beta_\pm[g + h(\cdot,\theta)]\right) \qquad \textit{and} \qquad w_\pm \in L^p_0 \quad \textit{(resp. } L^p(\mathbb{R})\textit{)}. \tag{15}$$

Conversely, if $\theta \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ *and* $w \in L^p_0$ *(resp.* $L^p(\mathbb{R})$*) satisfy* (13) *and* (14) *respectively, then* $z_\pm = \theta_\pm + w_\pm \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$ *(resp.* $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$*) is a solution to* (6) *and* (8) *with* $a = \phi + g$ *and* $f = \varphi + h$.

Proof: Similarly to the proof of Lemma 8, for $w \in BC(\mathbb{R})$ define the operator T_θ as

$$T_\theta w(t) = G^\gamma_\pm[g + h(\cdot,\theta) + f(\cdot,\theta + w) - f(\cdot,\theta)](t).$$

By the assumptions it is clear that $T_\theta : B \to B$ is a contractive operator, where $B = \{z \in BC(\mathbb{R}) \mid \|z\| \leq M\}$. So, there exists a unique $w \in B$ such that $T_\theta w = w$ satisfying the estimate in (15). Hence, it is clear that $w \in L^p_0$ (resp. $L^p(\mathbb{R})$) by Lemma 4. ∎

Remark 4. Let us stress that the direct sum of a and f and the existence of the "almost periodic part" $\theta_\pm$ induce the direct sum of the solution z and the equation (8) in the direct sum (13) with (14), in these new classes of functions $\mathcal{AP}(\mathbb{R},\mathbb{C},p)$ or $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$. Furthermore, notice that we are not using that g and $h(\cdot,\theta)$ are bounded continuous functions, but we need $G^\gamma_\pm[g + h(\cdot,\theta)] \in BC(\mathbb{R},\mathbb{C})$. Notice that an analogous version of Lemma 10 could be also obtained for $a \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$ and $f \in \mathcal{AAP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ or $\mathcal{AAP}_0(\mathbb{R}\times\mathbb{C},\mathbb{C})$. In other words, we can study equation (13) in $\mathcal{AP}(\mathbb{R},\mathbb{C})$ and then equation (14) in either $BC_0(\mathbb{R},\mathbb{C})$ or $C_{00}(\mathbb{R})$.

3 Main Results

In this section we shall present the main results concerning the linear second order differential equation (2). Recall that $\lambda_\pm$ are the roots of the polynomial $P(\lambda) = \lambda^2 + a_1\lambda + a_0$, $\gamma = \lambda_+ - \lambda_-$. Without loss of generality, we shall assume that $\operatorname{Re}\gamma > 0$. Let us notice that the Green's operator satisfies

$$\int_0^t G^\gamma_\pm[r](s)\,ds = \pm\frac{1}{\gamma}\int_0^t r(s)\,ds \mp \frac{1}{\gamma}\left(G^\gamma_\pm[r](t) - G^\gamma_\pm[r](0)\right). \tag{16}$$

Theorem 1. *Consider equation* (2) *with* $r_i \in \mathcal{AP}(\mathbb{R},\mathbb{C})$, $i = 0,1$. *Assume that*

$$\sqrt{\frac{8}{\operatorname{Re}\gamma}\left\|G^\gamma_\pm[r_0 + \lambda_\pm r_1]\right\|_\infty} + \left\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1]\right\|_\infty < 1. \tag{17}$$

Then there is a fundamental system of solutions $y_\pm : \mathbb{R} \to \mathbb{C}$ *to* (2) *such that*

$$y_\pm(t) = e^{\lambda_\pm t}\exp\left(\mp\frac{1}{\gamma}\int_0^t [r_0(s) + \lambda_\pm r_1(s) + r_1(s)z_\pm(s) + z^2_\pm(s)]\,ds \mp \frac{1}{\gamma}z_\pm(t)\right) \tag{18}$$

and for the derivative

$$y'_\pm(t) = (\lambda_\pm + z_\pm(t))y_\pm(t), \tag{19}$$

where $z_\pm \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ *satisfy the differential equation* (5), *and integral equation*

$$z_\pm = -G^\gamma_\pm[r_0 + \lambda_\pm r_1 + r_1 z_\pm + z^2_\pm] \qquad \textit{with} \qquad z_\pm = O(\mathcal{L}^\alpha_\pm[r_0 + \lambda_\pm r_1]) \tag{20}$$

for some $0 < \alpha < \operatorname{Re}\gamma$.

Proof: In view of Lemma 6, we apply Lemma 8 to equations (5). We take $-a = r_0 + \lambda_\pm r_1 \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and $f(t,z) = -[r_1 z + z^2]$, so that $f \in \mathcal{AP}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ and we choose $\xi_M = |r_1| + 2M$, with $M > 0$ given by

$$M = \frac{\operatorname{Re}\gamma}{4}\left(1 - \big\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1]\big\|_\infty + \sqrt{\left(1 - \big\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1]\big\|_\infty\right)^2 - \frac{8}{\operatorname{Re}\gamma}\big\|G^\gamma_\pm[r_0 + \lambda_\pm r_1]\big\|_\infty}\right).$$

Let us observe that by (17), M is well defined and positive. Thus, we have that

$$M < \frac{\operatorname{Re}\gamma}{2}\left(1 - \big\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1]\big\|_\infty\right) \quad \text{and choosing} \quad \varepsilon_0 = \|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1]\|_\infty + \frac{2M}{\operatorname{Re}\gamma}$$

we find that $\|G^\gamma_\pm[r_0 + \lambda_\pm r_1]\|_\infty \le (1-\varepsilon_0)M$ and $\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[\xi_M]\|_\infty \le \varepsilon_0 < 1$, so that, the assumptions of Lemma 8 are satisfied. Hence, there are $z = z_\pm$ satisfying (20) for some $0 < \alpha < \operatorname{Re}\gamma$. Notice that it holds that $\|z\|_\infty \le M$. Therefore, there are two solutions to (2) of the form (4) with $z = z_\pm$ satisfying (20). By using the integral equation in (20), the identity (16) and dividing by $e^{\mp\frac{1}{\gamma}z_\pm(0)}$ it follows (18). Also, (19) follows from (4).

Now, we show that y_+ and y_- are linearly independent by computing its Wronskian at $t = 0$. It is readily checked that $W[y_+, y_-](0) = -[\lambda_+ - \lambda_- + z_+(0) - z_-(0)]$. Hence, if $W[y_+, y_-](0) = 0$ then $|\lambda_+ - \lambda_-| = |z_+(0) - z_-(0)|$ and we get the following contradiction by using the choice of M, $|z_+(0) - z_-(0)| \le 2M < \operatorname{Re}\gamma \le |\lambda_+ - \lambda_-|$. Therefore, it follows that $W[y_+, y_-](0) \ne 0$. This completes the proof. ■

Remark 5. Since for any $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$ the Green's operators $G^\gamma_\pm$ satisfy $G^\gamma_\pm : BC(\mathbb{R},\mathbb{C}) \to BC(\mathbb{R},\mathbb{C})$, Theorem 1 can be established in the class of functions $BC(\mathbb{R},\mathbb{C})$, and sufficient conditions easier to check than (17) are also true. Indeed, for instance if $r_i \in BC(\mathbb{R},\mathbb{C})$ for $i = 0, 1$ and

$$\sqrt{8\|r_0 + \lambda_\pm r_1\|_\infty} + \|r_1\|_\infty < \operatorname{Re}(\lambda_+ - \lambda_-) \tag{21}$$

then (17) holds, since we find that

$$\sqrt{\frac{8\big\|G^\gamma_\pm[r_0 + \lambda_\pm r_1]\big\|_\infty}{\operatorname{Re}\gamma}} + \big\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1]\big\|_\infty \le \sqrt{\frac{8\|r_0 + \lambda_\pm r_1\|_\infty}{(\operatorname{Re}\gamma)^2}} + \frac{\|r_1\|_\infty}{\operatorname{Re}\gamma} < 1.$$

Remark 6. Define for $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$

$$\mathcal{D}(G^\gamma_\pm) = \{r : \mathbb{R} \to \mathbb{C} \mid G^\gamma_\pm[r](t) \text{ is well-defined for all } t \in \mathbb{R}\}$$

and for $\alpha > 0$

$$\mathcal{D}(\mathcal{L}_{\pm}^{\alpha}) = \{r : \mathbb{R} \to \mathbb{C} \mid \mathcal{L}_{\pm}^{\alpha}[r](t) \text{ is well-defined for all } t \in \mathbb{R}\}.$$

Notice that if $0 < \beta < \alpha$ then $\mathcal{D}(\mathcal{L}_{\pm}^{\beta}) \subset \mathcal{D}(\mathcal{L}_{\pm}^{\alpha})$. Furthermore, $BC(\mathbb{R}, \mathbb{C}) \subset \mathcal{D}(\mathcal{L}_{\pm}^{\operatorname{Re}\gamma}) \subset \mathcal{D}(G_{\pm}^{\gamma})$ and $L^p(\mathbb{R}) \subset \mathcal{D}(\mathcal{L}_{\pm}^{\alpha})$ for all $\alpha > 0$. Let us notice that Theorem 1 is also true if $r_i \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, $i = 0, 1$ is replaced by $r_1 \in \mathcal{D}(\mathcal{L}_{\pm}^{\operatorname{Re}\gamma - \alpha})$, $r_0 + \lambda_{\pm} r_1 \in \mathcal{D}(\mathcal{L}_{\pm}^{\alpha})$ for some $0 < \alpha < \operatorname{Re}\gamma$ and (17) is replaced by

$$\sqrt{\frac{8\left\|G_{\pm}^{\gamma}[r_0 + \lambda_{\pm} r_1]\right\|_{\infty}}{\operatorname{Re}\gamma - \alpha}} + \left\|\mathcal{L}_{\pm}^{\operatorname{Re}\gamma - \alpha}[r_1]\right\|_{\infty} < 1.$$

More precisely, (18)-(20) are true with $z_{\pm} \in BC(\mathbb{R}, \mathbb{C})$, instead of $z_{\pm} \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$. In other words, we just need integrability conditions to state our result. Furthermore, the map

$$\alpha \mapsto \left\|\mathcal{L}_{\pm}^{\operatorname{Re}\gamma - \alpha}[r_1]\right\|_{\infty} + \sqrt{\frac{8\left\|G_{\pm}^{\gamma}[r_0 + \lambda_{\pm} r_1]\right\|_{\infty}}{\operatorname{Re}\gamma - \alpha}}$$

is increasing in $\alpha \in [0, \operatorname{Re}\gamma)$ for fixed r_0, r_1.

Another kind of perturbations are $r_i \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$ for $i = 0, 1$. Let us recall that $\mathcal{AAP}(\mathbb{R}, \mathbb{C})$ is a closed subspace of $BC(\mathbb{R}, \mathbb{C})$. From the ideas used in previous Theorem and Corollary 1 it readily follows the next result. In particular, the decomposition of the coefficients r_i, $i = 0, 1$ induces the direct sum of the solution z. We omit its proof.

Theorem 2. *Consider equation* (2) *with* $r_i \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$ *(resp.* $\mathcal{AAP}_0(\mathbb{R}, \mathbb{C})$*),* $i = 0, 1$. *Assume that* (17) *holds. Then there is a fundamental system of solutions* $y_{\pm}$ *to* (2) *satisfying* (18) *and* (19)*, where* $z_{\pm} \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$ *(resp.* $\mathcal{AAP}_0(\mathbb{R}, \mathbb{C})$*) satisfy the differential equation* (5)*, the integral equation and the estimate in* (20) *for some* $0 < \alpha < \operatorname{Re}\gamma$. *Moreover, if* $r_i = \mu_i + \nu_i$, $i = 0, 1$ *with* $\mu_i \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ *and* $\nu_i \in BC_0(\mathbb{R}, \mathbb{C})$ *(resp.* $C_{00}(\mathbb{R}, \mathbb{C})$*) then* $z_{\pm} = \theta_{\pm} + \psi_{\pm}$*, where* $\theta_{\pm} \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ *and* $\psi_{\pm} \in BC_0(\mathbb{R}, \mathbb{C})$ *(resp.* $C_{00}(\mathbb{R}, \mathbb{C})$*) satisfy*

$$\theta_{\pm} = -G_{\pm}^{\gamma}[\mu_0 + \lambda_{\pm}\mu_1 + \mu_1\theta_{\pm} + \theta_{\pm}^2] \tag{22}$$

and

$$\psi_{\pm} = -G_{\pm}^{\gamma}[\nu_0 + \lambda_{\pm}\nu_1 + \nu_1\theta_{\pm} + (\mu_1 + \nu_1 + 2\theta_{\pm})\psi_{\pm} + \psi_{\pm}^2]. \tag{23}$$

Conversely, if $\theta_{\pm} \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ *and* $\psi_{\pm} \in BC_0(\mathbb{R}, \mathbb{C})$ *(resp.* $C_{00}(\mathbb{R}, \mathbb{C})$*) are solutions of the previous equations respectively, then* $z_{\pm} = \theta_{\pm} + \psi_{\pm} \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$ *(resp.* $\mathcal{AAP}_0(\mathbb{R}, \mathbb{C})$*) is a solution to* (5).

Notice that also, if $r_i \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$ (resp. $\mathcal{AAP}_0(\mathbb{R}, \mathbb{C})$), $i = 0, 1$ satisfy (21) then the Theorem 2 applies. In section 4, we shall see the advantages of studying separately equations (22) and (23).

A natural question is whether we could perturb $r_i \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, $i = 0, 1$, with L^p-functions in equation (2), see [16, 18]. In other words, equation (2) with

$r_i \in \mathcal{AP}(\mathbb{R},\mathbb{C},p)$, $i=0,1$. Let us stress that $L_0^p \cap BC(\mathbb{R},\mathbb{C})$ and $L^p(\mathbb{R}) \cap BC(\mathbb{R},\mathbb{C})$ are not closed subspaces of $BC(\mathbb{R},\mathbb{C})$, so that, we cannot obtain readily a version of Theorem 2 in the subspaces $\mathcal{AP}(\mathbb{R},\mathbb{C},p)$ or $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$. Despite of this loss of completeness, we shall find such solutions in $\mathcal{AP}(\mathbb{R},\mathbb{C},p)$ or $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$ by exploiting a decomposition property of the coefficients r_0 and r_1. More precisely, by using ideas presented in Theorem 1 (see also [16]) and Lemmata 4, 5, 9 and 10 we get the following result, which states again a decomposition property of $z_\pm$ solution to (5). In other words, we study equation (23), assuming the existence of a solution of (22). Notice that we can also consider $r_i \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ or $r_i \in \mathcal{AAP}_0(\mathbb{R},\mathbb{C})$, $i=0,1$, namely, we can study equation (22) in $\mathcal{AP}(\mathbb{R},\mathbb{C})$ and then equation (23) in either $BC_0(\mathbb{R},\mathbb{C})$ or $C_{00}(\mathbb{R})$. We also omit its proof.

Theorem 3. *Consider equation* (2) *with* $r_i = \mu_i + \nu_i \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ *(resp.* $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$, $\mathcal{AP}(\mathbb{R},\mathbb{C},p)$ *or* $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$*), with* $\mu_i \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ *and* $\nu_i \in BC_0(\mathbb{R},\mathbb{C})$ $i=0,1$ *(resp.* $C_{00}(\mathbb{R})$, L_0^p *or* $L^p(\mathbb{R})$*). Assume that there are solutions* $\theta_\pm \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ *for equations* (22), *so that*

$$\sqrt{\frac{8}{\operatorname{Re}\gamma}\left\|G_\pm^\gamma[\nu_0+\lambda_\pm\nu_1+\nu_1\theta_\pm]\right\|_\infty}+\left\|\mathcal{L}_\pm^{\operatorname{Re}\gamma}[r_1+2\theta_\pm]\right\|_\infty<1.$$

Then there is a fundamental system of solutions $y_\pm$ *to* (2) *satisfying* (18) *and* (19), *where* $z_\pm = \theta_\pm + \psi_\pm \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ *(resp.* $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$, $\mathcal{AP}(\mathbb{R},\mathbb{C},p)\cap\mathcal{AAP}(\mathbb{R},\mathbb{C})$ *or* $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)\cap\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$*), with* $\theta_\pm \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ *and* $\psi_\pm \in BC_0(\mathbb{R},\mathbb{C})$ *(resp. or* $C_{00}(\mathbb{R})$, $L_0^p \cap BC_0(\mathbb{R},\mathbb{C})$ *or* $L^p(\mathbb{R})\cap C_{00}(\mathbb{R},\mathbb{C})$*) satisfy* (23) *and for some* $0<\beta<\operatorname{Re}\gamma$

$$\psi_\pm = O(\mathcal{L}_\pm^\beta[\nu_0+\lambda_\pm\nu_1+\nu_1\theta_\pm]). \tag{24}$$

Remark 7. To the best of our knowledge, this is a first result concerning the study of functions of the type p-almost periodic, namely, $f=\phi+g \in BC(\mathbb{R})$ with $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ and either $g \in L_0^p$ or $g \in L^p(\mathbb{R})$. See [10, section 4.3; page 46]. Notice that actually we do not need that ν_0 and ν_1 are bounded, but it is enough that $\nu_0+\lambda_\pm\nu_1 \in \mathcal{D}(G_\pm^\gamma)$, $\nu_1 \in \mathcal{D}(\mathcal{L}_\pm^{\operatorname{Re}\gamma})$ and $G_\pm^\gamma[\nu_0+\lambda_\pm\nu_1], \mathcal{L}_\pm^{\operatorname{Re}\gamma}[\nu_1] \in BC(\mathbb{R},\mathbb{C})$.

In order to assure the existence of $\theta_\pm$, by studying equation (22), sufficient conditions can be found. More precisely, we have the following result ensuring the existence of both $\theta_\pm$ and $\psi_\pm$.

Theorem 4. *Consider equation* (2) *with* $r_i = \mu_i+\nu_i \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ *(resp.* $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$, $\mathcal{AP}(\mathbb{R},\mathbb{C},p)$ *or* $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p)$*), with* $\mu_i \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ *and* $\nu_i \in BC_0(\mathbb{R},\mathbb{C})$ $i=0,1$ *(resp.* $C_{00}(\mathbb{R},\mathbb{C})$, L_0^p *or* $L^p(\mathbb{R})$*). Denoting* $A := \dfrac{8}{\operatorname{Re}\gamma}\left\|G_\pm^\gamma[\mu_0+\lambda_\pm\mu_1]\right\|_\infty$, $B := \left\|\mathcal{L}_\pm^{\operatorname{Re}\gamma}[\mu_1]\right\|_\infty$, $C := \dfrac{8}{\operatorname{Re}\gamma}\left\|G_\pm^\gamma[\nu_0+\lambda_\pm\nu_1]\right\|_\infty$ *and* $D := \left\|\mathcal{L}_\pm^{\operatorname{Re}\gamma}[\nu_1]\right\|_\infty$, *assume that* $A^2+B^2 \neq 0$,

$$\sqrt{A}+B<1 \tag{25}$$

and

$$B+2D+\sqrt{(1-B)^2-A}+2\sqrt{C+2D\left(1-B+\sqrt{(1-B)^2-A}\right)}<1. \tag{26}$$

Then there is a fundamental system of solutions $y_\pm$ *to* (2) *satisfying* (18) *and* (19), *where* $z_\pm = \theta_\pm + \psi_\pm \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ *(resp.* $\mathcal{AAP}_0(\mathbb{R},\mathbb{C})$, $\mathcal{AP}(\mathbb{R},\mathbb{C},p) \cap \mathcal{AAP}(\mathbb{R},\mathbb{C})$ *or* $\mathcal{AP}_0(\mathbb{R},\mathbb{C},p) \cap \mathcal{AAP}_0(\mathbb{R},\mathbb{C})$*), with* $\theta_\pm \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ *satisfy* (22) *and for some* $0 < \alpha < \operatorname{Re}\gamma$

$$\theta_\pm = O(\mathcal{L}^\alpha_\pm[\mu_0 + \lambda_\pm\mu_1]) \tag{27}$$

and $\psi_\pm \in BC_0(\mathbb{R},\mathbb{C})$ *(resp.* $C_{00}(\mathbb{R})$, $L^p_0 \cap BC_0(\mathbb{R},\mathbb{C})$ *or* $L^p(\mathbb{R}) \cap C_{00}(\mathbb{R},\mathbb{C})$*) satisfy* (23) *and* (24) *for some* $0 < \beta < \operatorname{Re}\gamma$.

Proof: Notice that, similarly to the proof of Theorem 1, by (25) there are $\theta_\pm \in \mathcal{AP}(\mathbb{R},\mathbb{C})$ satisfying (22) and (27) for some $0 < \alpha < \operatorname{Re}\gamma$. Moreover, we also have that

$$\|\theta_\pm\|_\infty \le \frac{\operatorname{Re}\gamma}{4}\left(1 - B + \sqrt{(1-B)^2 - A}\right),$$

see proof of Theorem 1. Now, we shall verify that the conditions of Theorem 3 are satisfied in order to assure the existence of $\psi_\pm$. So, note that

$$\begin{aligned}\frac{8}{\operatorname{Re}\gamma}\|G^\gamma_\pm[\nu_0 + \lambda_\pm\nu_1 + \nu_1\theta_\pm]\|_\infty &\le C + \frac{8}{\operatorname{Re}\gamma}D\|\theta_\pm\|_\infty \\ &\le C + 2D\left(1 - B + \sqrt{(1-B)^2 - A}\right)\end{aligned}$$

and

$$\begin{aligned}\left\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1 + 2\theta_\pm]\right\|_\infty &\le B + D + \frac{2}{\operatorname{Re}\gamma}\|\theta_\pm\|_\infty \\ &\le B + D + \frac{1}{2}\left(1 - B + \sqrt{(1-B)^2 - A}\right).\end{aligned}$$

Hence, it is clear that

$$\begin{aligned}&\sqrt{\frac{8}{\operatorname{Re}\gamma}\left\|G^\gamma_\pm[\nu_0 + \lambda_\pm\nu_1 + \nu_1\theta_\pm]\right\|_\infty} + \left\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1 + 2\theta_\pm]\right\|_\infty \\ &\le \frac{1}{2} + \frac{1}{2}\left[B + 2D + \sqrt{(1-B)^2 - A} + 2\sqrt{C + 2D\left(1 - B + \sqrt{(1-B)^2 - A}\right)}\right]\end{aligned}$$

and the conclusion follows. ■

Notice that if $\mu_0 = \mu_1 = 0$, namely, $A = B = 0$ then $\theta_\pm = 0$ and (26) is not really useful. Actually, in this case it is enough that $\sqrt{C} + D < 1$. Furthermore, inequality (26) is not the most general condition. For instance, we could estimate better $\|\mathcal{L}^{\operatorname{Re}\gamma}_\pm[r_1 + 2\theta_\pm]\|_\infty$, by improving the estimate for $\theta_\pm$, by using its integral equation (22) and Lemma 2.

On the other hand, observe that in Theorem 2 we have obtained the decomposition $z_\pm = \theta_\pm + \psi_\pm$ and a estimate for $z_\pm$. In Theorem 3, we have found $\psi_\pm$ and its estimate. In Theorem 4, we have obtained $\theta_\pm$ and $\psi_\pm$ and estimates for both $\theta_\pm$ and $\psi_\pm$ simultaneously. In subsections 4.3 and 4.4 we shall discuss an example for Theorems 2 and 4.

Remark 8. An important consequence in case $\nu_i \in L_0^p$ or $L^p(\mathbb{R})$ for $i = 0, 1$ is that we can express $\psi_\pm$ as a sum of $L^{p/k}$ functions for $k = 1, \ldots, [p]$, where $[p]$ is the integer part of p (see [16], Theorem 3). Indeed, denote $\hat{\mu} = \mu_1 + \theta_\pm$ and $\hat{\nu} = \nu_0 + \lambda_\pm \nu_1 + \nu_1 \theta_\pm$ and note that $\hat{\mu} \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ and $\hat{\nu} \in L^p$. Then $\psi_\pm = \psi \in L^q$ (with either $L^q = L_0^q$ or $L^q = L^q(\mathbb{R})$) for all $q \geq p$, since ψ is bounded and $\psi = O(\mathcal{L}_\pm^\alpha[\hat{\nu}])$. If $p \in (1, 2]$ then it is clear that $\nu_1\psi, \psi^2 \in L^1$ so that $\psi = -G_\pm^\gamma[\hat{\nu} + \hat{\mu}\psi] - G_\pm^\gamma[\nu_1\psi + \psi^2] := \psi_1 + w_2$, with $\psi_1 \in L^p$ and $w_2 \in L^1$. Moreover, if $p \in (m, m+1]$ with $m \in \mathbb{N}$ and $m \geq 2$ then we write $\psi_\pm$ in the following form

$$\psi_\pm = \sum_{l=1}^{m-1} \psi_l^\pm + w_m^\pm, \tag{28}$$

where $\psi_1^\pm = -G_\pm^\gamma\left[\nu_0 + \lambda_\pm\nu_1 + \nu_1\theta_\pm + (\mu_1 + 2\theta_\pm)\psi_1^\pm\right]$ and for $l > 1$

$$\psi_l^\pm = -G_\pm^\gamma\left[\nu_1\psi_{l-1}^\pm + (\mu_1 + 2\theta_\pm)\psi_l^\pm + \sum_{j=1}^{l-1}\psi_j^\pm\psi_{l-j}^\pm\right], \tag{29}$$

with $\psi_k^\pm \in L_0^{p/k}$, (resp $L^{p/k}(\mathbb{R})$) $k = 1, \ldots, m-1$ and $w_m^\pm \in L_0^{p/m}$ (resp. $L^{p/m}(\mathbb{R})$). By following the previous lines, it is clear that $\nu_1\psi, \psi^2 \in L^{p/2}$ and we have that $\psi = \psi_1 + w_2$, with $\psi_1 \in L^p$ and $w_2 \in L^{p/2}$. Upon inserting in (23), we find that

$$\psi = -G_\pm^\gamma\left[\hat{\nu} + \hat{\mu}\psi_1\right] - G_\pm^\gamma[\hat{\mu}w_2 + \nu_1(\psi_1 + w_2) + (\psi_1 + w_2)^2].$$

Hence, we choose $\psi_1 = -G_\pm^\gamma\left[\hat{\nu} + \hat{\mu}\psi_1\right]$ and $w_2 = -G_\pm^\gamma[\hat{\mu}w_2 + \nu_1(\psi_1 + w_2) + (\psi_1 + w_2)^2]$. Note that we can write

$$w_2 = -G_\pm^\gamma[\hat{\mu}w_2 + \nu_1\psi_1 + \psi_1^2] - G_\pm^\gamma[\nu_1 w_2 + 2\psi_1 w_2 + w_2^2] = \psi_2 + w_3,$$

with $\psi_2 \in L^{p/2}$ and $w_3 \in L^{p/3}$. Replacing $w_2 = \psi_2 + w_3$ in the equation for w_2, then we choose ψ_2 according to (29) and w_3. The conclusion (28) and (29) follows in a recursive way. Finally, we conclude the following decomposition for the logarithmic derivative of a solution to (2)

$$\frac{y_\pm'}{y_\pm} = \lambda_\pm + \theta_\pm + \sum_{l=1}^{m-1}\psi_l^\pm + w_m^\pm.$$

4 Comments and examples

In this section we will discuss about the conditions of the results and we will present an example in order to illustrate each one of our results.

The method is general and shows its effectiveness, by the next examples and applications. For instance, we assume a dichotomy condition of the unperturbed linear part, so that we are able to manage the convolutions or Green's operators. Usually in asymptotic integration these tools play an important role, see for instance [2, 6, 7, 12]. Let us point out that our results rely on the fixed point argument by a contraction mapping, so that, we need conditions (17) and

(25)-(26) in order to have contraction mappings that assure the existence of the unique bounded (or almost periodic, or asymptotically almost periodic) solution to the corresponding Riccati equation (5), (22) or (23), depending on the case. Thus, we are lead to ask that r_0 and r_1 are small in a suitable way. In other words, we expect that the solutions to (2) are close, in some sense, to the solutions to the unperturbed linear part, i.e., $r_0 = r_1 \equiv 0$, because otherwise we will not find such solutions. For instance, if r_0 and r_1 are constants and big with respect to γ, the solutions to (2) could have a very different behavior. This is shown by the following simple example. Consider the equation

$$y'' + r_1(t)y' - (1 - r_0(t))\, y = 0, \tag{30}$$

where we have $\lambda_+ = 1$, $\lambda_- = -1$ and $\gamma = 2$. Assume that $r_1 \equiv 0$. If $r_0 \equiv 0$ then the solutions are $y_\pm(t) = e^{\pm t}$ and this is the unperturbed case. Now, notice that if $r_0(t) = 2$ then the solutions are $y_\pm(t) = e^{\pm\sqrt{3}t}$ while if $r_0(t) = -2$ then the solutions are $y_\pm(t) = e^{\pm it}$, where i is the unit imaginary number.

Concerning possible generalizations, on one hand, it would be interesting to know if similar results (Theorems 1-4) or formula (18) are still valid for higher order equations (1), namely, for $n \geq 3$. In other words, if it is possible to generalize Poincaré's and Perron's classical problem of approximation (1) to the class of almost periodic type functions independently of n, the order of the equation. On the other hand, let us notice that we are not using in a strong way that (5) is scalar to prove Lemma 8, so that we could state an equivalent version of this result in higher dimensions, namely, for $z \in \mathbb{R}^n$. This fact lead us to expect results for system of differential equations. In particular, by using the transformation in [31], we could look for solutions to an almost diagonal linear system

$$y' = [\Lambda(t) + R(t)]y, \;\; y = y(t) \in \mathbb{R}^n, \;\; t \in \mathbb{R},$$

where $\Lambda(t) = \mathrm{diag}(\lambda_1(t), \ldots, \lambda_n(t))$ and $R(t)$ is a $n \times n$ matrix, by studying a generalized Riccati equation.

Now, for simplicity, for the rest of this section we shall assume that $r_1 = 0$. The case $r_1 \neq 0$ can be readily addressed by following Theorems 1-4 and the analysis showed here. Let us stress that estimates or developments for $z_\pm$ will be present in the formulae for both $y_\pm$ and $y'_\pm$.

4.1 Application of Theorem 1

Notice that (17) is equivalent to $4\|G^2_\pm[r_0]\|_\infty < 1$, in view of $r_1 = 0$. Hence, there is a fundamental system of solutions $y_\pm$ to (2) satisfying (18). However, direct computations, (16), integration by parts and dividing by suitable constants lead us to find the following expression

$$y_\pm(t) = e^{\pm t} \exp\left(\mp\frac{1}{2}\int_0^t r_0(s)\, ds - \frac{1}{4}\int_0^t r_0(s)G^2_\pm[r_0](s)\, ds \pm \frac{1}{2}G^2_\pm[r_0](t) + u_\pm(t) \right),$$
$$y'_\pm(t) = (\lambda_\pm + z_\pm(t))y_\pm(t),$$

where

$$u_\pm(t) = \frac{3}{8}z_\pm^2(t) + \frac{1}{2}G_\pm^2[r_0 z_\pm \mp 2z_\pm^2 + z_\pm^3](t) - \frac{1}{4}\int_0^t \left[r_0(s)G_\pm^2[z_\pm^2](s) - z_\pm^3(s)\right] ds.$$

Indeed, by integration by parts and using (5)

$$z_\pm = -G_\pm^2[r_0 + z_\pm^2] = -G_\pm^2[r_0] \mp \frac{z_\pm^2}{2} \mp G_\pm^2[r_0 z_\pm \mp 2z_\pm^2 + z_\pm^3]$$

and by properties of the Green's operator

$$\begin{aligned}\int_0^t z_\pm^2(s)\,ds &= \mp\frac{1}{2}\int_0^t \left[r_0(s)z_\pm(s) + z_\pm^3(s)\right] ds \mp \frac{1}{2}\int_0^t z_\pm(s)z_\pm'(s)\,ds \\ &= \pm\frac{1}{2}\int_0^t \left[r_0(s)G_\pm^2[r_0](s) + r_0(s)G_\pm^2[z_\pm^2](s) - z_\pm^3(s)\right] ds \\ &\qquad \mp \frac{1}{4}[z_\pm^2(t) - z_\pm^2(0)].\end{aligned}$$

4.2 Application of Theorem 2

Notice that if $r_0 = \mu_0 + \nu_0 \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ with $\mu_0 \in \mathcal{AP}(\mathbb{R},\mathbb{C})$, $\nu_0 \in BC_0(\mathbb{R},\mathbb{C})$ and it holds $4\|G_\pm^2[\mu_0+\nu_0]\|_\infty < 1$ then there exist $z_\pm = \theta_\pm + \psi_\pm \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ with $\theta_\pm \in \mathcal{AP}(\mathbb{R},\mathbb{C})$, $\psi_\pm \in BC_0(\mathbb{R},\mathbb{C})$

$$\theta_\pm = -G_\pm^2[\mu_0 + \theta_\pm^2] \quad \text{and} \quad \psi_\pm = -G_\pm^2[\nu_0 + 2\theta_\pm\psi_\pm + \psi_\pm^2]. \tag{31}$$

Furthermore, by the previous computations it also follows that to (30) with $r_0 = \mu_0 + \nu_0$ there is a fundamental system of solutions $y_\pm$ satisfying

$$\begin{aligned} y_\pm(t) &= e^{\pm t}\exp\Bigg(\mp\frac{1}{2}\int_0^t \mu_0(s)\,ds - \frac{1}{4}\int_0^t \mu_0(s)G_\pm^2[\mu_0](s)\,ds \pm \frac{1}{2}G_\pm^2[\mu_0](t) \\ &\qquad \mp \frac{1}{2}\int_0^t \nu_0(s)\,ds - \frac{1}{4}\int_0^t \left[\mu_0(s)G_\pm^2[\nu_0](s) + \nu_0(s)G_\pm^2[\mu_0+\nu_0](s)\right] ds \\ &\qquad \pm \frac{1}{2}G_\pm^2[\nu_0](t) + u_\pm^\theta(t) + v_\pm^\psi(t)\Bigg), \\ y_\pm'(t) &= (\lambda_\pm + \theta_\pm(t) + \psi_\pm(t))y_\pm(t), \end{aligned} \tag{32}$$

where

$$u_\pm^\theta(t) = \frac{3}{8}\theta_\pm^2(t) + \frac{1}{2}G_\pm^2[\mu_0\theta_\pm \mp 2\theta_\pm^2 + \theta_\pm^3](t) - \frac{1}{4}\int_0^t \left[\mu_0(s)G_\pm^2[\theta_\pm^2](s) - \theta_\pm^3(s)\right] ds \tag{33}$$

and

$$\begin{aligned} v_\pm^\psi(t) &= \frac{1}{2}G_\pm^2\left[\mu_0\psi_\pm + \nu_0(\theta_\pm + \psi_\pm) \mp 2(2\theta_\pm\psi_\pm + \psi_\pm^2) + 3\theta_\pm^2\psi_\pm + 3\theta_\pm\psi_\pm^2 + \psi_\pm^3\right](t) \\ &\quad - \frac{1}{4}\int_0^t \Big[\mu_0(s)G_\pm^2[2\theta_\pm\psi_\pm + \psi_\pm^2](s) + \nu_0(s)G_\pm^2[z_\pm^2](s) + \\ &\qquad \theta_\pm^2(s)G_\pm^2[\nu_0 + 2\theta_\pm\psi_\pm + \psi_\pm^2](s) \\ &\qquad - z_\pm(s)[2\theta_\pm(s)\psi_\pm(s) + \psi_\pm^2(s)]\Big] ds + \frac{3}{8}(2\theta_\pm\psi_\pm + \psi_\pm^2)(t). \end{aligned} \tag{34}$$

4.3 Example for Theorems 1 and 2

Suppose that $r_0 = \mu_0 + \nu_0$ with $\mu_0(t) = \eta_1 \left[2 + \cos t + \cos(\sqrt{2}t)\right]$ and $\nu_0(t) = \dfrac{\eta_2}{1+t^2}$, where $\eta_1, \eta_2 \geq 0$. It is clear that μ_0 and ν_0 are non-negative functions, $\mu_0 \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, $\nu_0 \in C_{00}(\mathbb{R}, \mathbb{C}) \cap L^1(\mathbb{R})$, $\|\mu_0\|_\infty = 4\eta_1$, $\|\nu_0\|_\infty = \eta_2$ and $\|r_0\|_\infty = 4\eta_1 + \eta_2$. Hence, if $0 < \eta_1 < \dfrac{1}{8}$ and $\eta_2 = 0$, so that $r_0 = \mu_0$ then $8\|r_0\|_\infty = 32\eta_1 < 4 = \gamma^2$ and by using (21), Theorem 1 applies. On the other hand, it is straightforward to verify that

$$G_\pm^2[r_0](t) = \pm\eta_1 + \frac{\eta_1}{5}[\pm 2\cos t + \sin t] + \frac{\eta_1}{6}[\pm 2\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t)], \quad \eta_2 = 0,$$

since for any $\lambda \in \mathbb{R}$ and integration by parts

$$\begin{aligned}\int_t^{\mp\infty} e^{\mp 2(t-s)} \cos(\lambda s)\, ds &= \mp\frac{\cos(\lambda t)}{2} \pm \frac{\lambda}{2}\int_t^{\mp\infty} e^{\mp 2(t-s)} \sin(\lambda s)\, ds \\ &= \frac{\mp 2\cos(\lambda t) - \lambda \sin(\lambda t)}{4+\lambda^2}.\end{aligned}$$

Thus, we find that $\|G_\pm^2[r_0]\|_\infty \leq \eta_1\Big(1 + \dfrac{\sqrt{5}}{5} + \dfrac{\sqrt{6}}{6}\Big)$, with $\eta_2 = 0$. Hence, if also $\eta_1\Big(1 + \dfrac{\sqrt{5}}{5} + \dfrac{\sqrt{6}}{6}\Big) < \dfrac{1}{4}$ then Theorem 1 applies. Let us stress that this last condition is weaker than the previous one found by (21), since $1 + \dfrac{\sqrt{5}}{5} + \dfrac{\sqrt{6}}{6} < 2$. In other words, conditions in (21) are stronger than (17) in the Theorem 1. Furthermore, since $\|G_\pm^2[\mu_0]\|_\infty \leq \eta_1\Big(1 + \dfrac{\sqrt{5}}{5} + \dfrac{\sqrt{6}}{6}\Big)$ and $\|G_\pm^2[\nu_0]\|_\infty \leq \dfrac{1}{2}\|\nu_0\|_\infty = \dfrac{\eta_2}{2}$ for $\eta_2 > 0$, if

$$\left\|G_\pm^2[\mu_0 + \nu_0]\right\|_\infty \leq \eta_1\left(1 + \frac{\sqrt{5}}{5} + \frac{\sqrt{6}}{6}\right) + \frac{\eta_2}{2} < \frac{1}{4}$$

then Theorem 2 applies.

4.4 Example for Theorem 4

Notice that if $r_1 \equiv 0$ then $B = D = 0$ and inequalities (25) and (26) get re-written

$$4\left\|G_\pm^2[\mu_0]\right\|_\infty < 1 \qquad \text{and} \qquad \sqrt{1 - 4\left\|G_\pm^2[\mu_0]\right\|_\infty} + 2\sqrt{4\|G_\pm^2[\nu_0]\|_\infty} < 1. \quad (35)$$

Thus, the assumptions of Theorem 4 are satisfied. Therefore, there exist $\theta_\pm$ and $\psi_\pm$ satisfying equations (31), and a fundamental system of solutions $y_\pm$ satisfying (32). Notice that there exist η_1 and η_2, so that, inequalities (35) are satisfied, but

$$4\left\|G_\pm^2[\mu_0 + \nu_0]\right\|_\infty = 4\left\|G_\pm^2[\mu_0]\right\|_\infty + 4\left\|G_\pm^2[\nu_0]\right\|_\infty \geq 1.$$

In other words, Theorem 4 gives us solutions not directly deduced from the previous analysis using Theorem 2, cf. subsection 4.3.

Finally, notice that $\nu_0 \in L^1(\mathbb{R})$, so that, we find that $\psi_\pm \in C_{00}(\mathbb{R}, \mathbb{C}) \cap L^1(\mathbb{R})$ and up to divide for a suitable constant, from (32) the following asymptotic formula as $t \to +\infty$ is true

$$\begin{aligned} y_\pm(t) &= (1+o(1))e^{\pm t} \exp\left(\mp\frac{1}{2}\int_0^t \mu_0(s)\,ds - \frac{1}{4}\int_0^t \mu_0(s) G_\pm^2[\mu_0](s)\,ds \right. \\ &\qquad \left. \pm \frac{1}{2} G_\pm^2[\mu_0](t) + u_\pm^\theta(t) \right), \\ y'_\pm(t) &= (\lambda_\pm + \theta_\pm(t) + o(1)) y_\pm(t), \end{aligned}$$

where $u_\pm^\theta$ is given by (33), in view of $v_\pm^\psi \in L^1(\mathbb{R})$ defined in (34). The same behavior as $t \to -\infty$ is also true with a possibly different $o(1)$.

It is worth to mention that similar analysis for equation (30) could be performed in case $r_1 \neq 0$ following Theorem 4.

Acknowledgements

The first author has been supported by grants Fondecyt Postdoctorado 3120039 and Fondecyt Iniciación 11130517, Chile. The second author has been partially supported by Fondecyt 1120709, Chile.

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