Realizing homotopy group actions*

David Blanc   Debasis Sen

Abstract

For any finite group G, we define the notion of a *Bredon homotopy action* of G, modelled on the diagram of fixed point sets $(\mathbf{X}^H)_{H\leq G}$ for a G-space $\mathbf{X}$, together with a pointed homotopy action of the group $N_G H/H$ on $\mathbf{X}^H/(\bigcup_{H<K}\mathbf{X}^K)$. We then describe a procedure for constructing a suitable diagram $\underline{X}:\mathcal{O}_G^{\mathrm{op}}\to\mathcal{T}op$ from this data, by solving a sequence of elementary lifting problems. If successful, we obtain a G-space $\mathbf{X}'$ realizing the given homotopy information, determined up to Bredon G-homotopy type. Such lifting methods may also be used to understand other homotopy questions about group actions, such as transferring a G-action along a map $f:\mathbf{X}\to\mathbf{Y}$.

Introduction

The naive notion of a homotopy action of a group G on a topological space $\mathbf{X}$ can be described as the choice of a homotopy class of a map $\mathbf{B}G\to\mathbf{B}\,\mathrm{haut}(\mathbf{X})$, where $\mathrm{haut}(\mathbf{X})$ is the monoid of self-homotopy equivalences (see §1.1). This always lifts to a strict action, unique up to Borel equivalence (see §1.8). However, the G-actions we obtain in this way will be free, so the more delicate aspects of equivariant topology are not visible in this way.

A more informative approach to equivariant homotopy theory, due to Bredon, studies G-spaces $\mathbf{X}$ up to G-homotopy equivalence – that is, G-maps having G-homotopy inverses (see [Br]). This is equivalent to the homotopy theory of diagrams $\underline{X}:\mathcal{O}_G^{\mathrm{op}}\to\mathcal{T}op$ (where $\mathcal{O}_G$ is the orbit category of G and $\underline{X}(G/H)$ is the fixed point set $\mathbf{X}^H$ – see §1.4 and [E]). Dwyer and Kan showed that this

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in turn is equivalent to a homotopy theory of a certain diagram of fibrations (see [DK1, DK2]).

The purpose of this paper is to define a notion of homotopy action in Bredon equivariant homotopy theory, and describe an associated inductive procedure for realizing such an action by a continuous one.

One might be tempted to say that a homotopy action of G should simply be a homotopy-commutative diagram $\underline{X}' : \mathcal{O}_G^{\mathrm{op}} \to \mathrm{ho}\,\mathcal{T}op$. We then have available the obstruction theory of Dwyer, Kan, and Smith for rectifying general homotopy-commutative diagrams (cf. [DKS2, DK3]), which we can use to try to lift $\underline{X}'$ to a strict diagram $\underline{X} : \mathcal{O}_G^{\mathrm{op}} \to \mathcal{T}op$, yielding a G-space, unique up to Bredon equivalence.

However, the orbit category $\mathcal{O}_G$ can be quite complicated: it includes various isomorphisms $G/H \cong G/H^a$ for $a \in G$, and in particular an action of $N_G H$ as the automorphisms of G/H for each $H \leq G$. In the Dwyer-Kan-Smith approach, all the morphisms of $\mathcal{O}_G$ are treated on an equal footing, and must all be made to fit together at one time (with increasing levels of coherence). In particular, it does not allow us to interpret the initial data in terms of homotopy actions of each $N_G H$ on $\underline{X}'(G/H)$.

The version of homotopy action that we define here involves an ordinary diagram of spaces (with no group actions), which we assume for simplicity to be strict. We do require a certain amount of equivariant rectification in addition, but we keep this to the minimum, and in a form that reduces to an elementary lifting problem, in the spirit of [C] and [DK2], starting with certain ordinary homotopy actions of $N_G H$.

0.1. Bredon homotopy actions. We let Λ denote the partially ordered set of subgroups of G, and define a *Bredon homotopy action* of G to consist of:

- A diagram $\underset{\sim}{X} : \Lambda^{\mathrm{op}} \to \mathcal{T}op$;
- For each conjugacy class $\langle H \rangle$, a pointed homotopy action of $W_H := N_G H/H$ on the homotopy cofiber $\underset{\sim}{\mathbf{X}}{}^H_H$ of the obvious map $\mathrm{hocolim}_{K>H}\, \underset{\sim}{X}(K) \to \underset{\sim}{X}(H)$ for some representative $H \in \langle H \rangle$.

If H' and H are conjugate in G, we must have a homotopy-commuting square:

$$\begin{tikzcd}
\mathrm{hocolim}_{K>H}\, \underset{\sim}{X}(K) \arrow[r] \arrow[d, "\simeq"] & \underset{\sim}{X}(H) \arrow[d, "\simeq"] \\
\mathrm{hocolim}_{K'>H'}\, \underset{\sim}{X}(K') \arrow[r] & \underset{\sim}{X}(H')
\end{tikzcd} \tag{0.2}$$

with vertical homotopy equivalences.

0.3. Realizing Bredon homotopy actions. We wish to realize such a Bredon homotopy action by a topological action, using descending induction on the subgroups of G: without specifying the G-space $\mathbf{X}$ itself, assume that for some $H \leq G$

we have constructed a partial diagram $\underline{X}$ consisting of spaces $\underline{X}(K) \simeq \underset{\sim}{X}(K)$ (to be thought of as of "fixed point sets" $\mathbf{X}^K$ for the putative G-space $\mathbf{X}$) for all groups $H < K \leq G$, together with inclusions $i^* : \underline{X}(L) \hookrightarrow \underline{X}(K)$ for $i : K \hookrightarrow L$, compatible with an action of G on $\underline{X}$ by $\underline{X}(K) \mapsto \underline{X}(K^a)$.

Note that we may filter the collection of subgroups of G (or the objects of $\mathcal{O}_G$) by letting $\mathcal{F}_k$ consist of those subgroups H for which there is a chain of proper inclusions $H = H_0 < H_1 < \ldots < H_k = G$. If we set $\underline{\mathbf{X}}_H := \bigcup_{H<K} \underline{X}(K)$, by induction on this filtration we assume that we have actions of W_H on $\underline{\mathbf{X}}_H$ and $\underline{\mathbf{X}}_H^H$ (the latter realizing the given pointed homotopy action on $\underset{\sim H}{\mathbf{X}}^H$ – see Appendix). These fit into a homotopy cofibration sequence:

$$\underline{\mathbf{X}}_H \to \underset{\sim}{X}(H) \to \underline{\mathbf{X}}_H^H . \tag{0.4}$$

The key ingredient in the inductive procedure for realizing a Bredon homotopy action as above is the "interpolation" problem: given two W_H-spaces such as $\underline{X}_H$ and $\underline{\mathbf{X}}_H^H$, and two maps as in (0.4), how to obtain a compatible W_H-action on the middle space $\underset{\sim}{X}(H)$. This can be reduced to a lifting problem (see Propositions 3.6 and 3.10). If we succeed in solving it, we have extended our diagram $\underline{X}$ to H, too.

Our main result shows that if this procedure can be completed for all $H \leq G$, we obtain a full $\mathcal{O}_G^{\mathrm{op}}$-diagram $\underline{X}$, and thus a G-space $\mathbf{X}$ realizing the given Bredon homotopy action (cf. [E]):

Theorem A. *A Bredon homotopy action* $\mathcal{A} := \langle \underset{\sim}{X}, (\Phi_H^*)_{H \leq G} \rangle$ *for a finite group* G *can be realized by a* G*-space* $\mathbf{X}$ *if and only if one can inductively construct a sequence of cofibrant diagrams* $(\underline{X}_k : \mathcal{F}_k \to \mathcal{T}op)$ *realizing* $\mathcal{A}$*. Moreover, one can extend* $\underline{X}_k$ *to* $\underline{X}_{k+1}$ *if and only if for each* $H \subseteq \mathcal{F}_{k+1} \setminus \mathcal{F}_k$*, one can find a map* Ψ_H *making the following diagram of topological spaces commute:*

$$\begin{tikzcd}
 & & \mathbf{B}\,\mathrm{haut}(\underline{\mathbf{X}}_H) \\
\mathbf{B}W_H \arrow[urr, "\mathbf{B}\zeta_{\underline{\mathbf{X}}_H}"] \arrow[r, dotted, "\Psi_H"] \arrow[drr, "\mathbf{B}\zeta_{\underline{\mathbf{X}}_H^H}"'] & \mathbf{B}\mathcal{Q}_{j_H,q_H} \arrow[ur, "\mathbf{B}\delta_{j_H}\circ\mathbf{B}\mu"] \arrow[dr, "\mathbf{B}\varepsilon_{q_H}\circ\mathbf{B}\nu"'] & \\
 & & \mathbf{B}\,\mathrm{haut}(\underline{\mathbf{X}}_H^H)
\end{tikzcd}$$

[See Theorem 4.13 below; the topological monoid $\mathcal{Q}_{f,g}$ is defined in §3.4].

0.5. Related lifting problems. Along the way we discuss three related but simpler questions, of independent interest, and show how they too may be reduced to lifting problems for appropriate fibrations:

(i) How to extend a G-action on a space $\mathbf{X}$ along a map $f : \mathbf{X} \to \mathbf{Y}$;

(ii) How to lift a G-action on $\mathbf{Y}$ along a map $f : \mathbf{X} \to \mathbf{Y}$;

(iii) How to make a map $f : \mathbf{X} \to \mathbf{Y}$ between two G-spaces into a G-map.

See Propositions 2.7 and 2.17.

0.6. Obstructions. All these lifting problems have associated obstruction theories, described in terms of (Moore-)Postnikov towers (see Proposition 3.14), and thus similar in spirit to Cooke's original approach to the realization problem for homotopy actions (see [C]). These differ from the obstruction theory of [DKS2], though unfortunately neither version is easily computable.

We may thus conclude from Theorem A that the obstructions of Proposition 3.14 are the only ones to realizing a Bredon homotopy action, and the difference obstructions distinguish between the resulting realizations up to G-homotopy equivalence (see Corollary 4.15).

0.7 Remark. The question of realizing homotopy actions is an old one, going back to work of Cooke in [C] (see also [LSm, O, Z, SV1]). Many approaches to this and related problems appear in the literature: since (homotopy) actions induce maps between classifying spaces of groups and monoids, any information about the latter is relevant to the question at hand. Methods for analyzing maps between classifying spaces were developed by Dwyer, Zabrodsky, Jackowski, McClure, Oliver, and others in the 1980's (cf. [DM, DZ, JMO]), and later by Grodal and Smith for actions on spheres, in [GS], based on Lannes theory (cf. [La]). Our approach here is more elementary, and perhaps more conceptual, although the machinery for calculating our obstructions is not as well-developed.

0.8 Notation. The category of topological spaces will be denoted by $\mathcal{T}op$, and its objects will be denoted by boldface letters: $\mathbf{X}$, $\mathbf{Y}\ldots$. The category of pointed topological spaces $\mathbf{X}_* = (\mathbf{X}, x_0)$ is denoted by $\mathcal{T}op_*$.

A *G-space* is a topological space $\mathbf{X}$ equipped with a left G-action, and the category of G-spaces with *G-maps* (i.e., G-equivariant continuous maps) will be denoted by G-$\mathcal{T}op$. We write $\mathbf{X}^H$ for the *fixed point set* $\{x \in \mathbf{X} : hx = x \ \forall h \in H\}$ of X under a subgroup $H \leq G$.

An important example is a *G-CW complex*, obtained by attaching *G-cells* of the form $G/H \times \mathbf{D}^{n+1}$ for $n \geq -1$ (see [I]). For finite G, this is equivalent to $\mathbf{X}$ being a CW-complex on which G acts cellularly (see [tD, II, §1]).

An action of a (discrete) group G on $\mathbf{X}$ is given by a homomorphism $\varphi_{\mathbf{X}} : G \to \mathrm{Aut}(\mathbf{X})$, which we call the *action map* of $\mathbf{X}$. We call the composite $\zeta_{\mathbf{X}} := i_{\mathbf{X}} \circ \varphi_{\mathbf{X}} : G \to \mathrm{haut}(\mathbf{X})$ the *monoid action map* of $\mathbf{X}$, where $i_{\mathbf{X}} : \mathrm{Aut}(\mathbf{X}) \hookrightarrow \mathrm{haut}(\mathbf{X})$ is the inclusion. Similarly, a pointed action of G on $\mathbf{X}_*$, given by the *pointed action map* $\varphi^*_{\mathbf{X}_*} : G \to \mathrm{Aut}_*(\mathbf{X}_*)$, has a *pointed monoid action map* $\zeta^*_{\mathbf{X}} := i_{\mathbf{X}_*} \circ \varphi^*_{\mathbf{X}_*} : G \to \mathrm{haut}_*(\mathbf{X}_*)$.

0.9. Organization. In Section 1 we provide some basic background on G-spaces, the orbit category, and equivariant homotopy theory. In Section 2 we address the question of transferring group actions along a map (cf. §0.5), as preparation for the interpolation problem, discussed in Section 3 (both for arbitrary groups). In Section 4 we define the notion of a Bredon homotopy action and prove our main result (for finite G). In the Appendix, we review the notion of a pointed homotopy action.

1 G-Spaces and the Orbit Category

In this section we recall some basic facts about G spaces, and the Borel and Bredon approaches to equivariant homotopy theory.

1.1. Homotopy actions. Let $\operatorname{haut}(\mathbf{X})$ denote the strictly associative topological monoid of self-homotopy equivalences of a topological space $\mathbf{X}$. If G is a group, any monoid map $\zeta_{\mathbf{X}} : G \to \operatorname{haut}(\mathbf{X})$ factors through the submonoid $\operatorname{Aut}(\mathbf{X})$ of invertible elements (self-homeomorphisms) in $\operatorname{haut}(\mathbf{X})$, so it makes $\mathbf{X}$ into a *G-space*, equipped with a continuous *G-action*.

However, $\operatorname{Aut}(\mathbf{X})$ is not a homotopy invariant of $\mathbf{X}$, while $\operatorname{haut}(\mathbf{X})$ is. A *homotopy action* of G on $\mathbf{X}$ is therefore defined to be the homotopy class of a map $\Phi : \mathbf{B}G \to \mathbf{B}\operatorname{haut}(\mathbf{X})$ (see [DDK, DW] and compare [Su]). In particular, a group action determines a homotopy action, by setting $\Phi := B\zeta_{\mathbf{X}}$.

If $\mathbf{X}_* = (\mathbf{X}, x_0)$ is pointed, $\operatorname{haut}(\mathbf{X})$ has a sub-monoid $\operatorname{haut}_*(\mathbf{X}_*)$ consisting of the pointed self-homotopy equivalences of $\mathbf{X}$, and a *pointed homotopy action* of G on $\mathbf{X}_*$ is (the homotopy class of) a map $\Phi^* : \mathbf{B}G \to \mathbf{B}\operatorname{haut}_*(\mathbf{X}_*)$.

The inclusion $j : \operatorname{haut}_*(\mathbf{X}_*) \hookrightarrow \operatorname{haut}(\mathbf{X})$ fits into a homotopy fibration sequence:

$$\operatorname{haut}_*(\mathbf{X}_*) \xrightarrow{j} \operatorname{haut}(\mathbf{X}) \xrightarrow{\mathrm{ev}_{x_0}} \mathbf{X} \xrightarrow{k} \mathbf{B}\operatorname{haut}_*(\mathbf{X}_*) \xrightarrow{\mathbf{B}j} \mathbf{B}\operatorname{haut}(\mathbf{X}) \,, \tag{1.2}$$

where $\mathbf{B}j$ is universal for Hurewicz fibrations with homotopy fiber $\mathbf{X}$ (cf. [A, St], and see [BGM, Theorem 5.6] & [DFZ, Proposition 4.1]).

Note that a *free* G-space $\mathbf{X}$ is the total space of a principal G-bundle over the orbit space $\mathbf{X}/G$, which is classified by a map $\vartheta : \mathbf{X}/G \to \mathbf{B}G$. If we let $\mathbf{X}_1 \hookrightarrow E_\theta \xrightarrow{\theta} \mathbf{B}G$ denote the pullback of $\mathbf{B}j$ along Φ, this fits into a commuting diagram of fibration sequences:

$$\begin{tikzcd}
\ast \arrow[r] \arrow[d] & G \arrow[r, "="] \arrow[d] & G \arrow[d] \\
\mathbf{X}_1 \arrow[r, hook, "\simeq"] \arrow[d, "="] & \mathbf{X}' \arrow[r, two heads] \arrow[d, "\xi"'] \arrow[dr, phantom, "\boxed{\mathrm{PB}}"] & \mathbf{E}G \arrow[d] \\
\mathbf{X}_1 \arrow[r, hook] & E_\theta \arrow[r, two heads, "\theta"] & \mathbf{B}G,
\end{tikzcd} \tag{1.3}$$

thus yielding a (free) topological G-action on $\mathbf{X}' \sim \mathbf{X}$ (with $\theta \in [\mathbf{X}'/G, \mathbf{B}G]$ corresponding to $\vartheta \in [\mathbf{X}/G, \mathbf{B}G]$ under this equivalence, if $\mathbf{X}$ is a free G-space). Since we cannot guarantee that G will act on $\mathbf{X}$ itself, this is sometimes refereed to as a *proxy* action (cf. [DW]).

Note that for every G-space $\mathbf{X}$ there is a G-map $\mathbf{X} \times \mathbf{E}G \to \mathbf{X}$ which is a homotopy equivalence (out of a free G-space). With this notion of G-weak equivalence, we obtain the Borel version of equivariant homotopy theory, which thus reduces to the study of principal G-bundles.

We say that the homotopy action Φ is *realized* by a free G-space $\mathbf{X}_1$ if the corresponding principal G-bundle (1.3) is classified by a map θ which is the pullback of (1.2) along Φ; we have just seen that any homotopy action is realizable. Similarly, any pointed homotopy action is realizable by a pointed topological action (see Appendix).

Let G be a fixed group. Bredon's approach to G-equivariant homotopy theory (cf. [Br, E]) reduces the study of a G-space $\mathbf{X}$ to the system of fixed point sets under the subgroups of G. To describe it, we recall the following:

1.4. The orbit category. The *orbit category* $\mathcal{O}_G$ of G has the cosets G/H (for each $H \leq G$) as objects, and G-equivariant maps as morphisms.

Any map $G/H \to G/K$ in $\mathcal{O}_G$ can be factored as $i_* : G/H \to G/K^{a^{-1}}$ (induced by the inclusion $i : H \hookrightarrow K^{a^{-1}}$), followed by an isomorphism $\phi_a^{K^{a^{-1}}} : G/K^{a^{-1}} \to G/K$, where $K^{a^{-1}} := aKa^{-1}$, for $a \in G$, and $\phi_a^{K^{a^{-1}}}$ is induced by the right translation $g \mapsto ag$. Two maps $\phi_a^{K^{a^{-1}}} \circ i_*$ and $\phi_b^{K^{b^{-1}}} \circ j_*$ from G/H to G/K are the same in $\mathcal{O}_G$ if and only if $a^{-1}b \in K$. Thus the automorphism group $W_H := \mathrm{Aut}_{\mathcal{O}_G}(G/H)$ of $G/H \in \mathcal{O}_G$ is $N_G H/H$ (where $N_G H$ is the normalizer of H in G).

1.5. $\mathcal{O}_G$-diagrams. An $\mathcal{O}_G^{\mathrm{op}}$*-diagram* in $\mathcal{T}op$ is a functor $\Psi : \mathcal{O}_G^{\mathrm{op}} \to \mathcal{T}op$, and the category of all such will be denoted by $\mathcal{T}op^{\mathcal{O}_G^{\mathrm{op}}}$. The main example we have in mind is the fixed point set diagram $\underline{X}$ associated a G-space $\mathbf{X}$, defined $\underline{X}(G/H) := \mathbf{X}^H$.

Since $\mathcal{T}op$ is a simplicial model category, $\mathcal{T}op^{\mathcal{O}_G^{\mathrm{op}}}$ has a projective simplicial model category structure in which a map $f : \Psi \to \Psi'$ of $\mathcal{O}_G^{\mathrm{op}}$-diagrams is a weak equivalence (respectively, a fibration) if for each $H \leq G$, $f(G/H) : \Psi(G/H) \to \Psi'(G/H)$ is a weak equivalence (respectively, a fibration). See [Hi, Theorem 11.7.3].

There is an analogous simplicial model category structure on G-$\mathcal{T}op$, in which a G-map $f : \mathbf{X} \to \mathbf{Y}$ is a weak equivalence (respectively, fibration) if for each $H \leq G$, the map $f|_{\mathbf{X}^H}$ is a weak equivalence (respectively, fibration). See [DK1] and compare [Pi].

The following result of Elmendorf explains the central role of fixed-point sets in Bredon equivariant homotopy theory:

1.6 Theorem ([E, Theorem 1])**.** *The fixed point set functor sending a G-space $\mathbf{X}$ to the diagram $\underline{X} : \mathcal{O}_G^{\mathrm{op}} \to \mathcal{T}op$ has a right adjoint $C : \mathcal{T}op^{\mathcal{O}_G^{\mathrm{op}}} \to G\text{-}\mathcal{T}op$.*

1.7 Remark. In fact, this adjoint pair constitutes a simplicial Quillen equivalence between G-$\mathcal{T}op$ and $\mathcal{T}op^{\mathcal{O}_G^{\mathrm{op}}}$. Moreover, for any G-space $\mathbf{X}$, Elmendorf shows that $C\underline{X}$ is a G-CW complex. We therefore may (and shall) assume from now on that all our G-spaces are G-CW complexes.

1.8 Definition. A G-map $h : \mathbf{X} \to \mathbf{Y}$ which at the same time is a (non-equivariant) homotopy equivalence will be called a *Borel G-equivalence*. If $x_0 \in \mathbf{X}$ and $y_0 = h(x_0) \in \mathbf{Y}$ are G-base-points (fixed under the G action) and h is a pointed homotopy equivalence, it will be called a *pointed Borel G-equivalence*.

1.9 Lemma. *For any homotopy equivalence* $h : \mathbf{X} \to \mathbf{Y}$ *between CW complexes, there is a CW complex* $\mathbf{Z}$ *with homotopy equivalences* $i : \mathbf{X} \to \mathbf{Z}$ *and* $i' : \mathbf{Y} \to \mathbf{Z}$ *such that* $i \sim h \circ i'$*, inducing strictly multiplicative monic homotopy equivalences* $i_\star : \mathrm{haut}(\mathbf{X}) \to \mathrm{haut}(\mathbf{Z})$ *and* $i'_\star : \mathrm{haut}(\mathbf{Y}) \to \mathrm{haut}(\mathbf{Z})$.

Proof. Factoring h as $p' \circ i = h$ with i a cofibration and p' a fibration, and using the cofibrancy of $\mathbf{X}$ and $\mathbf{Y}$, we obtain a diagram of homotopy equivalences

(1.10)
$$\begin{tikzcd} & \mathbf{Z} \arrow[dl, bend right, "p"'] \arrow[dr, "p'"] & \\ \mathbf{X} \arrow[ur, "i"] \arrow[rr, "h"] & & \mathbf{Y} \arrow[ul, bend right, "i'"'] \end{tikzcd}$$

with $p \circ i = \mathrm{Id}_{\mathbf{X}}$ and $p' \circ i' = \mathrm{Id}_{\mathbf{Y}}$. Define $i_\star : \mathrm{haut}(\mathbf{X}) \to \mathrm{haut}(\mathbf{Z})$ by $\varphi \mapsto i \circ \varphi \circ p$ (for any homotopy equivalence $\varphi : \mathbf{X} \to \mathbf{X}$). Because $p \circ i = \mathrm{Id}_{\mathbf{X}}$, the map $i_\star$ is monic, preserves compositions, and has a (non-monoidal) homotopy inverse $p_* : \mathrm{haut}(\mathbf{Z}) \to \mathrm{haut}(\mathbf{X})$. Similarly for i'. ∎

1.11 Remark. Any homotopy equivalence $h : \mathbf{X} \to \mathbf{Y}$ induces a homotopy equivalence $\mathbf{B}\,\mathrm{haut}(\mathbf{X}) \simeq \mathbf{B}\,\mathrm{haut}(\mathbf{Y})$ (cf. [F1, Satz 7.7]). In fact, we can apply the classifying space functor $\mathbf{B}$ to the maps $i_\star$ and $i'_\star$, obtaining homotopy equivalences:

$$\mathbf{B}\,\mathrm{haut}(\mathbf{X}) \xrightarrow{\mathbf{B}i_\star} \mathbf{B}\,\mathrm{haut}(\mathbf{Z}) \xrightarrow{(\mathbf{B}i'_\star)^{-1}} \mathbf{B}\,\mathrm{haut}(\mathbf{Y}),$$

whose composite is denoted by $\mathbf{B}h_*$ (well-defined up to homotopy). Similarly in the pointed case.

2 Transferring group actions

In this and the following section G can be any topological group. Given a map $f : \mathbf{X} \to \mathbf{Y}$, consider the questions of:

- Transferring a given G-action on $\mathbf{X}$ along f to $\mathbf{Y}$. or conversely.
- Making f equivariant with respect to given actions on both $\mathbf{X}$ and $\mathbf{Y}$.

In the spirit of [DK2], we shall show how they can be reduced to suitable lifting problems. First, we make the questions more precise:

2.1 Definition. Given any map $f : \mathbf{X} \to \mathbf{Y}$, a G-map $f' : \mathbf{X}' \to \mathbf{Y}'$ is:

(i) a *right transfer* of a G-action on $\mathbf{X}$ *along* f if we have a homotopy-commutative diagram

(2.2)
$$\begin{tikzcd} \mathbf{X} \arrow[r, "f"] & \mathbf{Y} \arrow[d, "h", "\simeq"'] \\ \mathbf{X}' \arrow[u, "k", "\simeq"'] \arrow[r, "f'"] & \mathbf{Y}' \end{tikzcd}$$

in which h is a homotopy equivalence, and k is a Borel G-equivalence.

(ii) a *left transfer* of a G-action on $\mathbf{Y}$ *along* f if we have a diagram

(2.3)
$$\begin{tikzcd}
\mathbf{X} \arrow[r, "f"] & \mathbf{Y} & \mathbf{Y}'' \arrow[l, "m"', "\simeq"] \arrow[dl, "n", "\simeq"'] \\
\mathbf{X}' \arrow[u, "k", "\simeq"'] \arrow[r, "f'"] & \mathbf{Y}' &
\end{tikzcd}$$

in which k is a homotopy equivalence, m and n are Borel G-equivalences, which becomes homotopy-commutative after inverting m or n (up to homotopy).

(iii) a *compatible G-map* for f *with respect to G-actions* on $\mathbf{X}$ and $\mathbf{Y}$ if we have a diagram

(2.4)
$$\begin{tikzcd}
\mathbf{X} \arrow[r, "f"] & \mathbf{Y} & \mathbf{Y}'' \arrow[l, "m"', "\simeq"] \arrow[dl, "n", "\simeq"'] \\
\mathbf{X}' \arrow[u, "k", "\simeq"'] \arrow[r, "f'"] & \mathbf{Y}' &
\end{tikzcd}$$

in which k, m, and n are all Borel G-equivalences, which becomes homotopy-commutative after inverting m or n.

In order to describe the conditions under which such transfers exist, we require also the following construction:

2.5 Definition. For any map $f : \mathbf{X} \to \mathbf{Y}$, let $\mathcal{P}_f$ denote the homotopy pullback:

(2.6)
$$\begin{tikzcd}
\mathcal{P}_f \arrow[r, "\varepsilon"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\boxed{\mathrm{PB}}"] & \mathrm{haut}(\mathbf{Y}) \arrow[d, "f^*"] \\
\mathrm{haut}(\mathbf{X}) \arrow[r, "f_*"'] & \mathrm{Map}(\mathbf{X}, \mathbf{Y}) .
\end{tikzcd}$$

This can be constructed explicitly in two ways: if we change f into a cofibration, the map $f^* : \mathrm{Map}(\mathbf{Y}, \mathbf{Y}) \to \mathrm{Map}(\mathbf{X}, \mathbf{Y})$ is a fibration, so its restriction to $\mathrm{haut}(\mathbf{Y})$ is a fibration, too (since the latter is just a union of path components of $\mathrm{Map}(\mathbf{Y}, \mathbf{Y})$). In this case, the strict pullback is actually the homotopy pullback. Similarly when f is a fibration, so f_* is a fibration.

Using such a strict model, we see that $\mathcal{P}_f$ is a sub-monoid of the strictly associative monoid $\mathrm{haut}(\mathbf{X}) \times \mathrm{haut}(\mathbf{Y})$. Moreover, it is grouplike, since $(g, h) \in \mathcal{P}_f$ means that $f \circ g = h \circ f$ (for $g \in \mathrm{haut}(\mathbf{X})$ and $h \in \mathrm{haut}(\mathbf{Y})$), and thus $f \circ g^{-1} \sim h^{-1} \circ f$. If f is either a fibration or a cofibration, we can use [BJT, Lemma 4.16] to change h^{-1} (respectively, g^{-1}) up to homotopy to get $f \circ g^{-1} = h^{-1} \circ f$ and thus $(g^{-1}, h^{-1}) \in \mathcal{P}_f$, too. The maps δ and ε (the restrictions of the structure maps for $\check{\mathcal{P}}_f$) are monoid maps. Evidently $\mathcal{P}_f$ is a homotopy invariant of f.

With these notions we then have the following:

2.7 Proposition. *Let* $f : \mathbf{X} \to \mathbf{Y}$ *be any map in* $\mathcal{T}op$.

(i) There is a right transfer of a G-action on $\mathbf{X}$ *(with monoid action map* $\zeta_{\mathbf{X}} : G \to \operatorname{haut}(\mathbf{X})$*) along f if and only if there is a map* Ψ *making the following diagram commute up to homotopy:*

$$\begin{tikzcd} & \mathbf{B}\mathcal{P}_f \arrow[d, "\mathbf{B}\delta"] \\ \mathbf{B}G \arrow[ur, "\Psi"] \arrow[r, "\mathbf{B}\zeta_{\mathbf{X}}"'] & \mathbf{B}\operatorname{haut}(\mathbf{X}). \end{tikzcd} \tag{2.8}$$

(ii) There is a left transfer of a G-action on $\mathbf{Y}$ *(with monoid action map* $\zeta_{\mathbf{Y}} : G \to \operatorname{haut}(\mathbf{Y})$*) along f if and only if there is a map* Ψ *making the following diagram commute up to homotopy:*

$$\begin{tikzcd} & \mathbf{B}\mathcal{P}_f \arrow[d, "\mathbf{B}\varepsilon"] \\ \mathbf{B}G \arrow[ur, "\Psi"] \arrow[r, "\mathbf{B}\zeta_{\mathbf{Y}}"'] & \mathbf{B}\operatorname{haut}(\mathbf{Y}). \end{tikzcd} \tag{2.9}$$

(iii) There is a compatible G-map for f with respect to G-actions on $\mathbf{X}$ *and* $\mathbf{Y}$ *(with monoid action maps* $\zeta_{\mathbf{X}} : G \to \operatorname{haut}(\mathbf{X})$ *and* $\zeta_{\mathbf{Y}} : G \to \operatorname{haut}(\mathbf{Y})$*) if and only if there is a map* Ψ *making the following diagram commute up to homotopy:*

$$\begin{tikzcd} & \mathbf{B}\mathcal{P}_f \arrow[d, "(\mathbf{B}\delta{,}\mathbf{B}\varepsilon)"] \\ \mathbf{B}G \arrow[ur, "\Psi"] \arrow[r, "(\mathbf{B}\zeta_{\mathbf{X}}{,}\mathbf{B}\zeta_{\mathbf{Y}})"'] & \mathbf{B}\operatorname{haut}(\mathbf{X}) \times \mathbf{B}\operatorname{haut}(\mathbf{Y}). \end{tikzcd} \tag{2.10}$$

Proof. (i) If $\mathbf{X}$ is a G-space, and we have a right transfer $f' : \mathbf{X}' \to \mathbf{Y}'$, of the G-action along f, we may change f' into a G-cofibration, and the monoid action maps $\zeta_{\mathbf{X}'} : G \to \operatorname{haut}(\mathbf{X}')$ and $\zeta_{\mathbf{Y}'} : G \to \operatorname{haut}(\mathbf{Y}')$ then fit together to define a monoid map $z : G \to \check{\mathcal{P}}_{f'}$ in (2.6), which actually lands in $\mathcal{P}_{f'}$, since G is a group. Because $\mathbf{B}\zeta_{\mathbf{X}}$ is just $\mathbf{B}\zeta_{\mathbf{X}'}$, up to homotopy, $\mathbf{B}z : \mathbf{B}G \to \mathbf{B}\mathcal{P}_{f'}$ is the required lift in (2.8).

Conversely, given a lift Ψ in (2.8), by applying Kan's G-functor to (2.8), realizing, and then taking cofibrant replacement in the model category of strictly associative topological monoids (see [SV2, Theorem B]), we obtain a diagram of cofibrant (and grouplike) topological monoids:

$$\begin{tikzcd} & \widehat{\mathcal{P}}_f \arrow[d, "\widehat{\delta}"] \arrow[dr, "\widehat{\varepsilon}"] & \\ \widehat{G} \arrow[ur, "\widehat{\rho}"] & \widehat{\operatorname{haut}}(\mathbf{X}) & \widehat{\operatorname{haut}}(\mathbf{Y}) , \end{tikzcd} \tag{2.11}$$

with $\widehat{\mathcal{P}}_f$ weakly equivalent to $\mathcal{P}_f$.

Since $\mathbf{B}\,\mathrm{haut}(\widehat{\mathbf{X}}) \simeq \mathbf{B}\,\mathrm{haut}(\mathbf{X})$, by pulling back (1.2) we obtain a monoid action of $\widehat{G}$ on $\widehat{\mathbf{X}} \simeq \mathbf{X}$. By [Pr, Theorem 5.8] (see also [DL, F2], [Ma2, §7,9], [Bo, §5] and [DDK]), this is classified by a map $\hat{\theta} : E_{\hat{\theta}} \to \mathbf{B}\widehat{G}$. Up to homotopy, $\hat{\theta}$ corresponds to the map θ in the fibre bundle sequence:

$$\text{(2.12)} \qquad \mathbf{X}' :\ = \mathbf{E}G \times \mathbf{X} \ \to\ E_\theta \ \xrightarrow{\theta}\ \mathbf{B}G$$

classifying the free G-action on $\mathbf{X}'$ (Borel equivalent to the given $\mathbf{X}$). Similarly, we get a free $\widehat{G}$-action on $\widehat{\mathbf{Y}} \simeq \mathbf{Y}$, classified by $\hat{\kappa} : E_{\hat{\kappa}} \to \mathbf{B}\widehat{G}$ Moreover, we have a map $\widehat{f} : \widehat{\mathbf{X}} \to \widehat{\mathbf{Y}}$ (which is just $f : X \to Y$, up to homotopy), and we may assume that $\widehat{f}$ is itself a cofibration (for example, by carrying out the above construction in simplicial sets, and replacing $\widehat{\mathbf{Y}}$ by $\widehat{\mathbf{Y}} \times \mathbf{C}\widehat{\mathbf{X}}$ before realizing).

As in (2.6), we obtain a commuting diagram

$$\text{(2.13)} \qquad \begin{tikzcd} \mathcal{P}_{\widehat{f}} \arrow[r, "\widehat{\varepsilon}'"] \arrow[d, "\widehat{\delta}'"'] & \mathrm{haut}(\widehat{\mathbf{Y}}) \arrow[d, "\widehat{f}^*"] \\ \mathrm{haut}(\widehat{\mathbf{X}}) \arrow[r, "\widehat{f}_*"'] & \mathrm{Map}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) , \end{tikzcd}$$

in which $\widehat{f}^*$ is a fibration, so $\widehat{\delta}'$ is, too.

Because $\widehat{\mathbf{X}} \simeq \mathbf{X}$, $\widehat{\mathrm{haut}}(\mathbf{X})$ and $\mathrm{haut}(\mathbf{X})$ are weakly equivalent, and since the former is cofibrant and the latter is fibrant, we have a weak equivalence of monoids $k : \widehat{\mathrm{haut}}(\mathbf{X}) \to \mathrm{haut}(\mathbf{X})$, and similarly $\ell : \widehat{\mathrm{haut}}(\mathbf{Y}) \simeq \mathrm{haut}(\mathbf{Y})$. Moreover, since (2.13) is a homotopy pullback, $\mathcal{P}_f$ and $\mathcal{P}_{\widehat{f}}$ are weakly equivalent, and again we have a weak equivalence of monoids $h : \widehat{\mathcal{P}}_f \xrightarrow{\simeq} \mathcal{P}_{\widehat{f}}$. Thus the strict diagram (2.13) fits into a homotopy commutative diagram:

$$\text{(2.14)} \qquad \begin{tikzcd} \widehat{\mathcal{P}}_f \arrow[r, "\widehat{\varepsilon}"] \arrow[d, "\widehat{\delta}"'] \arrow[drr, "h"] & \widehat{\mathrm{haut}}(\mathbf{Y}) \arrow[drr, "\ell"] \arrow[dr, phantom, "\boxed{\mathrm{B}}"] & & \\ \widehat{\mathrm{haut}}(\mathbf{X}) \arrow[drr, "k"'] \arrow[rr, phantom, "\boxed{\mathrm{A}}"] & & \mathcal{P}_{\widehat{f}} \arrow[r, "\widehat{\varepsilon}'"] \arrow[d, "\widehat{\delta}'"] & \mathrm{haut}(\widehat{\mathbf{Y}}) \arrow[d, "\widehat{f}^*"] \\ & & \mathrm{haut}(\widehat{\mathbf{X}}) \arrow[r, "\widehat{f}_*"'] & \mathrm{Map}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) . \end{tikzcd}$$

In the model category of strictly associative monoids, we can replace h by another weak equivalence of monoids making $\boxed{\mathrm{A}}$ commute on the nose (cf. [BJT, Lemma 4.16]), and then changing ℓ into a fibration, we may replace $\widehat{\varepsilon}$ by a map making $\boxed{\mathrm{B}}$ commute strictly, too, without changing $\widehat{\mathcal{P}}_f$.

Composing the monoid map $\widehat{\rho} : \widehat{G} \to \widehat{\mathcal{P}}_f$ of (2.11) with $k \circ \widehat{\delta} : \widehat{\mathcal{P}}_f \to \mathrm{haut}(\widehat{\mathbf{X}})$ and $\ell \circ \widehat{\varepsilon} : \widehat{\mathcal{P}}_f \to \mathrm{haut}(\widehat{\mathbf{Y}})$, we obtain monoid action maps $\widehat{\zeta}_{\widehat{\mathbf{X}}} : \widehat{G} \to \mathrm{haut}(\widehat{\mathbf{X}})$ and $\widehat{\zeta}_{\widehat{\mathbf{Y}}} : \widehat{G} \to \mathrm{haut}(\widehat{\mathbf{Y}})$ making $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$ into strict $\widehat{G}$-spaces, with

$\widehat{f}:\widehat{\mathbf{X}}\hookrightarrow\widehat{\mathbf{Y}}$ a $\widehat{G}$-map (which is a cofibration). It therefore fits into a commuting diagram of principal $\widehat{G}$-bundles

$$\begin{tikzcd} \widehat{\mathbf{X}} \arrow[rr, hook, "\widehat{f}"] \arrow[d] & & \widehat{\mathbf{Y}} \arrow[d] \\ E_{\hat{\theta}} \arrow[rr, "\hat{\beta}"] \arrow[dr, "\hat{\theta}"'] & & E_{\hat{\kappa}} \arrow[dl, "\hat{\kappa}"] \\ & \mathbf{B}\widehat{G}\,. & \end{tikzcd} \tag{2.15}$$

Here $\hat{\beta}$ is obtained by realizing the bar construction for the $\widehat{G}$-actions on $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$, respectively (see [Pr, §5]), so it commutes up to homotopy with the classifying maps $\hat{\theta}$ and $\hat{\kappa}$ for the two bundles, where (as noted above) up to homotopy $\hat{\theta}$ is just $\theta:E_\theta\to\mathbf{B}G$, classifying the free G-space $\mathbf{X}'$.

Let $\kappa:E_{\hat{\kappa}}\to\mathbf{B}G$ denote the composite of $\hat{\kappa}$ with $\mathbf{B}\widehat{G}\simeq\mathbf{B}G$, classifying a free G-bundle $\mathbf{Y}'\to E_{\hat{\kappa}}$ with $\mathbf{Y}'\simeq\widehat{\mathbf{Y}}\simeq\mathbf{Y}$. If we also let $\beta:E_\theta\to E_{\hat{\kappa}}$ denote the composite of $\hat{\beta}$ with $E_\theta\simeq E_{\hat{\theta}}$, then $\kappa\circ\beta\simeq\theta$, so we have a map

$$\begin{tikzcd} \mathbf{X}' \arrow[rr, "f'"] \arrow[d] & & \mathbf{Y}' \arrow[d] \\ E_{\theta} \arrow[rr, "\beta"] \arrow[dr, "\theta"'] & & E_{\hat{\kappa}} \arrow[dl, "\kappa"] \\ & \mathbf{B}G & \end{tikzcd} \tag{2.16}$$

of principal G-bundles, so in particular, f' is a G-map.

Statements (ii) and (iii) are proven analogously. ■

Compare [Z, Proposition 2.2] for the compatibility version for homotopy actions.

2.17 Proposition. *Let $f:\mathbf{X}\to\mathbf{Y}$ be any map. In a right transfer of a G-action on $\mathbf{X}$ along f, we may assume that k in (2.2) is a homeomorphism; in a left transfer of a G-action on $\mathbf{Y}$ along f, we may assume that m and n in (2.3) are homeomorphisms; and in a compatible G-map for f with respect to G-actions on $\mathbf{X}$ and $\mathbf{Y}$, we may assume that either k, or m and n, are homeomorphisms in (2.4).*

Proof. If the action of G on $\mathbf{X}$ is *free* (and (2.8) holds), we can replace $\mathbf{X}':=\mathbf{E}G\times\mathbf{X}$ by $\mathbf{X}$ in (2.12), and therefore also in (2.16), so we have a right transfer $f':\mathbf{X}\hookrightarrow\mathbf{Y}'$ (along $f:\mathbf{X}\to\mathbf{Y}$) which is a G-map.

The same argument shows that if the action of G on $\mathbf{Y}$ is free (and (2.9) holds), it has a left transfer along f to a fibration $f':\mathbf{X}'\twoheadrightarrow\mathbf{Y}$ which is a G-map. Moreover, given free G-actions on both $\mathbf{X}$ and $\mathbf{Y}$, any $f:\mathbf{X}\to\mathbf{Y}$ has a compatible G-map $f':\mathbf{X}\to\mathbf{Y}$ with the same source and target (if (2.10) holds).

Since every G-space $\mathbf{Y}$ has a Borel G-equivalence $h:\mathbf{Y}'\to\mathbf{Y}$, where $\mathbf{Y}'$ is a free G-space, we do not actually need to assume that the action on $\mathbf{Y}$ is free, because we can compose the left transfer or compatible map f' with this h. Thus we may always assume that ℓ, m, and n are homeomorphisms.

Finally, given a G-space $\mathbf{X}$, we may replace it by the free G-space $\mathbf{X}' := \mathbf{X} \times \mathbf{E}G$ and produce a G-map $f' : \mathbf{X}' \hookrightarrow \mathbf{Y}'$ which is a cofibration. Then taking the pushout:

(2.18)
$$\begin{tikzcd} \mathbf{X}' \arrow[r, hook, "f'"] \arrow[d, "\simeq"', "q"] & \mathbf{Y}' \arrow[d, "\simeq"', "r"] \\ \mathbf{X} \arrow[r, hook, "f''"'] \arrow[ur, phantom, "\boxed{\mathrm{PO}}"] & \mathbf{Y}'', \end{tikzcd}$$

we see that all maps are G-maps; f', and thus f'', are cofibrations, so this is a homotopy pushout in $\mathcal{T}op$, and since q is a homotopy equivalences, so is r. ∎

2.19. Applications. In general, the lifting problems of Proposition 2.7 are hard to solve. However, in certain cases the obstructions to obtain the relevant liftings may be computable, or may vanish for dimension reasons. For example:

(i) When $\mathbf{X} = K(\pi, n)$ is an Eilenberg-Mac Lane space, then:

(2.20)
$$\operatorname{haut}(\mathbf{X}) \simeq K(\pi, n) \times \operatorname{Aut}(\pi)$$

(as a monoid) is a semi-direct product of the Eilenberg-Mac Lane space itself and the discrete group $\operatorname{Aut}(\pi)$, while $\operatorname{haut}_*(X) \simeq \operatorname{Aut}(\pi) \cong \pi_0 \operatorname{haut}_*(X)$ is homotopically discrete (see [Ma1, Proposition 25.2]).

Thus if both $\mathbf{X}$ and $\mathbf{Y}$ are Eilenberg-Mac Lane spaces, all but the pullback itself in (2.6) are generalized Eilenberg-Mac Lane spaces, so each of the lifting problems (2.8), (2.9), and (2.10) reduces to an algebraic question about certain classes in the cohomology of $\mathbf{B}G$ (as expected).

(ii) A more interesting example is when each of $\mathbf{X}$ and $\mathbf{Y}$ has only two non-trivial homotopy groups (see §3.16 below). In this case the homotopy groups of $\operatorname{haut}(\mathbf{X})$ and $\operatorname{haut}(\mathbf{Y})$ are completely known by [Di, §3] (see also [Mo]), and in particular if $\pi_i\mathbf{X} = 0$ for $i \neq k, m$ $(k < m)$, then $\pi_i \operatorname{haut}(\mathbf{X}) = 0$ unless $k \leq i \leq m$. Using the Postnikov tower for $\mathbf{Y}$ we can also determine $\pi_* \operatorname{Map}(\mathbf{X}, \mathbf{Y})$, up to extension. Therefore the homotopy groups of $\mathcal{P}_f$ may also be determined, up to extension.

Since we need not assume G is finite, $H^i(\mathbf{B}G; \pi)$ may vanish for large enough i, at least when π is one of the groups $\pi_*\mathcal{P}_f$ Thus in certain cases we can show that there is no obstruction to solving the lifting problems.

(iii) The case when $\mathbf{X}$ and $\mathbf{Y}$ are spheres has been the subject of intense study over the years, beginning with [PSm]. Moreover, much is known about $\operatorname{haut}(\mathbf{S}^n)$ and $\operatorname{haut}_*(\mathbf{S}^n)$ (see, e.g., [Ha1, Ha2]). Therefore, one might be able to compute obstructions to extending or lifting certain group actions on spheres along some map $f : \mathbf{S}^n \to \mathbf{S}^m$, or making f equivariant.

3 Interpolating group actions

For our approach to the realization problem for Bredon homotopy actions, we need to consider a slightly more complicated situation than that studied in the previous section: assume given a sequence of maps

$$\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{Z} \tag{3.1}$$

with given G-actions on $\mathbf{X}$ and $\mathbf{Z}$, for which we want to find a compatible G-action on $\mathbf{Y}$.

3.2 Definition. A *G-interpolation* for two G-spaces $\mathbf{X}$ and $\mathbf{Z}$ and maps as in (3.1) is a pair of G-maps $f' : \mathbf{X}' \to \mathbf{Y}'$ and $g' : \mathbf{Y}' \to \mathbf{Z}'$ fitting into a diagram

$$\begin{tikzcd}
 & \mathbf{X} \arrow[r, "f"] & \mathbf{Y} \arrow[r, "g"] \arrow[d, "h"] & \mathbf{Z} & \mathbf{Z}'' \arrow[l, "m"', "\simeq"] \arrow[dl, "\simeq"', "n"] \\
\mathbf{X}'' \arrow[ur, "k", "\simeq"'] \arrow[r, "\simeq", "\ell"'] & \mathbf{X}' \arrow[r, "f'"] & \mathbf{Y}' \arrow[r, "g'"] & \mathbf{Z}' &
\end{tikzcd} \tag{3.3}$$

in which k, ℓ, m, and n are all homotopy equivalences and G-maps, and h is a homotopy equivalence, which becomes homotopy-commutative after inverting m or n (up to homotopy).

3.4 Definition. Given two composable maps as in (3.1), let $\mathcal{Q}_{f,g}$ denote the homotopy pullback in:

$$\begin{tikzcd}
\mathcal{Q}_{f,g} \arrow[r, "\nu_g"] \arrow[d, "\mu_f"'] \arrow[dr, phantom, "\boxed{\text{PB}}"] & \mathcal{P}_g \arrow[d, "\delta_g"] \\
\mathcal{P}_f \arrow[r, "\varepsilon_f"'] & \mathrm{haut}(\mathbf{Y})\,.
\end{tikzcd} \tag{3.5}$$

(see §2.5). If f and g are cofibrations, both $\mathcal{P}_f$ and $\mathcal{P}_g$ are actually pullbacks, and the map $\delta : \mathcal{P}_g \to \mathrm{haut}(\mathbf{Y})$ is a fibration, so $\mathcal{Q}_{f,g}$ is the ordinary pullback. Furthermore, it is a grouplike strictly associative monoid, and the maps μ and ν are monoid maps.

3.6 Proposition. *Two maps as in (3.1) for G-spaces $\mathbf{X}$ and $\mathbf{Z}$ (with monoid action maps $\zeta_{\mathbf{X}} : G \to \mathrm{haut}(\mathbf{X})$ and $\zeta_{\mathbf{Z}} : G \to \mathrm{haut}(\mathbf{Z})$, respectively) have a G-interpolation if and only if there is a map Ψ making the following diagram commute up to homotopy:*

$$\begin{tikzcd}
 & & \mathbf{B}\,\mathrm{haut}(\mathbf{X}) \\
\mathbf{B}G \arrow[urr, "\mathbf{B}\zeta_{\mathbf{X}}"] \arrow[r, dotted, "\Psi"] \arrow[drr, "\mathbf{B}\zeta_{\mathbf{Z}}"'] & \mathbf{B}\mathcal{Q}_{f,g} \arrow[ur, "\mathbf{B}\delta_f \circ \mathbf{B}\mu"] \arrow[dr, "\mathbf{B}\varepsilon_g \circ \mathbf{B}\nu"'] & \\
 & & \mathbf{B}\,\mathrm{haut}(\mathbf{Z}).
\end{tikzcd} \tag{3.7}$$

Proof. Given a G-interpolation, the diagram (3.7) is obtained by applying $\mathbf{B}$ to the corresponding monoid action maps.

Conversely, a homotopy commutative diagram (3.7) may be lifted (together with (3.5)) to a commuting diagram of topological monoids:

(3.8)
$$\begin{tikzcd}
 & \widehat{G} \arrow[r, "\widehat{\rho}"] & \widehat{\mathcal{Q}}_{f,g} \arrow[dl, "\widehat{\mu}_f"'] \arrow[dr, "\widehat{\nu}_g"] & & \\
 & \widehat{\mathcal{P}}_f \arrow[dl, "\widehat{\delta}_f"'] \arrow[dr, "\widehat{\varepsilon}_f"] & & \widehat{\mathcal{P}}_g \arrow[dl, "\widehat{\delta}_g"'] \arrow[dr, "\widehat{\varepsilon}_g"] & \\
\widehat{\mathrm{haut}}(\mathbf{X}) & & \widehat{\mathrm{haut}}(\mathbf{Y}) & & \widehat{\mathrm{haut}}(\mathbf{Z})
\end{tikzcd}$$

and we have spaces $\widehat{\mathbf{X}} \simeq \mathbf{X}$, $\widehat{\mathbf{Y}} \simeq \mathbf{Y}$, and $\widehat{\mathbf{Z}} \simeq \mathbf{Z}$ on which $\widehat{\mathrm{haut}}(\mathbf{X})$, $\widehat{\mathrm{haut}}(\mathbf{Y})$, and $\widehat{\mathrm{haut}}(\mathbf{Z})$, respectively act. Moreover, we have maps $\widehat{f} : \widehat{\mathbf{X}} \to \widehat{\mathbf{Y}}$ and $\widehat{g} : \widehat{\mathbf{Y}} \to \widehat{\mathbf{Z}}$ (corresponding up to homotopy to f and g, respectively), and as in the proof of Proposition 2.7, we may assume $\widehat{f}$ and $\widehat{g}$ are cofibrations.

Therefore, (3.8) fits into a diagram:

(3.9)
$$\begin{tikzcd}
 & & \widehat{\mathcal{Q}}_{f,g} \arrow[dl, "\widehat{\mu}_f"'] \arrow[dr, "\widehat{\nu}_g"] & & \\
 & \widehat{\mathcal{P}}_f \arrow[dl, "\widehat{\delta}_f"'] \arrow[dr, "\widehat{\varepsilon}_f"] \arrow[dd, dotted, "h"] & & \widehat{\mathcal{P}}_g \arrow[dl, "\widehat{\delta}_g"'] \arrow[dr, "\widehat{\varepsilon}_g"] \arrow[dd, dotted, "k"] & \\
\widehat{\mathrm{haut}}(\mathbf{X}) \arrow[dd, "\ell"] & & \widehat{\mathrm{haut}}(\mathbf{Y}) \arrow[dd, "m"] & & \widehat{\mathrm{haut}}(\mathbf{Z}) \arrow[dd, "n"] \\
 & \mathcal{P}_{\widehat{f}} \arrow[dl, "\delta_{\widehat{f}}"'] \arrow[dr, "\varepsilon_{\widehat{f}}"] & & \mathcal{P}_{\widehat{g}} \arrow[dl, "\delta_{\widehat{g}}"'] \arrow[dr, "\varepsilon_{\widehat{g}}"] & \\
\mathrm{haut}(\widehat{\mathbf{X}}) & & \mathrm{haut}(\widehat{\mathbf{Y}}) & & \mathrm{haut}(\widehat{\mathbf{Z}})
\end{tikzcd}$$

in which $\mathcal{P}_{\widehat{f}}$ and $\mathcal{P}_{\widehat{g}}$ are homotopy limits, so the homotopy equivalences ℓ, m, and n induce homotopy equivalences h and k making (3.9) homotopy-commutative. Since $\mathcal{Q}_{\widehat{f},\widehat{g}}$ is also a homotopy limit, h and k induce a homotopy equivalence $p : \widehat{\mathcal{Q}}_{f,g} \to \mathcal{Q}_{\widehat{f},\widehat{g}}$.

Composing $p \circ \widehat{\rho} : \widehat{G} \to \mathcal{Q}_{\widehat{f},\widehat{g}}$ of diagram (3.8) with the appropriate structure maps for $\mathcal{Q}_{\widehat{f},\widehat{g}}$ yields monoid action maps $\widehat{\zeta}_{\widehat{\mathbf{X}}} : \widehat{G} \to \mathrm{haut}(\widehat{\mathbf{X}})$, $\widehat{\zeta}_{\widehat{\mathbf{Y}}} : \widehat{G} \to \mathrm{haut}(\widehat{\mathbf{Y}})$, and $\widehat{\zeta}_{\widehat{\mathbf{Z}}} : \widehat{G} \to \mathrm{haut}(\widehat{\mathbf{Z}})$ making $\widehat{f}$ and $\widehat{g}$ into $\widehat{G}$-equivariant maps (by definition of $\mathcal{Q}_{\widehat{f},\widehat{g}}$), with $\widehat{\zeta}_{\widehat{\mathbf{X}}}$ and $\widehat{\zeta}_{\widehat{\mathbf{Z}}}$ corresponding up to homotopy to the given monoid action maps $\zeta_{\mathbf{X}} : G \to \mathrm{haut}(\mathbf{X})$ and $\zeta_{\mathbf{Z}} : G \to \mathrm{haut}(\mathbf{Z})$.

Passing to the classifying maps for the corresponding principle G- and $\widehat{G}$-bundles as in the proof of Proposition 2.7, we obtain the required G-interpolation. ∎

We now have the following analogue of Proposition 2.17:

3.10 Proposition. *If $p := g \circ f : \mathbf{X} \to \mathbf{Z}$ is a G-map in (3.1), in any G-interpolation for we may assume that k, ℓ, m, and n are homeomorphisms in (3.3), and that $g' \circ f' = p$.*

Proof. In the proof of Proposition 3.6 we saw that the lifting Ψ in (3.7) allows us to factor the map of free G-spaces (i.e., total spaces of principle G-bundles)

$$\mathbf{X}' := \mathbf{X} \times \mathbf{E}G \xrightarrow{p' := p \times \mathrm{Id}_{\mathbf{E}G}} \mathbf{Z} \times \mathbf{E}G =: \mathbf{Z}'$$

as the composite of two maps of free G-spaces $\mathbf{X}' \xrightarrow{f'} \mathbf{Y}' \xrightarrow{g'} \mathbf{Z}'$ with $\mathbf{Y} \simeq \mathbf{Y}'$ (using a homotopy-factorization of the corresponding classifying maps of the bundles). Applying the (homotopy) pushout (2.18) we obtain a commutative diagram of G-spaces:

(3.11)
$$\begin{tikzcd}
\mathbf{X}' \arrow[r, hook, "f'"] \arrow[d, "\simeq"', "q_{\mathbf{X}}"] & \mathbf{Y}' \arrow[r, "g'"] \arrow[d, "\simeq"', "r"] & \mathbf{Z}' \arrow[r, "q_{\mathbf{Z}}"] & \mathbf{Z} \\
\mathbf{X} \arrow[r, hook, "f''"'] \arrow[ur, phantom, "\boxed{\mathrm{PO}}"] & \mathbf{Y}'' \arrow[urr, dotted, "g''"'] & &
\end{tikzcd}$$

where the map g'' out of the pushout is induced by $q_{\mathbf{Z}} \circ g' : \mathbf{Y}' \to \mathbf{Z}$ and $p := f \circ g : \mathbf{X} \to \mathbf{Z}$, which agree on $\mathbf{X}'$ since $p' = g' \circ f' = (g \circ f) \times \mathrm{Id}_{\mathbf{E}G}$. ∎

3.12 Definition. Given two maps as in (3.1) for G-spaces $\mathbf{X}$ and $\mathbf{Z}$, let $\rho : \mathbf{B}\mathcal{Q}_{f,g} \to \mathbf{B}\,\mathrm{haut}(\mathbf{X}) \times \mathbf{B}\,\mathrm{haut}(\mathbf{Z})$ be the map $(\mathbf{B}\delta_f \circ \mathbf{B}\mu,\ \mathbf{B}\varepsilon_g \circ \mathbf{B}\nu)$ of (3.7), and let:

(3.13)
$$\begin{tikzcd}
\mathbf{B}\mathcal{Q}_{f,g}\ \dots \arrow[r] \arrow[rrrr, bend left, "\rho"] & W_{n+1} \arrow[r, "p_{n+1}"] & W_n \arrow[r, "p_n"] & W_{n-1} & \dots \mathbf{B}\,\mathrm{haut}(\mathbf{X}) \times \mathbf{B}\,\mathrm{haut}(\mathbf{Z})
\end{tikzcd}$$

be the Moore-Postnikov tower for ρ, with $W_0 := \mathbf{B}\,\mathrm{haut}(\mathbf{X}) \times \mathbf{B}\,\mathrm{haut}(\mathbf{Z})$ (cf. [GJ, VI, 3.9]).

If F denotes the homotopy fiber of ρ, then up to homotopy each map $p_{n+1} : W_{n+1} \to W_n$ is a fibration with fiber $K(\pi_{n+1}F, n+1)$, which is classified by a map $\tilde{k}_n : W_n \to K_{\pi_1 Q_n}(\pi_{n+1}F, n+2)$ (see [R, Theorem 3.4]). Assume by induction on $n \geq 0$ that we have constructed a lift $g_n : \mathbf{B}G \to W_n$ for $g_0 := (\mathbf{B}\zeta_{\mathbf{X}}, \mathbf{B}\zeta_{\mathbf{Z}}) : \mathbf{B}G \to W_0 = \mathbf{B}\,\mathrm{haut}(\mathbf{X}) \times \mathbf{B}\,\mathrm{haut}(\mathbf{Z})$. Then the *$n$-th obstruction class* for this lift is

$$[\tilde{k}_n \circ g_n] \in [\mathbf{B}G,\ K_{\pi_1 Q_n}(\pi_{n+1}F, n+2)] \ \cong\ H^{n+2}(G;\ \pi_{n+1}F)\ ,$$

where G acts on $\pi_{n+1}F$ via $(g_n)_\# : G \to \pi_1 W_n$.

From Proposition 3.6 we deduce:

3.14 Proposition. *Two maps as in (3.1) for G-spaces $\mathbf{X}$ and $\mathbf{Z}$ (with monoid action maps $\zeta_{\mathbf{X}} : G \to \mathrm{haut}(\mathbf{X})$ and $\zeta_{\mathbf{Z}} : G \to \mathrm{haut}(\mathbf{Z})$, respectively) have a G-interpolation if and only if the obstruction classes $[\tilde{k}_n \circ g_n] \in H^{n+2}(G; \pi_{n+1}F)$ successively vanish, for some sequence of lifts. There is also a sequence of* difference obstructions *in $H^{n+1}(G; \pi_{n+1}F)$ for distinguishing between non-homotopic lifts in (3.7).*

3.15 Remark. As with any obstruction theory, non-vanishing of a cohomology class $[\tilde{k}_n \circ g_n]$ merely requires that we back-track to an earlier stage and try different choices, so in reality we have tree of obstructions, and the G-interpolation exists if and only if *some* branch extends to infinity.

3.16 Example. We can use the method described here to study G-actions on a space $\mathbf{Y}$ if $\pi_i\mathbf{Y} = 0$ for $i \neq k, m$ $(k < m)$: In this case we can choose in $\mathbf{X} := K(\pi, m)$ and $\mathbf{Z} := K(\pi', k)$ in (3.1), with given actions of G on π and π', and use Proposition 3.6 to interpolate a G-action on $\mathbf{Y}$. As noted in §2.19, for suitable choices of G the obstructions of Proposition 3.14 will vanish (e.g., for reasons of dimension).

4 Realizing diagrammatic homotopy actions

From now on we assume that G is finite (but see §4.16 below). Given a G-space $\mathbf{X}$, the associated fixed point set diagram $\underline{X}$ encodes the Bredon G-homotopy type of $\mathbf{X}$, by Theorem 1.6. This diagram consists of the various fixed point sets $\mathbf{X}^H$ $(H \leq G)$, the inclusions $i^* : \mathbf{X}^K \hookrightarrow \mathbf{X}^H$ induced by $i : H \hookrightarrow K$, and the G-action by conjugation: $\mathbf{X}^H \to \mathbf{X}^{H^a}$. Our goal is to provide a "homotopy version" of $\underline{X}$, and describe a procedure for realizing it by attempting to solve a sequence of simpler lifting problems as in Section 3.

4.1. Filtering $\mathcal{O}_G^{\mathrm{op}}$. For any subgroup H of G, we define the *length* of H in G, denoted by $\mathrm{len}_G\, H$, to be the maximal $0 \leq k < \infty$ such that there exists a sequence of proper inclusions of subgroups:

$$H = H_0 < H_1 < H_2 < \ldots < H_{k-1} < H_k = G\,. \tag{4.2}$$

This induces a filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \mathcal{F}_k \subset \ldots \subset \mathcal{O}_G^{\mathrm{op}} \tag{4.3}$$

by full subcategories, where $\mathrm{Obj}\ \mathcal{F}_k := \{G/H \in \mathcal{O}_G^{\mathrm{op}} : \mathrm{len}_G\, H \leq k\}$ (so $\mathrm{Obj}\ \mathcal{F}_0 = \{G/G\}$).

Since G is finite, the filtration is exhaustive: if $\mathrm{len}_G\{e\} = N$ – that is, the longest possible sequence (4.2) in G has N inclusions of proper subgroups – then $\mathcal{F}_N = \mathcal{O}_G^{\mathrm{op}}$. We let $\widehat{\mathcal{F}}_k$ denote the collection of subgroups $H < G$ such that $G/H \in \mathcal{F}_k$.

Let $\langle H \rangle := \{H^a : a \in G\}$ denote the conjugacy class of a subgroup $H \leq G$: note that if $H \in \widehat{\mathcal{F}}_k$, then $\langle H \rangle \subseteq \widehat{\mathcal{F}}_k$.

4.4 Definition. Let Λ denote the partially ordered set of subgroups of G; we can think o f the opposite category Λ^{op} as a subcategory of $\mathcal{O}_G$. The full subcategory Λ_H consists of all subgroups K with $H < K \leq G$, and Λ_k is the full subcategory of objects in filtration $\mathcal{F}_k$.

4.5 Definition. A *Bredon homotopy action* of G $\langle \underset{\sim}{X}, (\Phi^*_H)_{\langle H \rangle \subseteq \Lambda} \rangle$ consists of:

(i) A diagram $\underset{\sim}{X} : \Lambda^{\mathrm{op}} \to \mathcal{T}op$.

(ii) A choice of a representative H in each conjugacy class $\langle H \rangle \subseteq \Lambda$, equipped with a pointed homotopy W_H-action $\Phi^*_H : \mathbf{B}W_H \to \mathbf{B}\,\mathrm{haut}_*(\underset{\sim H}{\mathbf{X}}^H)$ on $\underset{\sim H}{\mathbf{X}}^H$, defined by the homotopy cofibration sequence:

(4.6)
$$\underset{\sim H}{\mathbf{X}} \to \underset{\sim}{X}(H) \to \underset{\sim H}{\mathbf{X}}^H ,$$

where $\underset{\sim H}{\mathbf{X}} := \mathrm{hocolim}_{\Lambda^{\mathrm{op}}_H} \underset{\sim}{X}(K)$.

We require that if H' and H are conjugate, their homotopy cofibration sequences (4.6) fit into a homotopy-commuting square (0.2).

4.7 Definition. A cofibrant diagram $\underline{X}_k : \mathcal{F}_k \to \mathcal{T}op$ (in the projective model category $\mathcal{T}op^{\mathcal{F}_k}$ – cf. [Hi, §11.6]) *realizes* a Bredon homotopy action $\langle \underset{\sim}{X}, (\Phi^*_H)_{H \leq G} \rangle$ *in the k-th filtration* if:

(a) The corresponding homotopy diagram $(\gamma \circ \underline{X}_k)|_{\Lambda^{\mathrm{op}}_k} : \Lambda^{\mathrm{op}}_k \to \mathrm{ho}\,\mathcal{T}op$ is weakly equivalent to $\gamma \circ \underset{\sim}{X}|_{\Lambda^{\mathrm{op}}_k}$, for $\gamma : \mathcal{T}op \to \mathrm{ho}\,\mathcal{T}op$ the quotient functor.

(b) For each $H \in \mathcal{F}_k$, the pointed action of W_H on the cofiber of

(4.8)
$$\mathrm{colim}_{K>H}\, \underline{X}_k(G/K) \to \underline{X}_k(G/H)$$

realizes the pointed homotopy action Φ^*_H.

Note that because $\underline{X}_k$ is cofibrant, this colimit is a homotopy colimit and (4.8) is a cofibration, and because of (0.2), the homotopy action Φ^*_H is defined for *every* $H \leq G$, not only our chosen representatives.

A sequence $(\underline{X}_k : \mathcal{F}_k \to \mathcal{T}op)_{k=0}^{\infty}$ of such diagrams is *coherent* if $\underline{X}_k|_{\mathcal{F}_{k-1}} = \underline{X}_{k-1}$ for each $k \geq 1$.

4.9 Example. If $\mathbf{X}$ is a G-CW complex, let $\underset{\sim}{X}$ be the restriction of $\underline{X} : \mathcal{O}^{\mathrm{op}}_G \to \mathcal{T}op$ to the subcategory Λ^{op}. For any $H \leq G$, $\underset{\sim H}{\mathbf{X}} := \mathrm{hocolim}_{\Lambda^{\mathrm{op}}_H} \underset{\sim}{X}(K)$ is simply $\mathbf{X}_H := \bigcup_{H<K} \mathbf{X}^K$, which is a sub-$W_H$-complex of $\mathbf{X}^H$. The quotient $\mathbf{X}^H_H := \mathbf{X}^H/\mathbf{X}_H$ is the cofiber of the inclusion $j_H : \mathbf{X}_H \hookrightarrow \mathbf{X}^H$, which is a free pointed W_H-space (unless $\mathbf{X}_H = \emptyset$), with monoid action map $\zeta^*_{\mathbf{X}^H_H} : W_H \to \mathrm{haut}_*(\mathbf{X}^H_H)$ and $\Phi^*_H := \mathbf{B}\zeta^*_{\mathbf{X}^H_H} : \mathbf{B}W_H \to \mathbf{B}\,\mathrm{haut}_*(\mathbf{X}^H_H)$. Evidently $\underline{X} : \mathcal{F}_\infty = \mathcal{O}^{\mathrm{op}}_G \to \mathcal{T}op$ realizes the Bredon homotopy action $\langle \underline{X}|_{\Lambda^{\mathrm{op}}}, (\Phi^*_H)_{H \leq G} \rangle$ we have just defined (in all filtrations). In this case, we also say that the G-space $\mathbf{X}$ *realizes* $\langle \underline{X}|_{\Lambda^{\mathrm{op}}}, (\Phi^*_H)_{H \leq G} \rangle$.

4.10 Definition. If $\underline{X}_k : \mathcal{F}_k \to \mathcal{T}op$ realizes a Bredon homotopy action $\langle \underset{\sim}{X}, (\Phi^*_H)_{H \leq G}\rangle$ in the k-th filtration, we may realize the pointed homotopy action Φ^*_H of W_H on $\underset{\sim}{\mathbf{X}}^H_H$ by a topological pointed action of W_H on a space $\underline{\mathbf{X}}^H_H \simeq \underset{\sim}{\mathbf{X}}^H_H$ (see Proposition 5.8 below). This fits into a homotopy cofibration sequence:

$$\underline{\mathbf{X}}_H \xrightarrow{j_H} \underset{\sim}{X}(H) \xrightarrow{q_H} \underline{\mathbf{X}}^H_H\,, \tag{4.11}$$

where $\underline{\mathbf{X}}_H ; = \mathrm{hocolim}_{K > H}\, \underline{X}_k(G/K)$ (homotopic to (4.6)). Note that the action of $N_G H$ on $\mathcal{F}_k$ by conjugation defines a W_H-action on $\underline{\mathbf{X}}_H$.

The *H-lifting problem* for $\underline{X}_k$ is to find a map Ψ_H making the following diagram commute up to homotopy:

$$\begin{tikzcd}
 & & \mathbf{B}\,\mathrm{haut}(\underline{\mathbf{X}}_H) \\
\mathbf{B}W_H \arrow[urr, "\mathbf{B}\zeta_{\underline{\mathbf{X}}_H}"] \arrow[r, dotted, "\Psi_H"] \arrow[drr, "\mathbf{B}\zeta_{\underline{\mathbf{X}}^H_H}"'] & \mathbf{B}\mathcal{Q}_{j_H,q_H} \arrow[ur, "\mathbf{B}\delta_{j_H}\circ\mathbf{B}\mu"'] \arrow[dr, "\mathbf{B}\varepsilon_{q_H}\circ\mathbf{B}\nu"] & \\
 & & \mathbf{B}\,\mathrm{haut}(\underline{\mathbf{X}}^H_H)
\end{tikzcd} \tag{4.12}$$

in the notation of (3.5).

We are now in a position to state our main result:

4.13 Theorem. *A Bredon homotopy action $\mathcal{A} := \langle \underset{\sim}{X}, (\Phi^*_H)_{H\leq G}\rangle$ for a finite group G can be realized by a G-space $\mathbf{X}$ if and only if one can inductively construct a coherent sequence of cofibrant diagrams $(\underline{X}_k : \mathcal{F}_k \to \mathcal{T}op)^\infty_{k=0}$ realizing $\mathcal{A}$, where one can extend $\underline{X}_k$ to $\underline{X}_{k+1}$ if and only if for each $\langle H\rangle \subseteq \widehat{\mathcal{F}}_{k+1} \setminus \widehat{\mathcal{F}}_k$, there is an $H \in \langle H\rangle$ for which the H-lifting problem (4.12) can be solved.*

Proof. If $\mathcal{A}$ can be realized by a G-space $\mathbf{X}$, the corresponding diagrams $\underline{X}_k$ were described in Example 4.9.

To see that solving the H-lifting problem suffices to extend an inductively-defined $\underline{X}_k$ to $\underline{X}_{k+1}$, we start with $\underline{X}_0(G/G) := \underset{\sim}{X}(G)$ (which we denote by $\mathbf{Y}$). To construct $\underline{X}_1$, we must consider all maximal proper subgroups $M \in \widehat{\mathcal{F}}_1$, which are of two types:

(a) If $N_G M = M$, then $W_M = \{e\}$ and the correspondence $aM \mapsto M^a$ is a bijection between G/M and $\langle M\rangle$. In this case we change $\mathbf{Y} = \underset{\sim}{X}(G/G) \to \underset{\sim}{X}(G/M)$ into a cofibration $i : \mathbf{Y} \hookrightarrow \mathbf{Z}_{(M)}$ (with no group action), and form a diagram consisting of a copy $i_{(M^a)} : \mathbf{Y} \hookrightarrow \mathbf{Z}_{(M^a)}$ of i for each coset $M^a \in \langle M\rangle$, with $\underline{X}_1((\widetilde{\phi}^M_a)^{\mathrm{op}})$ the homeomorphism identifying $\mathbf{Z}$ with $\mathbf{Z}_{(M^a)}$ (relative to the fixed subspace $\mathbf{Y}$).

(b) Otherwise $N_G M = G$, so $W_M = G/M$ and $\langle M\rangle$ is a singleton. We then apply Proposition and 3.6 to obtain a W_M-action on $\mathbf{Z}_{(M)} \simeq \underset{\sim}{X}(G/M)$, extending the trivial action on $\mathbf{Y} := \underset{\sim}{X}(G/G)$. This is possible since we assume that the M-lifting problem can be solved. The action map $\zeta_{\mathbf{Z}_{(M)}} : G/M \to \mathrm{Aut}(\mathbf{Z}_{(M)})$ lifts to a G-action via the homomorphism $G \twoheadrightarrow G/M$.

Since all the conjugation G-actions we have described agree on $\mathbf{Y}$ (where they are trivial), we obtain a diagram $\underline{X}_1 : \mathcal{F}_1 \to \mathcal{T}op$, whose restriction to Λ_1^{op} consists of the inclusions $\mathbf{Y} \hookrightarrow \mathbf{Z}_{(M)}$ for all $M \in \widehat{\mathcal{F}}_i \setminus \widehat{\mathcal{F}}_{i-1}$.

At the k-th stage of the induction, we assume given a cofibrant diagram $\underline{X}_{k-1} : \mathcal{F}_{k-1} \to \mathcal{T}op$ realizing $\underset{\sim}{X}$ up to filtration $k-1$. In particular, for each $H \in \widehat{\mathcal{F}}_k \setminus \widehat{\mathcal{F}}_{k-1}$ we have a space $\underline{X}_H := (\underline{X}_{k-1})_H$ as in §4.4, on which $N_G H$ acts (by conjugation), with $H \subseteq N_G H$ acting trivially. Thus $\underline{X}_H$ has a W_H-action compatible with the structure maps of $\underline{X}_{k-1}$.

For each conjugacy class $\langle H\rangle \subseteq \widehat{\mathcal{F}}_k \setminus \widehat{\mathcal{F}}_{k-1}$, we have a specified representative H. We use Proposition 5.8 to lift the given pointed homotopy action of W_H on $\underset{\sim H}{\mathbf{X}}^H$ to a (free) pointed action on $\mathbf{X}_H^H \simeq \underset{\sim H}{\mathbf{X}}^H$. Next, use Proposition 3.6 to produce a W_H-interpolation of the given W_H-actions on $\underline{X}_H$ and $\mathbf{X}_H^H$ for the homotopy cofibration sequence (4.6). Denote the new W_H-space we have produced by $\mathbf{Z}_{(H)} \simeq \underset{\sim}{X}(G/H)$. By Proposition 3.10, we may assume that the inclusion $i_{(H)} : \underline{X}_H \hookrightarrow \mathbf{Z}_{(H)}$ is W_H-equivariant (with respect to the given conjugation action on $\underline{X}_{k-1}$).

For any conjugate $H^a \in \langle H\rangle$, choose a fixed element $a \in G$ representing the coset $aN_G H \in G/N_G H \cong \langle H\rangle$, and let $\underline{X}_k(G/H^a) := \mathbf{Z}_{(H)}$. The W_{H^a}-action on $\underline{X}_k(G/H^a)$ is the composite of the action map $W_H \to \mathrm{Aut}(\mathbf{Z}_{(H)})$ with the isomorphism $(\rho_a^H)_*^{-1} : W_{H^a} \to W_H$ induced by $\rho_a^H : N_G H \to N_G H^a$ (conjugation by a).

We define $i_{H^a} : \underline{X}_{H^a} \hookrightarrow \mathbf{Z}_{(H)}$ to be the composite $i_{(H)} \circ (\widetilde{\phi}_a^H)^{\mathrm{op}}$. This is W_{H^a}-equivariant because $(\widetilde{\phi}_a^H)^{\mathrm{op}}$ is induced by ρ_a^H, so we have extended $\underline{X}_{k-1}$ to a diagram $\underline{X}_k : \mathcal{F}_k \to \mathcal{T}op$.

At the end of the process we have a full $\mathcal{O}_G^{\mathrm{op}}$ diagram $\underline{X}_\infty := \mathrm{colim}_{k\to\infty} \underline{X}_k$, and thus (by Theorem 1.6) a G-space $\mathbf{X}$ realizing the given Bredon homotopy action $\mathcal{A}$.

Note that homotopic maps $\Phi^* \sim (\Phi')^* : \mathbf{B}W_H \to \mathbf{B}\,\mathrm{haut}_*(\underset{\sim H}{\mathbf{X}}^H)$ induce pointed Borel W_H-equivalences $\mathbf{X}_H^H \to (\mathbf{X}')_H^H$ (assuming both are W_H-CW complexes), and homotopic lifts $\Psi \sim \Psi' : \mathbf{B}W_H \to \mathbf{B}\mathcal{Q}_{j_H, q_H}$ in (4.12) yield Borel equivalent W_H-spaces $\mathbf{Z}_H$ and $\mathbf{Z}'_H$, which implies that we have a weak equivalence of the resulting $\mathcal{F}_{k+1}$-diagrams $\underline{X}_{k+1}$ and $\underline{X}'_{k+1}$, since all the structure maps which are not inclusions can be described in terms of the conjugation action of G. ∎

4.14 Definition. If $\underline{X}_k : \mathcal{F}_k \to \mathcal{T}op$ realizes a Bredon homotopy action $\mathcal{A}$ in the k-th filtration, for each conjugacy class $\langle H\rangle \subseteq \widehat{\mathcal{F}}_{k+1} \setminus \widehat{\mathcal{F}}_k$, choose any representative $H \in \langle H\rangle$. The *$\langle H\rangle$-sequence of obstructions* $(e_n)_{n=1}^\infty$ to extending $\underline{X}_k$ to $\underline{X}_{k+1}$

is defined by letting $e_n \in H^{n+2}(W_H; \pi_{n+1}F)$ denote the n-th obstruction of Proposition 3.14 for the H-lifting problem (4.12).

The *difference* obstructions $f_n \in H^{n+1}(W_H; \pi_{n+1}F)$ for distinguishing between different extensions of $\underline{X}_k$ to $\underline{X}_{k+1}$ are defined analogously.

4.15 Corollary. *For any finite group G, a Bredon homotopy action $\mathcal{A}$ can be realized by a G-space $\mathbf{X}$ if and only if for each $k \geq 0$ and $\langle H \rangle \subseteq \widehat{\mathcal{F}}_{k+1} \setminus \widehat{\mathcal{F}}_k$, (some branch of) the inductively defined $\langle H \rangle$-sequence of obstructions $(e_n)_{n=1}^{\infty}$ vanishes. Moreover, two such realizations $\mathbf{X}$ and $\mathbf{X}'$ (by G-CW complexes) are G-homotopy equivalent if the corresponding sequence of difference obstructions vanish.*

Remark 3.15 applies here too, of course.

4.16. Generalizations. The procedure described above extends to some infinite groups G, as long as we have a class function $\ell : \Lambda \to \kappa$ into some ordinal κ with $\ell(K) \preceq \ell(H)$ for $H \leq K$. In this case we have a filtration corresponding to (4.3) of length κ, and thus a transfinite inductive procedure as in the proof of Theorem 4.13.

For example, if $G = \mathbb{Z}$ then $\ell : \Lambda \to \omega + 1$ assigns to $n\mathbb{Z} \leq \mathbb{Z}$ the number of (not necessarily distinct) prime factors of n, with $\ell(\{0\}) = \omega$. On the other hand, there is no such function ℓ for $G = \mathbb{Z}^2$ or S^1.

4.17. Some simple examples. The approach to realizing homotopy actions described here is quite complicated, in general, even for cyclic groups. Nevertheless, in certain cases the theory simplifies to some extent:

I. In a semi-free action all fixed points are global. In terms of a Bredon homotopy action this implies that the maps $j_H : \underset{\sim}{\mathbf{X}}_H \to \underset{\sim}{X}(H)$ are homotopy equivalences for $\{e\} \neq H$, and thus $\underset{\sim}{\mathbf{X}}_H^H$ is contractible – but $\underset{\sim}{X}(G) =: \mathbf{Y}$ need not be contractible. However, we do have a trivial G-action on $\mathbf{Y}$. Thus we are left with the obstructions of Proposition 3.14 for interpolating the given G-actions in the homotopy cofibration sequence $\mathbf{Y} \to \underset{\sim}{X}(\{e\}) \to \mathbf{Z}$. If these vanish, we obtain the required semi-free G-action an a space $\mathbf{X} \simeq \underset{\sim}{X}(\{e\})$.

Note that in this case G need not be finite, so the examples mentioned in §2.19 are relevant here.

II. A necessary condition in order for our obstruction theory to be effectively computable is that the (homotopy groups of) the spaces of self-equivalences of $\underset{\sim}{\mathbf{X}}_H$ and $\underset{\sim}{\mathbf{X}}_H^H$ in (4.6) are known for each $H \leq G$.

One simple case where this holds is when each of the above spaces is an Eilenberg-Mac Lane space (cf. (2.20)). If the groups $\pi_* \operatorname{haut}(\underset{\sim}{X}(H))$ are know – e.g., if $\underset{\sim}{X}(H)$ is also an Eilenberg-Mac Lane space – then the homotopy groups of

$$\operatorname{Map}(\underset{\sim}{\mathbf{X}}_H, \underset{\sim}{X}(H)) \quad \text{and} \quad \operatorname{Map}(\underset{\sim}{X}(H), \underset{\sim}{\mathbf{X}}_H^H) \tag{4.18}$$

are known (by [T]), so we may determine the homotopy groups of $\mathcal{P}_{j_H}$ and $\mathcal{P}_{q_H}$ up to an extension from the pullback diagram (2.6), (2.20), and (4.18), respectively, and these determine the homotopy groups of $\mathcal{Q}_{j_H,q_H}$ – and thus of the fibers F – up to extensions from and (3.5) and (3.13).

III. The discussion above can also be extended to the case of two-stage Postnikov systems (see §2.19 and Example 3.16 above).

Appendix: Pointed homotopy actions

For convenience, we collect here some basic facts about pointed homotopy actions. These are well-known, but we have not found a suitable reference in the literature.

5.1 Definition. A *pointed homotopy action* of a group G on a pointed space $\mathbf{X}_* = (\mathbf{X}, x_0)$ is (the homotopy class of) a map $\Phi^* : \mathbf{B}G \to \mathrm{haut}_*(\mathbf{X}_*)$. It is *realized* by a pointed G-action $\varphi^*_{\mathbf{Y}_*} : G \to \mathrm{Aut}_*(\mathbf{Y}_*)$ if there is a (pointed) homotopy equivalence $h : \mathbf{X}_* \to \mathbf{Y}_*$ such that

$$\text{(5.2)} \qquad \begin{tikzcd} & & \mathbf{B}\,\mathrm{haut}_*(\mathbf{X}_*) \arrow[d, "\mathbf{B}h_*"] \\ \mathbf{B}G \arrow[urr, "\Phi^*"] \arrow[r, "\mathbf{B}\varphi^*_{\mathbf{Y}_*}"'] & \mathbf{B}\,\mathrm{Aut}_*(\mathbf{Y}_*) \arrow[r, "\mathbf{B}i"'] & \mathbf{B}\,\mathrm{haut}_*(\mathbf{Y}_*) \end{tikzcd}$$

commutes up to homotopy.

5.3 Definition. A G-action $\varphi : G \to \mathrm{Aut}(\mathbf{X})$ on a space $\mathbf{X}$ *lifts weakly to a pointed action* $\varphi^* : G \to \mathrm{Aut}_*(\mathbf{Y}_*)$ if we have Borel G-equivalences $p : \mathbf{Z} \to \mathbf{X}$ and $p' : \mathbf{Z} \to \mathbf{Y}$, with sections $i : \mathbf{X} \to \mathbf{Z}$ and $i' : \mathbf{Y} \to \mathbf{Z}$ as in Lemma 1.9, such that the diagram of associative topological monoids (and multiplicative maps):

$$\text{(5.4)} \qquad \begin{tikzcd} & \mathrm{Aut}_*(\mathbf{Y}_*) \arrow[r, hook] & \mathrm{haut}_*(\mathbf{Y}_*) \arrow[r, hook] & \mathrm{haut}(\mathbf{Y}) \arrow[r, "i'_\star"] & \mathrm{haut}(\mathbf{Z}) \\ G \arrow[ur, "\varphi^*_{\mathbf{Y}_*}"] \arrow[r, "\varphi_{\mathbf{X}}"] & \mathrm{Aut}(\mathbf{X}) \arrow[rrr, hook] & & & \mathrm{haut}(\mathbf{X}) \arrow[u, "i_\star"'] \end{tikzcd}$$

commutes up to homotopy after inverting the homotopy equivalence $i_\star$.

Since we assumed that $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are CW complexes, any homotopy equivalence between them can be made into a pointed homotopy equivalence by choosing appropriate (non-degenerate) base-points (cf. [Do, Theorem 3.6]). As a result, we may assume that $\mathbf{X}$ and $\mathbf{Y}$ in Definition 5.3 are pointed.

If $\mathbf{X}_* = (\mathbf{X}, x_0)$ is a G-space with chosen base point x_0, and the G-action is free on $\mathbf{X} \setminus \{x_0\}$, we call $\mathbf{X}_*$ a *free pointed G-space*. For any pointed G-space, the *associated free pointed G-space* is the quotient

$$\mathbf{E}G \ltimes \mathbf{X} := \mathbf{E}G \times \mathbf{X}/\mathbf{E}G \times \{x_0\},$$

with G-action induced from the diagonal action on $\mathbf{E}G \times \mathbf{X}$. A *homotopy fixed point* for a G-space $\mathbf{X}$ is a G-map $f : \mathbf{E}G \to \mathbf{X}$. and we have:

5.5 Lemma. *Any pointed G-space* $\mathbf{X}_*$ *has a G-map* $r : \mathbf{E}G \ltimes \mathbf{X} \to \mathbf{X}$ *which is a pointed homotopy equivalence; if* $\mathbf{X}_*$ *is a* free *pointed G-space, the map r is a G-homotopy equivalence.*

Proof. We have a diagram of G-spaces:

$$
\begin{tikzcd}
\mathbf{E}G \times \{x_0\} \arrow[r, hook, "j"] \arrow[d, "p", "\text{h.e.}"'] & \mathbf{E}G \times \mathbf{X} \arrow[r, two heads, "s"] \arrow[d, "q", "\text{h.e.}"'] \arrow[l, bend right, "\varphi"'] & \mathbf{E}G \ltimes \mathbf{X} \arrow[d, "r"] \\
\{x_0\} \arrow[r, hook] & \mathbf{X} \arrow[r, "\mathrm{Id}"] & \mathbf{X}
\end{tikzcd}
\tag{5.6}
$$

where the vertical maps are projections onto the second factor. Since each row is a cofibration sequence and p and q are Borel G-equivalences, so is r. If the pointed action on $\mathbf{X}_*$ is free, r induces homotopy equivalences on all fixed point sets (which consist only of the basepoint for all $\{e\} \neq H \leq G$), so by [JS, Theorem (1.1)] r is in fact a G-homotopy equivalence. ∎

5.7 Lemma. *A G-space* $\mathbf{X}$ *with action* $\varphi : G \to \mathrm{Aut}(\mathbf{X})$ *has a homotopy fixed point corresponding to each weak lift of* φ *to a pointed action* $\varphi^* : G \to \mathrm{Aut}_*(\mathbf{Y}_*)$.

Proof. A weak lift of φ to a pointed action $\varphi^* : G \to \mathrm{Aut}_*(\mathbf{Y}_*)$ yields a fixed point $y_0 \in \mathbf{Y}$, and thus a homotopy fixed point for $\mathbf{Y}_* = (\mathbf{Y}, y_0)$ given by the constant map $c_{y_0} : \mathbf{E}G \to \mathbf{Y}$ (which is a G-map). This lifts to a homotopy fixed point $\hat{f} : (\mathrm{Id}, c_{y_0}) : \mathbf{E}G \to \mathbf{E}G \times \mathbf{Y}$. Since $\mathrm{Id} \times h : \mathbf{E}G \times \mathbf{Z} \to \mathbf{E}G \times \mathbf{Y}$ is a G-map of free G-CW complexes (§1.7) which is also a homotopy equivalence, it is actually a G-homotopy equivalence by [JS, Theorem (1.1)], with G-inverse $h^{-1} : \mathbf{E}G \times \mathbf{Y} \to \mathbf{E}G \times \mathbf{Z}$. The G-map $k \circ h^{-1} \circ \hat{f} : \mathbf{E}G \to \mathbf{X}$ is the corresponding homotopy fixed point for $\mathbf{X}$.

Conversely, if $f : \mathbf{E}G \to \mathbf{X}$ is a G-map, we may factor f in the model category $G\text{-}\mathcal{T}op$ (see §1.5) as a G-cofibration followed by a G-fibration weak equivalence: $\mathbf{E}G \xrightarrow{\tilde{f}} \mathbf{Z} \overset{p}{\twoheadrightarrow} \mathbf{X}$,. If we let $\mathbf{Y} := \mathbf{Z}/\mathbf{E}G$ denote the (homotopy) cofiber of $\tilde{f}$, with quotient G-map $p' : \mathbf{Z} \to \mathbf{Y}$, then $\mathbf{Y}$ has a basepoint y_0 (corresponding to $\mathbf{E}G \subseteq \mathbf{Z}$), fixed under the G-action, and p' is a Borel G-equivalence since $\mathbf{E}G$ is contractible. Thus the G-action on $\mathbf{Y}_* = (\mathbf{Y}, y_0)$ yields the required pointed lift. ∎

5.8 Proposition. *Any pointed homotopy action* $\Phi^* : \mathbf{B}G \to \mathbf{B}\,\mathrm{haut}_*(\mathbf{X}_*)$ *can be realized by a (free) pointed G-action.*

Proof. Pulling back the universal fibration $\mathbf{B}j$ of (1.2) along $\Phi := i' \circ \Phi^*$ yields the following (homotopy) pullback square:

$$
\begin{tikzcd}
\mathbf{B}G \arrow[dr, dotted, "\sigma"] \arrow[drr, bend left, "\Phi^*"] \arrow[ddr, bend right, "="'] & & \\
& E_\theta \arrow[r] \arrow[d, "\theta"] \arrow[dr, phantom, "\boxed{\mathrm{PB}}"] & \mathbf{B}\,\mathrm{haut}_*(\mathbf{X}_*) \arrow[d, "\mathbf{B}j"] \\
& \mathbf{B}G \arrow[r, "\Phi"] & \mathbf{B}\,\mathrm{haut}(\mathbf{X})
\end{tikzcd}
\tag{5.9}
$$

so we can use Φ^* to obtain a homotopy section $\sigma : \mathbf{B}G \to E_\theta$ as indicated.

We now use the lower left homotopy pullback square in (1.3) to obtain a homotopy fixed point $f : \mathbf{E}G \to \mathbf{X}_1$:

(5.10)
$$\begin{tikzcd}
\mathbf{E}G \arrow[dr, dotted, "f"] \arrow[drr, bend left, "="] \arrow[ddr, bend right, "\sigma\circ q"'] & & \\
& \mathbf{X}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\mathrm{PB}}"] & \mathbf{E}G \arrow[d, "q"] \\
& E_\theta \arrow[r, "\theta"] & \mathbf{B}G ,
\end{tikzcd}$$

where $\mathbf{X}_1 \simeq \mathbf{X}$. Hence by Lemma 5.7 we obtain a pointed G-action on a pointed space $\mathbf{Y}_*$ homotopy equivalent to $\mathbf{X}$. ∎

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Department of Mathematics
University of Haifa
31905 Haifa, Israel
email:blanc@math.haifa.ac.il

Department of Mathematics and Statistics,
Indian Institute of Technology, Kanpur
Uttar Pradesh 208016, India
email:debasis@iitk.ac.in