Volume differences of mixed complex projection bodies

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Abstract

Recently, Abardia and Bernig introduced the notion of mixed complex projection bodies and established a number of important geometric inequalities for them. In the present paper we prove several new isoperimetric type inequalities for volume differences of mixed complex projection bodies.

1 Introduction

Projection bodies in $\mathbb{R}^n$ have and a long history and are widely studied. An extensive article that details this is by Bolker [9]. Bolker's article, prompted even more intensive investigations of projection bodies and also generalizations to the L_p Brunn-Minkowski theory (see, e.g., [6], [8], [11-13], [15], [17], [20-21], [27], [31], [33-34], [37], [39-41], [48] and [51])). New applications have appeared in combinatorics (see Stanley [49]), in stereology (see Betke-McMullen [8]), in stochastic geometry (see Schneider [42]), and even in the study of random determinants (see Vitale [50]). In 1988, a fascinating paper of Alexander [5] demonstrates a close relationship between the study of projection bodies and work on Hilbert's fourth problem. We also refer to Goodey and Weil [16], Martini [36] and Schneider and Weil [43] for related results.

Mixed projection bodies are related to projection bodies in the same way as mixed volumes are related to ordinary volume. The definition and elementary

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properties of mixed projection bodies can be found in [10]. The support functions of mixed projection bodies were studied by Chakerian [14]. Lutwak had systematically studied mixed projection bodies and their polars and obtained a number of elegant results (see, for example, [26-31]). Many recent important results have appeared in [3], [19], and [32].

Moreover, it is well-known that the projection operator is a Minkowski valuation. In fact, Ludwig [23] characterized the projection body map as the unique continuous Minkowski valuation which is contravariant with respect to non-degenerate linear transformations (see [1], [18], [25] and [47]). See the references [23-24] and [44-45] for more information on Minkowski valuations.

Let V be a real vector space of dimension n. Let $\mathcal{K}(V)$ denote the space of non-empty compact convex bodies in V, endowed with the Hausdorff topology.

The projection body of $K \in K(V)$ is the convex body $\Pi K \in K(V^*)$ whose support function is defined by

$$h(\Pi K, u) = \frac{n}{2} V(K[n-1], J_u), u \in V.$$

Here $V(K[n-1], J_u) = V(K, \ldots, K, J_u)$ is the mixed volume of $(n-1)$ copies of K and one copy of the segment $J_u = [-u, u]$ joining $-u$ and u. The support function of $K \in \mathcal{K}(V)$ is the function $h(K, \xi) : V^* \to \mathbb{R}$ defined by

$$h(K, \xi) = \sup_{x \in K} \langle \xi, x \rangle,$$

where $\langle \xi, x \rangle$ denotes the pairing of $\xi \in V^*$ and $x \in V$.

In more intuitive terms, suppose that V is endowed with a Euclidean scalar product. Then we can identify V^* with V and the support function of ΠK in the direction $u \in S^{n-1}$ is the volume of the orthogonal projection of K onto the hyperplane $u^{\perp}$.

In [2], Abardia and Bernig studied projection bodies in complex vector spaces: The real vector space V of real dimension n is replaced by a complex vector space W of complex dimension m and the group $\mathrm{SL}(V) = \mathrm{SL}(n, \mathbb{R})$ is replaced by the group $\mathrm{SL}(W, \mathbb{C}) = \mathrm{SL}(m, \mathbb{C})$. Note that $\mathrm{SL}(m, \mathbb{C}) \subset \mathrm{SL}(2m, \mathbb{R})$, so that each element in $\mathrm{SL}(m, \mathbb{C})$ is volume preserving. A complex version of Ludwig's characterization theorem of the projection operator (see [23]) was established by Abardia and Bernig.

Theorem A *Let W be a complex vector space of complex dimension $m \geq 3$. A map $Z : K(W) \to K(W^*)$ is a continuous translation invariant and $\mathrm{SL}(W, \mathbb{C})$-contravariant Minkowski valuation if and only if there exists a convex body $C \subset \mathbb{C}$ such that $Z = \Pi_C$, where $\Pi_C K \in K(W^*)$ is the convex body with support function*

$$h(\Pi_C K, w) = V(K[2m-1], C \cdot w), \ \forall\, w \in W, \tag{1.1}$$

where $C \cdot w := \{cw | c \in C\} \subset W$, and C is unique up to translations.

The mixed complex projection bodies of $K_1, \ldots, K_{2m-1}$ were also defined by Abardia and Bernig:

Definition 1.1 Let $K_1, \ldots, K_{2m-1} \in \mathcal{K}(W)$ and $C \subset \mathbb{C}$. The mixed complex projection body $\Pi_C(K_1, \ldots, K_{2m-1}) \in \mathcal{K}(W^*)$ is the convex body whose support function is given by

$$h\Big(\Pi_C(K_1, \ldots, K_{2m-1}), w\Big) = V(K_1, \ldots, K_{2m-1}, C \cdot w), \ \forall\, w \in W. \tag{1.2}$$

In this paper we also fix a Euclidean scalar product on W, and denote its unit ball by B. Let $K_1,\ldots,K_{2m-1} \in \mathcal{K}(W)$ and $0 \leq i \leq 2m-1$. If $K_1 = \cdots = K_{2m-1-i} = K$, $K_{2m-i} = \cdots = K_{2m-1} = L, K_{2m} = M$, then the mixed volume $V(K_1,\ldots,K_{2m})$ will be written as $V(K[2m-1-i], L[i], M)$. In particular, when $L = B$, $W_i(K,M)$ denotes the mixed volume $V(K[2m-1], B[i], M)$. Moreover $W_i(K[2m-i], B[i])$ will be written as $W_i(K)$ and is also called the i-th quermass-integral of K.

If $K_i \in \mathcal{K}(W), 1 \leq i \leq 2m-1$, then the mixed complex projection body of K_i is denoted by $\Pi_C(K_1,\ldots,K_{2m-1})$. If $K_1 = \cdots = K_{2m-1-i} = K$ and $K_{2m-i} = \cdots = K_{2m-1} = L$, then $\Pi(K_1,\ldots,K_{2m-1})$ will be written as $\Pi_C(K[2m-i], L)$.

Abardia and Bernig [2] also showed geometric inequalities of Brunn-Minkowski, Aleksandrov-Fenchel and Minkowski type.

Theorem B (Brunn-Minkowski type inequality) *If $K, L \in \mathcal{K}(W)$, then*

$$V\Big(\Pi_C(K+L)\Big)^{1/2m(2m-1)} \geq V\Big(\Pi_C K\Big)^{1/2m(2m-1)} + V\Big(\Pi_C L\Big)^{1/2m(2m-1)}. \quad (1.3)$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic.

Theorem C (Aleksandrov-Fenchel type inequality) *If $K_1,\ldots,K_{2m-1} \in \mathcal{K}(W)$, $0 \leq i \leq 2m-1$ and $2 \leq r \leq 2m-2$, then*

$$W_i\Big(\Pi_C(K_1,\ldots,K_{2m-1})\Big)^r \geq \prod_{j=1}^{r} W_i\Big(\Pi_C(K_j[r], K_{r+1},\ldots,K_{2m-1})\Big). \quad (1.4)$$

Theorem D (Minkowski type inequality) *If $K, L \in \mathcal{K}(W)$ and $0 \leq i < 2m-1$, then*

$$W_i\Big(\Pi_C(K[2m-2], L)\Big)^{2m-1} \geq W_i(\Pi_C K)^{2m-2} W_i(\Pi_C L). \quad (1.5)$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic.

Indeed, Lutwak's seminal work on Brunn-Minkowski type inequalities for the classical projection bodies was generalized to the much more general class of Minkowski valuations intertwining rigid motions (see [4], [38] and [46]).

In 2004 Leng [22] defined the volume difference function of two compact domains D and K, where $D \subseteq K$. The following Minkowski and Brunn-Minkowski type inequalities for volume difference functions were also established by Leng [22].

Theorem E *If K, L, D and D' are compact domains, $D \subseteq K, D' \subseteq L$, and D' is a homothetic copy of D, then*

$$(V_1(K,L) - V_1(D,D'))^n \geq (V(K) - V(D))^{n-1}(V(L) - V(D')),$$

and

$$(V(K+L) - V(D+D'))^{1/n} \geq (V(K) - V(D))^{1/n} + (V(L) - V(D'))^{1/n}.$$

In each case, equality holds if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

Recently, Lv [35] introduced the *dual volume difference function* for star bodies and established the following dual Minkowski and Brunn-Minkowski type inequalities for them:

Theorem F *If K, L, D and D' are star bodies in $\mathbb{R}^n$, and $D \subseteq K, D' \subseteq L$, and L is a dilation of K, then*

$$(\tilde{V}_1(K,L) - (\tilde{V}_1(D,D'))^n \geq (V(K) - V(D))^{n-1}(V(L) - V(D'))$$

with equality if and only if D and D' are dilates and $(K, D)) = \mu(L, D')$, where μ is a constant, and

$$(V(K\tilde{+}L) - (V(D\tilde{+}D'))^{1/n} \geq (V(K) - V(D))^{1/n} + (V(L) - V(D'))^{1/n}$$

with equality if and only if D and D' are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

Moreover, the Aleksandrov-Fenchel type inequalities for volume differences functions were established in [53]. Motivated by the work of Leng and Lv, in this paper we establish some new affine isoperimetric inequalities in complex vector space.

Theorem 1.1 *Let $K, L, D, D' \in \mathcal{K}(W)$. If D' is a homothetic copy of D, $V(\Pi_C D) \leq V(\Pi_C K)$ and $V(\Pi_C D') \leq V(\Pi_C L)$, then*

$$\Big[V\Big(\Pi_C(K+L)\Big) - V\Big(\Pi_C(D+D')\Big)\Big]^{1/2m(2m-1)}$$
$$\geq \Big[V\Big(\Pi_C K\Big) - V\Big(\Pi_C D\Big)\Big]^{1/2m(2m-1)} + \Big[V\Big(\Pi_C L\Big) - V\Big(\Pi_C D'\Big)\Big]^{1/2m(2m-1)}. \tag{1.6}$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic and $\Big(V(\Pi_C K), V(\Pi_C D)\Big) = \mu\Big(V(\Pi_C L), V(\Pi_C D')\Big)$, where μ is a constant.

If D and D' are singletons, then (1.6) becomes (1.3).

Theorem 1.2 *Let $K, L, D, D' \in \mathcal{K}(W)$. If D' is a homothetic copy of D, $W_i(\Pi_C D) \leq W_i(\Pi_C K)$ and $W_i(\Pi_C D') \leq W_i(\Pi_C L)$, then for $0 \leq i < 2m-1$,*

$$\Big[W_i\Big(\Pi_C(K[2m-2], L)\Big) - W_i\Big(\Pi_C(D[2m-2], D')\Big)\Big]^{2m-1}$$
$$\geq \Big[W_i(\Pi_C K) - W_i(\Pi_C D)\Big]^{2m-2}\Big[W_i(\Pi_C L) - W_i(\Pi_C D')\Big]. \tag{1.7}$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic and $\Big(W_i(\Pi_C K), W_i(\Pi_C D)\Big) = \mu\Big(W_i(\Pi_C L), W_i(\Pi_C D')\Big)$, where μ is a constant.

If D and D' are singletons, then (1.7) becomes (1.5).

Theorem 1.3 *For $i = 1, \ldots, 2m-1$, let $K_i, D_i \in \mathcal{K}(W)$.*
If $V(\Pi_C(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{2m-1})) \geq V(\Pi_C(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{2m-1}))$, and

D_j ($j = 1,\dots,r$) are homothetic copies of each other, then for $0 \le i \le 2m-1$ and $2 \le r \le 2m-2$,

$$\Big[V(\Pi_C(K_1,\dots,K_{2m-1})) - V(\Pi_C(D_1,\dots,D_{2m-1}))\Big]^r$$

$$\ge \prod_{j=1}^{r}\Big[V(\Pi_C(\underbrace{K_j,\dots,K_j}_{r},K_{r+1},\dots,K_{2m-1})) - V(\Pi_C(\underbrace{D_j,\dots,D_j}_{r},D_{r+1},\dots,D_{2m-1}))\Big]. \tag{1.8}$$

If D_j ($j = 1,\dots,r$) are singletons, then (1.8) becomes (1.4).

2 Auxiliary Results

The following results will be required to prove our theorems.

Lemma 2.1 ([7, p.38]) *Let*

$$\phi(x) = (x_1^p - x_2^p - \cdots - x_n^p)^{1/p},\ p > 1,$$

and suppose that

(a) $x_i \ge 0$,

(b) $x_1 \ge (x_2^p + x_3^p + \cdots + x_n^p)^{1/p}$.

Then for $x, y \in \mathbb{R}^n$, we have

$$\phi(x+y) \ge \phi(x) + \phi(y), \tag{2.1}$$

with equality if and only if $x = \mu y$ where μ is a constant.

Lemma 2.2 ([52]) *Let $a, b, c, d > 0, 0 < \alpha < 1, 0 < \beta < 1$ and $\alpha + \beta = 1$. If $a > b$ and $c > d$, then*

$$a^\alpha c^\beta - b^\alpha d^\beta \ge (a-b)^\alpha (c-d)^\beta, \tag{2.2}$$

with equality if and only if $a/b = c/d$.

Lemma 2.3 ([7, p.26]) *If $x_i > 0, y_i > 0$, then*

$$\left(\prod_{i=1}^{n}(x_i + y_i)\right)^{1/n} \ge \left(\prod_{i=1}^{n} x_i\right)^{1/n} + \left(\prod_{i=1}^{n} y_i\right)^{1/n}, \tag{2.3}$$

with equality if and only if $c_1/b_1 = c_2/b_2 = \cdots = c_n/b_n$.

3 Inequalities for mixed complex projection bodies

3.1 Brunn-Minkowski-type inequality

In the following we establish the Brunn-Minkowski-type inequality, Theorem 1.1, for complex projection bodies.

Theorem 3.1 *Let $K, L, D, D' \in \mathcal{K}(W)$. If D' is a homothetic copy of D, $V(\Pi_C D) \leq V(\Pi_C K)$ and $V(\Pi_C D') \leq V(\Pi_C L)$, then*

$$\Big[V\Big(\Pi_C(K+L)\Big) - V\Big(\Pi_C(D+D')\Big)\Big]^{1/2m(2m-1)}$$
$$\geq \Big[V\Big(\Pi_C K\Big) - V\Big(\Pi_C D\Big)\Big]^{1/2m(2m-1)} + \Big[V\Big(\Pi_C L\Big) - V\Big(\Pi_C D'\Big)\Big]^{1/2m(2m-1)}. \tag{3.1}$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic and $\Big(V(\Pi_C K), V(\Pi_C D)\Big) = \mu\Big(V(\Pi_C L), V(\Pi_C D')\Big)$, where μ is a constant.

Proof. If $K, L \in \mathcal{K}(W)$, then, by Theorem B,

$$V\Big(\Pi_C(K+L)\Big)^{1/2m(2m-1)} \geq V\Big(\Pi_C K\Big)^{1/2m(2m-1)} + V\Big(\Pi_C L\Big)^{1/2m(2m-1)}. \tag{3.2}$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic.

Notice that D' is a homothetic copy of D, thus

$$V\Big(\Pi_C(D+D')\Big)^{1/2m(2m-1)} = V\Big(\Pi_C D\Big)^{1/2m(2m-1)} + V\Big(\Pi_C D'\Big)^{1/2m(2m-1)}. \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$V\Big(\Pi_C(K+L)\Big) - V\Big(\Pi_C(D+D')\Big) \geq$$
$$\Big[V\Big(\Pi_C K\Big)^{1/2m(2m-1)} + V\Big(\Pi_C L\Big)^{1/2m(2m-1)}\Big]^{2m(2m-1)}$$
$$- \Big[V\Big(\Pi_C D\Big)^{1/2m(2m-1)} + V\Big(\Pi_C D'\Big)^{1/2m(2m-1)}\Big]^{2m(2m-1)}. \tag{3.4}$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic.

From (3.4) and Lemma 3.2, we now obtain

$$\Big[V\Big(\Pi_C(K+L)\Big) - V\Big(\Pi_C(D+D')\Big)\Big]^{1/2m(2m-1)}$$
$$\geq \Big\{\Big[V\Big(\Pi_C K\Big)^{1/2m(2m-1)} + V\Big(\Pi_C L\Big)^{1/2m(2m-1)}\Big]^{2m(2m-1)}$$
$$- \Big[V\Big(\Pi_C D\Big)^{1/2m(2m-1)} + V\Big(\Pi_C D'\Big)^{1/2m(2m-1)}\Big]^{2m(2m-1)}\Big\}^{1/2m(2m-1)}$$
$$\geq \Big[V\Big(\Pi_C K\Big) - V\Big(\Pi_C D\Big)\Big]^{1/2m(2m-1)} + \Big[V\Big(\Pi_C L\Big) - V\Big(\Pi_C D'\Big)\Big]^{1/2m(2m-1)}$$

In view of the equality conditions of inequalities (3.4) and (2.1), it follows that if K and L have non-empty interior and C is not a point, then equality in (3.1) holds if and only if K and L are homothetic and $\Big(V(\Pi_C K), V(\Pi_C D)\Big) = \mu\Big(V(\Pi_C L), V(\Pi_C D')\Big)$, where μ is a constant.

3.2 Minkowski-type inequality

In the following we establish the Minkowski-type inequality, Theorem 1.2, for mixed complex projection bodies.

Theorem 3.2 *Let $K, L, D, D' \in \mathcal{K}(W)$. If D' is a homothetic copy of D, $W_i(\Pi_C D) \leq W_i(\Pi_C K)$ and $W_i(\Pi_C D') \leq W_i(\Pi_C L)$, then for $0 \leq i < 2m-1$,*

$$\Big[W_i\Big(\Pi_C(K[2m-2], L)\Big) - W_i\Big(\Pi_C(D[2m-2], D')\Big)\Big]^{2m-1} \geq \Big[W_i(\Pi_C K) - W_i(\Pi_C D)\Big]^{2m-2}\Big[W_i(\Pi_C L) - W_i(\Pi_C D')\Big]. \tag{3.5}$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic and $\Big(W_i(\Pi_C K), W_i(\Pi_C D)\Big) = \mu\Big(W_i(\Pi_C L), W_i(\Pi_C D')\Big)$, where μ is a constant.

Proof. If $K, L \in \mathcal{K}(W)$, then, by Theorem D,

$$W_i\Big(\Pi_C(K[2m-2], L)\Big)^{2m-1} \geq W_i(\Pi_C K)^{2m-2} W_i(\Pi_C L). \tag{3.6}$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic.

Since D' is a homothetic copy of D, we have

$$W_i\Big(\Pi_C(D[2m-2], D')\Big)^{2m-1} = W_i(\Pi_C D)^{2m-2} W_i(\Pi_C D'), \tag{3.7}$$

hence

$$W_i\Big(\Pi_C(K[2m-2], L)\Big) - W_i\Big(\Pi_C(D[2m-2], D')\Big) \geq W_i(\Pi_C K)^{(2m-2)/(2m-1)} W_i(\Pi_C L)^{1/(2m-1)} - W_i(\Pi_C D)^{(2m-2)/(2m-1)} W_i(\Pi_C D')^{1/(2m-1)}. \tag{3.8}$$

If K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic.

Since $\frac{2m-2}{2m-1} + \frac{1}{2m-1} = 1$, it follows from Lemma 2.2, that

$$\begin{aligned}\Big[W_i\Big(\Pi_C(K[2m-2], L)\Big) - W_i\Big(\Pi_C(D[2m-2], D')\Big)\Big]^{2m-1} &\geq \Big[W_i(\Pi_C K)^{(2m-2)/(2m-1)} W_i(\Pi_C L)^{1/(2m-1)} \\ &\quad - W_i(\Pi_C D)^{(2m-2)/(2m-1)} W_i(\Pi_C D')^{1/(2m-1)}\Big]^{2m-1} \\ &\geq [W_i(\Pi_C K) - W_i(\Pi_C D)]^{2m-2}[W_i(\Pi_C L) - W_i(\Pi_C D')].\end{aligned}$$

From the equality conditions of inequalities (3.8) and (2.2), it follows that if K and L have non-empty interior and C is not a point, then equality holds if and only if K and L are homothetic and $\Big(W_i(\Pi_C K), W_i(\Pi_C D)\Big) = \mu\Big(W_i(\Pi_C L), W_i(\Pi_C D')\Big)$, where μ is a constant.

3.3 Aleksandrov-Fenchel-type inequality

Theorem 3.3 *For $i = 1,\ldots,2m-1$, let $K_i, D_i \in \mathcal{K}(W)$. If $V(\Pi_C(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_{2m-1})) \geq V(\Pi_C(\underbrace{D_j,\ldots,D_j}_{r}, D_{r+1},\ldots,D_{2m-1}))$, and D_j $(j = 1,\ldots,r)$ are homothetic copies of each other, then for $0 \leq i \leq 2m-1$ and $2 \leq r \leq 2m-2$,*

$$\Big[V(\Pi_C(K_1,\ldots,K_{2m-1})) - V(\Pi_C(D_1,\ldots,D_{2m-1}))\Big]^r$$
$$\geq \prod_{j=1}^{r}\Big[V(\Pi_C(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_{2m-1})) - V(\Pi_C(\underbrace{D_j,\ldots,D_j}_{r}, D_{r+1},\ldots,D_{2m-1}))\Big]. \tag{3.9}$$

Proof. For $0 \leq i \leq 2m-1$ and $2 \leq r \leq 2m-2$, we have by Theorem C

$$W_i\Big(\Pi_C(K_1,\ldots,K_{2m-1})\Big)^r \geq \prod_{j=1}^{r} W_i\Big(\Pi_C(K_j,\ldots,K_j,K_{r+1},\ldots,K_{2m-1})\Big). \tag{3.10}$$

Since D_j $(j = 1,\ldots,r)$ are homothetic copies of each other, we have

$$W_i\Big(\Pi_C(D_1,\ldots,D_{2m-1})\Big)^r = \prod_{j=1}^{r} W_i\Big(\Pi_C(D_j,\ldots,D_j,D_{r+1},\ldots,D_{2m-1})\Big). \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\begin{aligned} &V(\Pi_C(K_1,\ldots,K_{2m-1})) - V(\Pi_C(D_1,\ldots,D_{2m-1}))\\ &\qquad\geq \left(\prod_{j=1}^{r} V(\Pi_C(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_{2m-1}))\right)^{1/r}\\ &\qquad\quad - \left(\prod_{j=1}^{r} V(\Pi_C(\underbrace{D_j,\ldots,D_j}_{r}, D_{r+1},\ldots,D_{2m-1}))\right)^{1/r}. \end{aligned} \tag{3.12}$$

Thus using Lemma 2.3, we obtain

$$\begin{aligned} &\Big[V(\Pi_C(K_1,\ldots,K_{2m-1})) - V(\Pi_C(D_1,\ldots,D_{2m-1}))\Big]^r\\ &\qquad\geq \left[\left(\prod_{j=1}^{r} V(\Pi_C(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_{2m-1}))\right)^{1/r}\right.\\ &\qquad\quad \left. - \left(\prod_{j=1}^{r} V(\Pi_C(\underbrace{D_j,\ldots,D_j}_{r}, D_{r+1},\ldots,D_{2m-1}))\right)^{1/r}\right]^r\\ &\geq \prod_{j=1}^{r}\Big[V(\Pi_C(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_{2m-1})) - V(\Pi_C(\underbrace{D_j,\ldots,D_j}_{r}, D_{r+1},\ldots,D_{2m-1}))\Big]. \end{aligned}$$

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