Local well posedness of a 2D semilinear heat equation

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Abstract

We investigate the initial value problem for a semilinear heat equation with exponential-growth nonlinearity in two space dimension. First, we prove the local existence and unconditional uniqueness of solutions in the Sobolev space $H^1(\mathbb{R}^2)$. The uniqueness part is non trivial although it follows Brezis-Cazenave's proof [3] in the case of monomial nonlinearity in dimension $d \geq 3$. Next, we show that in the defocusing case our solution is bounded, and therefore exists for all time. In the focusing case, we prove that any solution with negative energy blows up in finite time.

1 Introduction

Consider the initial value problem for a semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + f(u) \\ u(0) = u_0 \end{cases} \tag{1}$$

where $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$ and $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ is a given function satisfying $f(0) = 0$. The Cauchy problem (1) has been extensively studied in the scale of Lebesgue spaces L^q, especially for polynomial type nonlinearities i.e

$$f(u) := \pm |u|^{\gamma-1} u, \quad \gamma > 1. \tag{2}$$

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In such a case, observe that the equation enjoys the interesting property of scaling invariance

$$u_\lambda(t,x) := \lambda^{2/\gamma}\, u(\lambda^2 t, \lambda x), \quad \lambda > 0 \tag{3}$$

i.e. if u solves (2) then also does u_λ. The Lebesgue space $L^{q_c}(\mathbb{R}^d)$ with exponent $q_c := \frac{d(\gamma-1)}{2}$ is also the only one invariant under the same scaling (3). This property defines a sort of trichotomy in the dynamic of solutions of (2), and basically one can notice the following three different regimes for initial data in L^q:

THE SUBCRITICAL CASE I.E. $q > q_c \geq 1$: Weissler in [28] proved the existence of a unique solution $u \in \mathcal{C}([0,T); L^q(\mathbb{R}^d)) \cap L^\infty_{loc}(]0,T]; L^\infty(\mathbb{R}^d))$. Later on, Brezis-Cazenave [3] proved the unconditional uniqueness of Weissler's solutions.[1]

THE CRITICAL CASE i.e. $q = q_c$ and $d \geq 3$: There are two sub-cases:

- If $q_c > \gamma + 1$, then we have local wellposedness of the Cauchy problem where the existence is also due to Weissler [28] and the unconditional uniqueness to Brezis-Cazenave [3].

- If $q = q_c = \gamma + 1$ or equivalently $q = \frac{d}{d-2}$ and $\gamma - 1 = \frac{2}{d-2}$ (double critical or energy critical case[2]): Weissler [29] proved the conditional wellposedness. When the underlying space is the unit ball of $\mathbb{R}^d$, Ni-Sacks [20] showed that the unconditional uniqueness fails. This result was extended to the whole space by C. Tarsi [24] for suitable initial data. See also [16] for general initial data.

THE SUPERCRITICAL CASE I.E. $q < q_c$: there are indications that there exists no (local) solution in any reasonable weak sense (cf. [3, 28, 29]). Moreover, it is known that uniqueness is lost for the initial data $u_0 = 0$ and $1 + \frac{1}{d} < \gamma < \frac{d+2}{d-2}$, see Haraux-Weissler [7].

The way in constructing solutions consists in using a fixed point argument in suitable spaces where the free solution lives and the nonlinear terms can be well estimated using the heat regularizing properties. Note that the solution can be written as

$$u(t) = \mathrm{e}^{t\Delta} u_0 + M(u)(t),$$

where M is the integral operator defined by $M(u)(t) := \int_0^t \mathrm{e}^{(t-s)\Delta} f(u(s))\, ds$. This operator behaves differently in the sub and critical cases. It is clearly continuous in $\mathcal{C}([0,T); L^q(\mathbb{R}^d))$ when the nonlinearity is subcritical, while it is discontinuous in the critical case (see [20] for more details).

In the energy critical case, the nice idea of Ni and Sacks [20] to prove the non-uniqueness is constructive and based on the fact that the Poisson equation does

[1] Uniqueness in the natural space where solutions exist, namely $\mathcal{C}(L^q)$.

[2] Observe that in such a case, the potential energy term is finite.

not regularize as much as the heat equation when the source term is only an integrable function. In the energy critical case, the potential term $|u|^{\gamma-1}u \in L_t^\infty(L^1)$. So, Ni and Sacks constructed a singular stationary solution in the punctured unit ball. The singularity holds only at the center of the ball and is weak enough to extend the singular solution (in the distributional sense) to the whole ball. Then, they constructed a local solution which will immediately enjoy a smoothing effect that the stationary singular solution will never have. This makes the two solutions different and the unconditional non-uniqueness immediately follows.

Let us finally mention that the well posedness in Sobolev and Besov spaces was investigated in [21, 17].

In two space dimension, observe that the energy[3] scaling index $q_c = \frac{d}{d-2}$ becomes infinite. So any power nonlinearity $1 < \gamma < \infty$ is subcritical in the sense that one can always choose a Lebesgue space L^q (other than L^∞) where one can prove the well-posedness for the Cauchy problem (4). However, when taking an infinite polynomial e.g exponential nonlinearity, the only Lebesgue space in which Weissler's result is applicable is L^∞. To this extent, the Cauchy problem (4) is always subcritical in L^∞ and one can wonder if there is any notion of criticality in two space dimensions. The loss of the scaling property for inhomogeneous nonlinearities also does not help in having any insight toward an answer.

The ultimate aim is to show that in $2D$, a kind of trichotomy (similar to the one described above in higher dimensions) can still be defined. It is based on the topology of the initial data. More precisely, consider the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u = \pm u(\mathrm{e}^{u^2} - 1) \quad \text{in} \quad \mathbb{R}^2 \\ \\ \qquad\qquad u(0) = u_0 \,. \end{cases} \tag{4}$$

However, our goal in this paper is to study whether or not there exists local/global solution to the Cauchy problem (4) when the data is no longer in L^∞.

First, observe that for an exponential nonlinearity, the largest Lebesgue type space in which the equation is meaningful in the distributional sense is of Orlicz kind. In this respect, Ruf and Terraneo [23] showed a local existence result for *small* initial data in Orlicz space. This result was extended to global existence by Norisuke Ioku [14]. In what follows, we will focus our attention only to the case $d = 2$.

We will show that we have a "good" H^1 theory for the Cauchy problem (4) i.e. finite time/global existence of solutions (depending on the sign of the nonlinearity), and unconditional uniqueness.

Our results show that even though there is no a scaling property for this problem, a sort of trichotomy analogous to the one described in higher dimension can still be defined. It is based on the topology of the initial data. In a forthcoming paper, we show the non-existence of solutions of the Cauchy problem (4) if the initial

[3] i.e. $q_c = \gamma + 1$.

data is in the Sobolev space $H^s(\mathbb{R}^2)$ with $s < 1$, and the loss of uniqueness if u_0 belongs to some Orlicz space.

This paper is organized as follows. In the next section, we state our main results. In Section 3, we recall some basic definitions and auxiliary lemmas. The fourth section deals with the H^1 regularity regime.

Finally, we mention that C will be used to denote an absolute constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2 Main results

First, we prove that without any restriction on the size of the initial data, the Cauchy problem (4) is locally well-posed in the Sobolev space $H^1(\mathbb{R}^2)$. To do so, we use a standard fixed point argument. The uniqueness part is non trivial although it follows the steps of Brezis-Cazenave's proof [3] in the case of monomial nonlinearity in dimension $d \geq 3$. Thanks to a standard blow-up criterion (see for example [3]), the parabolic regularization effect and the maximum principle, we prove that in the defocusing case, the maximal solution remains bounded and therefore can be extended for all positive time. Recall that the energy is given by

$$J(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)} - \int_{\mathbb{R}^2} F(u(t))\, dx, \quad \text{with} \quad F(u) = \int_0^u f(v)\, dv\,.$$

Our main result can be stated as follows.

Theorem 2.1. *Let $u_0 \in H^1(\mathbb{R}^2)$.*

1) *There exist $T > 0$ and a unique u solution to* (4) *in $\mathcal{C}([0,T]; H^1)$, moreover $u \in \mathcal{C}((0,T]; L^\infty)$.*

2) *If $f(u) = -u(\mathrm{e}^{u^2} - 1)$, then the (above) solution is global.*

3) *If $f(u) = u(\mathrm{e}^{u^2} - 1)$, then a data $u_0 \neq 0$ with $J(u_0) \leq 0$ gives a unique solution blowing up in finite time.*

Remark 2.2. *The first assertion of the above Theorem remains true for $f(u) = \pm u\mathrm{e}^{u^2}$, and the second one also extends to the case $f(u) = -u\mathrm{e}^{u^2}$. This means that we need to remove the quadratic term from the nonlinearity only for the blow up result.*

The previous Theorem shows that the H^1 regularity supports well the exponential nonlinearity. That is why we have obtained a "good" H^1-theory.
For small data, the solution to (4) given by the previous Theorem is global. In fact, following the proof of Nakamura-Ozawa [19] for the Schrödinger equation we have

Proposition 2.3. *There exists $\varepsilon > 0$ such that for any $u_0 \in H^1(\mathbb{R}^2)$ satisfying $\|u_0\|_{H^1(\mathbb{R}^2)} \leq \varepsilon$, the solution to* (4) *given by the previous Theorem is global.*

3 Background material

In this section we will fix the notation, state the basic definitions and recall some known and useful tools. First we recall the standard smoothing effect (see for example [3]).

Lemma 3.1. *There exists a positive constant C such that for all $1 \leq \beta \leq \gamma \leq \infty$, we have*

$$\|e^{t\Delta}\varphi\|_{L^\gamma} \leq \frac{C}{t^{\frac{1}{\beta}-\frac{1}{\gamma}}}\|\varphi\|_{L^\beta}, \quad \forall t > 0, \forall \varphi \in L^\beta(\mathbb{R}^2) \tag{5}$$

where $e^{t\Delta}\varphi := K_t * \varphi = \frac{1}{4\pi t}e^{-\frac{|.|^2}{4t}} * \varphi$.

Using Young and Hölder inequalities and the precedent Lemma with the following integral formula

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left(\partial_t u - \Delta u\right)(s)\, ds$$

we deduce the following estimates

Proposition 3.2.

$$\sup_{t\in[0,T]} \|u(t,.)\|_{H^1(\mathbb{R}^2)} \leq C\Big(\|u(t_0,.)\|_{H^1(\mathbb{R}^2)} + \|\partial_t u - \Delta u\|_{L^1([0,T],H^1(\mathbb{R}^2))}\Big). \tag{6}$$

$$\sup_{t\in[0,T]} \|u(t,.)\|_{L^\infty(\mathbb{R}^2)} \leq C\Big(\|u(t_0,.)\|_{L^\infty(\mathbb{R}^2)} + \|\partial_t u - \Delta u\|_{L^1([0,T],L^\infty(\mathbb{R}^2))}\Big). \tag{7}$$

We recall the following nonlinear estimates which are consequence of the mean value theorem and the convexity of the exponential function. See [10, 6].

Lemma 3.3. *For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|f(U_1) - f(U_2)| \leq C_\varepsilon |U_1 - U_2| \sum_{i=1}^{2}\Big(e^{(1+\varepsilon)U_i^2} - 1\Big). \tag{8}$$

$$|f'(U_1) - f'(U_2)| \leq C_\varepsilon |U_1 - U_2| \sum_{i=1}^{2}\Big(e^{2(1+\varepsilon)U_i^2} - 1\Big)^{1/2}. \tag{9}$$

In order to control the nonlinear part in $L^1_t(H^1_x)$, we will use the following Moser-Trudinger inequality [1, 18, 27].

Proposition 3.4. *Let $\alpha \in (0, 4\pi)$, a constant C_α exists such that for all $u \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$, we have*

$$\int_{\mathbb{R}^2}\Big(e^{\alpha|u(x)|^2} - 1\Big)dx \leq C_\alpha \|u\|^2_{L^2(\mathbb{R}^2)}. \tag{10}$$

Moreover, (10) *is false if $\alpha \geq 4\pi$.*

Let us mention that $\alpha = 4\pi$ becomes admissible if we require $\|u\|_{H^1(\mathbb{R}^2)} \leq 1$ rather than $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. Precisely

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)}\leq 1} \int_{\mathbb{R}^2} \left(e^{4\pi|u(x)|^2} - 1\right) dx < \infty \tag{11}$$

and this is false for $\alpha > 4\pi$. See [22] for more details.

Now, we give some technical results which will be useful later. The following lemma is classical (see for example [26]) but the proof seems to be new.

Lemma 3.5. *Let $u \in H^1(\mathbb{R}^2)$. Then for any $\alpha > 0$ and $1 \leq q < \infty$,*

$$e^{\alpha u^2} - 1 \in L^q(\mathbb{R}^2).$$

Proof of Lemma 3.5. Without loss of generality, we may assume that $\alpha = q = 1$ and u is radial. First, let us observe that thanks to the following well known radial estimate

$$|u(r)| \leq \frac{C}{\sqrt{r}} \|u\|_{H^1},$$

we obtain for any $a > 0$,

$$\int_{|x|\geq a} \left(e^{|u(x)|^2} - 1\right) dx \leq \int_{|x|\geq a} |u(x)|^2 e^{|u(x)|^2} dx \leq e^{\frac{C}{a}\|u\|_{H^1}^2} \|u\|_{L^2}^2 < \infty.$$

Therefore, to conclude the proof it is sufficient to show that for suitable $a > 0$, we have

$$\int_0^a e^{u^2(r)} r\, dr < \infty. \tag{12}$$

For $a > 0$ and $0 < r < a$, write

$$\begin{aligned} |u(r) - u(a)| &= \left| \int_r^a \sqrt{s}\, u'(s) \frac{ds}{\sqrt{s}} \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \|\nabla u\|_{L^2(|x|<a)} \left(-\log\left(\frac{r}{a}\right)\right)^{1/2}. \end{aligned}$$

Choosing $a > 0$ small enough such that

$$\|\nabla u\|_{L^2(|x|<a)}^2 < 2\pi,$$

and witting

$$\begin{aligned} e^{u^2(r)} r &\lesssim e^{2(u(r)-u(a))^2} r \\ &\lesssim r^{1-\beta}, \quad \beta := \frac{\|\nabla u\|_{L^2(|x|<a)}^2}{\pi}, \end{aligned}$$

we end up with (12). ∎

Proposition 3.6. *For any $T > 0$, and $u \in \mathcal{C}([0,T]; H^1(\mathbb{R}^2))$, we have*

$$e^{u^2} - 1 \ \in \ \mathcal{C}([0,T]; L^1(\mathbb{R}^2)).$$

Proof of Proposition 3.6. Let $t \in [0,T]$ and (t_n) be a sequence in $(0,T)$ such that $t_n \to t$. Denote by $u_n := u(t_n)$ and $u = u(t)$. We will prove that

$$e^{u_n^2} - 1 \to e^{u^2} - 1 \quad \text{in} \quad L^1(\mathbb{R}^2).$$

Set $v_n := u_n - u$. Clearly, we have

$$e^{u_n^2} - e^{u^2} = e^{u^2}\Big((e^{v_n^2} - 1)(e^{2v_n u} - 1) + (e^{2v_n u} - 1) + (e^{v_n^2} - 1)\Big).$$

Hence

$$\begin{aligned} \|e^{u_n^2} - e^{u^2}\|_{L^1} &\lesssim \|e^{u^2} - 1\|_{L^2}\, \|(e^{v_n^2} - 1)(e^{2v_n u} - 1) + (e^{2v_n u} - 1) + (e^{v_n^2} - 1)\|_{L^2} \\ &+ \|(e^{v_n^2} - 1)(e^{2v_n u} - 1) + (e^{2v_n u} - 1) + (e^{v_n^2} - 1)\|_{L^1}, \end{aligned} \tag{13}$$

and by Moser-Trudinger inequality we have

$$\lim_n \|e^{v_n^2} - 1\|_{L^p} = 0, \quad \text{for any} \quad \mathbf{p} \geq 1. \tag{14}$$

Now, it is sufficient to prove that

$$\lim_n \|e^{2|uv_n|} - 1\|_{L^p} = 0, \quad \text{for any} \quad \mathbf{p} \geq 1.$$

We prove the last inequality for $p = 1$. The general case follows by changing v_n by pv_n. Denoting by $\varepsilon_n = \|v_n\|_{H^1}$, we have

$$\begin{aligned} e^{2|uv_n|} - 1 &\leq e^{\varepsilon_n^{1/2}|u|^2 + \varepsilon_n^{-1/2}|v_n|^2} - 1 \\ &= e^{\varepsilon_n^{1/2}|u|^2} - 1 + e^{\varepsilon_n^{-1/2}|v_n|^2} - 1 + \left(e^{\varepsilon_n^{1/2}|u|^2} - 1\right)\left(e^{\varepsilon_n^{-1/2}|v_n|^2} - 1\right). \end{aligned}$$

Taking advantage of Lebesgue theorem, we obtain $\|e^{\varepsilon_n^{1/2}|u|^2} - 1\|_{L^1} \to 0$. For the second term, we use Moser-Trudinger inequality to get

$$\int \left(e^{\varepsilon_n^{-1/2}|v_n|^2} - 1\right) dx \lesssim \|\frac{v_n}{\sqrt{\varepsilon_n}}\|_{L^2}^2 \lesssim \varepsilon_n\,.$$

Finally, using Hölder inequality and arguing in the same manner, we deduce that the last term tends to zero in L^1. This together with (13), (14) ends the proof of Proposition 3.6. ∎

4 H^1-theory: proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. We divide the proof into several steps. First, we show the existence of a local solution to (4), regardless of the sign of the nonlinearity. In the second step, we prove the uniqueness in $\mathcal{C}([0,T); H^1)$. This result, is not straightforward, although it follows Brezis-Cazenave's steps. Then we show that in the defocusing case we can extend the solution globally in time. Finally, we establish a finite time blow-up result in the focusing case.

4.1 Local existence

We summarize the result in the following Theorem.

Theorem 4.1. *Let $u_0 \in H^1(\mathbb{R}^2)$. Then, there exist $T > 0$ and a solution u to (4) in the class*

$$\mathcal{C}([0,T); H^1(\mathbb{R}^2)).$$

Proof of Theorem 4.1. The idea here is similar to the one used in [12, 11, 10]. Indeed, we decompose the initial data to a regular part and a small one. We prove the existence of a local solution v to (4) associated to the regular initial data. Then to recover a solution of our original problem we solve a perturbed equation satisfied by $w := u - v$ with small data.

We start by giving the local existence in $(H^1 \cap L^\infty)(\mathbb{R}^2)$.

Proposition 4.2. *Let $u_0 \in (H^1 \cap L^\infty)(\mathbb{R}^2)$. Then, there exists $T > 0$ (depending upon u_0) and a solution u to (4) such that*

$$u - \mathbf{e}^{t\Delta} u_0 \in \mathcal{C}([0,T); (H^1 \cap L^\infty)(\mathbb{R}^2)).$$

Remark 4.3.

i) Recall that

$$e^{t\Delta} u_0 \in \mathcal{C}([0,T); H^1) \cap \mathcal{C}((0,T); L^\infty) \cap L^\infty(0,T) \times \mathbb{R}^2).$$

ii) For $T > 0$ sufficiently small (depending on $\|u_0\|_{L^\infty}$ and not on $\|u_0\|_{H^1}$), one has $\|u\|_{L^\infty([0,T],H^1 \cap L^\infty)} \leq 2\|u_0\|_{H^1}$.

We omit the proof of Proposition 4.2. Now we solve the perturbed problem. We decompose the initial data as follows $u_0 = (I - S_N)u_0 + S_N u_0$ where $S_N = \sum_{j \leq N-1} \triangle_j$, $(\triangle_j)$ being an inhomogeneous frequency localization, and N is a large integer to be fixed later. Recall that $\|(I - S_N)u_0\|_{H^1} \xrightarrow{N} 0$ and $S_N u_0 \in (H^1 \cap L^\infty)(\mathbb{R}^2)$.
By Proposition 4.2, there exist a time $T_N > 0$ and a solution $\mathbf{v}_N$ to the problem (4) with data $S_N u_0$. Now, we consider the perturbed problem satisfied by $w := u - \mathbf{v}_N$ and with data $(I - S_N)u_0$. Namely, let

$$\begin{cases} \partial_t w - \Delta w = -f(\mathbf{v}_N) + f(\mathbf{v}_N + w) \\ \qquad \mathbf{w}(0) = (I - S_N)u_0. \end{cases} \tag{15}$$

Using a standard fixed point argument, we shall prove that (15) has a local solution in the space $X_T := \mathcal{C}([0,T]; H^1(\mathbb{R}^2))$ for a suitable time $T > 0$ to be chosen.
We denote by $\|u\|_T := \|u\|_{L^\infty([0,T];H^1(\mathbb{R}^2))}$ and we recall that $(X_T, \|\,.\,\|_T)$ is a Banach space.
Set $w_l := \mathrm{e}^{t\Delta}(I - S_N)u_0$ and consider the map

$$\Psi : u \longmapsto \int_0^t \mathrm{e}^{(t-s)\Delta}(f(u + \mathbf{v}_N + w_l) - f(\mathbf{v}_N))(s)\, ds.$$

Let $B_T(r)$ be the ball in X_T of radius $r > 0$ and centered at the origin. We prove that for some $T, r > 0$, the map Ψ is a contraction from $B_T(r)$ into itself.
Applying the energy estimate (6) to $u_1, u_2 \in B_T(r)$ and using the smoothing effect (5), we infer

$$\begin{aligned}\|\Psi(u_1) - \Psi(u_2)\|_T &\lesssim \|f(u_1 + \mathbf{v}_N + w_l) - f(u_2 + \mathbf{v}_N + w_l)\|_{L^1([0,T],L^2(\mathbb{R}^2))} \\ &+ T^{\frac{3}{2}}\|\nabla(f(u_1 + \mathbf{v}_N + w_l) - f(u_2 + \mathbf{v}_N + w_l))\|_{L^\infty([0,T],L^1(\mathbb{R}^2))}.\end{aligned}$$

Set $w := u_1 - u_2$ and $v_i := u_i + \mathbf{v}_N + w_l$. Using Lemma 3.3, we obtain

$$\|f(v_1) - f(v_2)\|_{L^2(\mathbb{R}^2)} \lesssim \sum_{i=1,2} \|w(e^{2v_i^2} - 1)\|_{L^2(\mathbb{R}^2)}$$

Since $|v_i|^2 \leq 2(w_l + u_i)^2 + 2v_N^2$ and using the simple observation

$$e^{a+b} - 1 = (e^a - 1)(e^b - 1) + (e^a - 1) + (e^b - 1)$$

we have,

$$\begin{aligned}\|w(e^{2v_i^2} - 1)\|_{L^2} &\leq \|w(e^{4\mathbf{v}_N^2} - 1)\|_{L^2} + \|w(e^{4(u_i+w_l)^2} - 1)\|_{L^2} \\ &+ \|w(e^{4\mathbf{v}_N^2} - 1)(e^{4(1+\epsilon)(u_i+w_l)^2} - 1)\|_{L^2}.\end{aligned}$$

By Hölder inequality and Sobolev embedding, we have

$$\|w(e^{4\mathbf{v}_N^2} - 1)\|_{L^2} \leq \|w\|_{L^2} e^{4\|\mathbf{v}_N\|_{L^\infty}^2} \leq e^{4\|\mathbf{v}_N\|_{L^\infty}^2}\|w\|_{H^1},$$

and

$$\|w(e^{4(u_i+w_l)^2} - 1)\|_{L^2} \leq \|w\|_{L^6}\|e^{4(u_i+w_l)^2} - 1\|_{L^3} \lesssim \|w\|_{H^1}\|e^{4(u_i+w_l)^2} - 1\|_{L^3}.$$

Denoting $''_N := \|(I - S_N)u_0\|_{H^1}$, we have $\|\nabla(u_i + w_l)\|_{L^2} \leq r + \varepsilon_N \xrightarrow{r,N} 0$. Hence, for $\alpha > 0$, $p \geq 1$ and thanks to Moser-Trudinger inequality we derive that for **large** N **and small** $r > 0$,

$$\begin{aligned}\|e^{\alpha(u_i+w_l)^2} - 1\|_{L^p} &\leq \|e^{\alpha p(u_i+w_l)^2} - 1\|_{L^1}^{\frac{1}{p}} \\ &\lesssim (r + \varepsilon_N)^{2/p},\end{aligned} \tag{16}$$

and

$$\|w(e^{4(u_i+w_l)^2} - 1)\|_{L^2} \lesssim \|w\|_{H^1}(r + \varepsilon_N)^{2/3}.$$

Consequently,

$$\|w(e^{4\mathbf{v}_N^2} - 1)(e^{4(1+\epsilon)(u_i+w_l)^2} - 1)\|_{L^2} \lesssim e^{4\|\mathbf{v}_N\|_{L^\infty}^2}\|w\|_{H^1}(r + \varepsilon_N)^{2/3}.$$

Therefore,

$$\begin{aligned}\|f(v_1) - f(v_2)\|_{L^1([0,T],L^2)} \lesssim [&e^{4\|\mathbf{v}_N\|_{L^\infty}^2} + \\ &(1 + e^{4\|\mathbf{v}_N\|_{L^\infty}^2})(r + \varepsilon_N)^{2/3}]T\|w\|_{L^\infty([0,T],H^1(\mathbb{R}^2))},\end{aligned}$$

It remains to control $\|\nabla(f(v_1)-f(v_2))\|_{L^\infty([0,T],L^1(\mathbb{R}^2))}$. We have

$$\begin{aligned}\|\nabla(f(v_1)-f(v_2))\|_{L^1(\mathbb{R}^2)} & \\ &= \|\nabla v_1(f'(v_1)-f'(v_2))+(\nabla v_1-\nabla v_2)f'(v_2)\|_{L^1(\mathbb{R}^2)}\\ &\le \|\nabla v_1(f'(v_1)-f'(v_2))\|_{L^1(\mathbb{R}^2)}+\|\nabla w f'(v_2)\|_{L^1(\mathbb{R}^2)}\\ &\le \mathbf{E}+\mathbf{F}.\end{aligned}$$

Arguing as before, we have

$$\mathbf{E}\lesssim \|v_1\|_{H^1}\mathrm{e}^{4\|v_N\|^2_{L^\infty}}(1+(r+\varepsilon_N)^{1/2})\|w\|_{H^1},$$

and

$$\mathbf{F}\lesssim (\|v_N\|_{H^1}+2r+2\varepsilon_N)\mathrm{e}^{4\|v_N\|^2_{L^\infty}}\|w\|_{H^1}.$$

Therefore

$$\|\nabla(f(v_1)-f(v_2))\|_{L^\infty([0,T],L^1(\mathbb{R}^2))} \le \mathbf{C}_{0,r,N}\|w\|_T,$$

which implies that

$$\|\Psi(u_1)-\Psi(u_2)\|_T\le \mathbf{C}_{0,r,N}(1+T)T^{\frac{1}{2}}\|w\|_T. \tag{17}$$

Now we estimate $\|\Psi(u_1)\|_T$. Taking account of the energy estimate (6) and the smoothing effect (5), we get

$$\|\Psi(u_1)\|_T\le C\|f(v_1)-f(v_N)\|_{L^1([0,T],L^2(\mathbb{R}^2))}+ \\ T^{\frac{3}{2}}\|\nabla(f(v_1)-f(v_N))\|_{L^\infty([0,T],L^1(\mathbb{R}^2))},$$

where we set $v_1:=u_1+v_N+w_l$. Taking $v_2=v_N$, in the precedent computations, we have

$$\begin{aligned}\|\Psi(u_1)\|_T &\le \mathbf{C}_{0,r,N}(1+T)T^{\frac{1}{2}}\|u_1+w_l\|_T\\ &\le \mathbf{C}_{0,r,N}(r+\|u_0\|_{H^1(\mathbb{R}^2)})(1+T^{\frac{1}{2}})T^{\frac{1}{2}}\end{aligned}$$

In conclusion, for some fixed large N and small r, there exists $T>0$ small enough such that Ψ is a contraction of some ball of X_T. We obtain the desired solution by taking $u+w_l$ where u is the fixed point of Ψ. The proof is achieved. ■

4.2 Uniqueness in $\mathcal{C}([0,T[;H^1(\mathbb{R}^2))$

This subsection is devoted to the proof of the uniqueness part of Theorem 2.1. More precisely, we prove an unconditional uniqueness result.

Theorem 4.4. *The solution given in Theorem 4.1 is unique in the class*

$$\mathcal{C}([0,T);H^1(\mathbb{R}^2)).$$

Proof of Theorem 4.4. Let $u,\ v \in \mathcal{C}([0,T];H^1)$ be two solutions to (4) with same data u_0 and set $w := u - v$. Define the potential

$$a(t,x) := \begin{cases} \frac{f(u)-f(v)}{w}, & \text{if } w \neq 0 \\ f'(u), & \text{if } w = 0, \end{cases}$$

so that,

$$w(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} a(s) w(s)\ ds.$$

The following Lemma can be seen as an extension of Brezis-Cazenave's result [3] to the two dimensional case. The crucial point is to show the continuity of the potential term (continuity at $t = 0$). As pointed out in [3], this result seems to remain open if the potential is only L^∞ in time.

Lemma 4.5. *Let $a \in \mathcal{C}([0,T];L^p(\mathbb{R}^2))$ and $u \in L^\infty((0,T);L^q(\mathbb{R}^2))$ with $2 \leq q < \infty, 1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} < 1$ and such that*

$$u(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} a(s) u(s) ds, \quad \forall\ t \in [0,T].$$

Then $u = 0$ on $[0,T]$.

Proof. It is clear that $au \in L^\infty([0,T),L^r(\mathbb{R}^2))$ where $\dfrac{1}{r} = \dfrac{1}{q} + \dfrac{1}{p}$ $(1 < r < \infty)$, so that by maximal regularity $u \in L^{\widetilde{p}}((0,T),W^{2,r}(\mathbb{R}^2))$ for all $\widetilde{p} < \infty$ and satisfies for almost every $t \in (0,T)$ the next equation in $L^r(\mathbb{R}^2)$,

$$\partial_t u - \triangle u = au\,. \tag{18}$$

Let $t_0 \in [0,T]$, $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $a_n := \min\{n, \max\{a,-n\}\}$. Denote by v_n the solution to the dual problem

$$\begin{cases} -\partial_t v_n - \Delta v_n &= a_n v_n \quad \text{in } (0,t_0) \times \mathbb{R}^2, \\ v_n(t_0) = \psi. \end{cases}$$

Multiplying (18) by v_n and then integrating on $(0,t_0) \times \mathbb{R}^2$, we have

$$\int_0^{t_0} \int_{\mathbb{R}^2} (\partial_t u v_n - \Delta u v_n)\ dx\ dt = \int_0^{t_0} \int_{\mathbb{R}^2} a u v_n\ dx\ dt$$

Hence

$$\begin{aligned} \int_{\mathbb{R}^2} u(t_0)\psi\ dx &= \int_0^{t_0} \partial_t(u v_n)\ dx\ dt \\ &= \int_0^{t_0} \int_{\mathbb{R}^2} u \partial_t v_n + (\Delta u + au) v_n\ dx\ dt \\ &= \int_0^{t_0} \int_{\mathbb{R}^2} (a - a_n) u v_n\ dx\ dt. \end{aligned} \tag{19}$$

In order to prove that $u = 0$ on $[0,T]$ it is sufficient to observe that

$$a_n \overset{n\to\infty}{\longrightarrow} a \text{ in } \mathcal{C}([0,T];L^p(\mathbb{R}^2)) \tag{20}$$

and

$$\sup_{n\geq 0}\|v_n\|_{L^\infty([0,t_0),L^{r'}(\mathbb{R}^2))}\leq C_{r'}\|\psi\|_{L^{r'}(\mathbb{R}^2)}, \tag{21}$$

where $\frac{1}{r'}=1-\frac{1}{r}$. Now, we prove (21). We take $\widetilde{v}_n(t):=v_n(t_0-t)$, and $b_n(t)=: a_n(t_0-t)$, we have

$$\begin{cases} \partial_t\widetilde{v}_n-\Delta\widetilde{v}_n &= b_n\widetilde{v}_n,\\ \widetilde{v}_n(0)=\psi. \end{cases}$$

First, we multiply the precedent equation by $|\widetilde{v}_n|^{r'-2}\widetilde{v}_n$ then we integrate over $\mathbb{R}^2$, we obtain

$$\begin{aligned}\frac{1}{r'}\frac{d}{dt}\int_{\mathbb{R}^2}|\widetilde{v}_n(t,x)|^{r'}\,dx+\frac{4(r'-1)}{r'^2}\int_{\mathbb{R}^2}|\nabla|\widetilde{v}_n|^{r'/2}|^2\,dx &\leq \int_{\mathbb{R}^2}|b_n||\widetilde{v}_n|^{r'}\,dx\\ &\leq \int_{\mathbb{R}^2}|b||\widetilde{v}_n|^{r'}\,dx.\end{aligned}$$

In the last inequality we used $|b_n|\leq|b|$ because $|a_n|\leq|a|$, where $b=a(t_0-.)$ on $[0,t_0]$.
Using the fact that $|b_j|\leq j$ and Sobolev embedding, we get

$$\begin{aligned}\int_{\mathbb{R}^2}|b||\widetilde{v}_n|^{r'}\,dx &\leq \int_{\mathbb{R}^2}|b-b_j||\widetilde{v}_n|^{r'}\,dx+\int_{\mathbb{R}^2}|b_j||\widetilde{v}_n|^{r'}\,dx\\ &\leq \|b-b_j\|_{L^p}\|\widetilde{v}_n^{r'/2}\|^2_{L^{2p'}}+j\int_{\mathbb{R}^2}|\widetilde{v}_n|^{r'}\,dx,\\ &\leq C\|b-b_j\|_{L^p}\|\nabla|\widetilde{v}_n|^{r'/2}\|^2_{L^{2(1+\frac{1}{\varepsilon})}}+(j+C)\|\widetilde{v}_n(t)\|^{r'}_{L^{r'}}\end{aligned}$$

Since $b_j\overset{j\to\infty}{\longrightarrow}b$ in $\mathcal{C}([0,T];L^p(\mathbb{R}^2))$, we choose $j\geq 0$ large enough such that

$$C\|b-b_j\|_{L^p}\leq\frac{4(r'-1)}{r'^2}.$$

Therefore

$$\frac{1}{r'}\frac{d}{dt}\|\widetilde{v}_n(t)\|^{r'}_{L^{r'}}\leq(j+C)\|\widetilde{v}_n(t)\|^{r'}_{L^{r'}}.$$

Using Gronwall Lemma, il follows that

$$\|\widetilde{v}_n(t)\|^{r'}_{L^{r'}}\leq\|\psi\|^{r'}_{L^{r'}}e^{(j+C)r't}$$

which conclude the proof of (21).
The proof of the Lemma 4.5 is achieved. ∎

Now, in order to apply Lemma 4.5, we need to check that $a\in\mathcal{C}([0,T];L^2)$. We proceed by contradiction. Assume that there exists $\varepsilon>0$, $t\in[0,T]$ and a sequence of real numbers (t_n) in $[0,T]$ such that

$$t_n\to t\quad\text{and}\quad\|a(t_n)-a(t)\|_{L^2}>\varepsilon,\quad\forall n\in\mathbb{N}. \tag{22}$$

Denote $u_n:=u(t_n), v_n:=v(t_n)$ and $w_n:=w(t_n)$. Recall that $u,v\in\mathcal{C}([0,T];H^1)$. So up to extraction of a subsequence, we have

$$a(t_n)\to a(t)\quad\text{almost everywhere.}$$

Moreover, by a convexity argument

$$|a(t_n)| \leq e^{2u_n^2} - 1 + e^{2v_n^2} - 1.$$

Since $u \in \mathcal{C}([0,T]; H^1)$, using Proposition 3.6, we infer

$$e^{2u_n^2} - 1 \to e^{2u^2} - 1 \quad \text{and} \quad e^{2v_n^2} - 1 \to e^{2v^2} - 1 \quad \text{in} \quad L^2(\mathbb{R}^2).$$

Thus, there exists $\phi \in L^2$ such that

$$|a(t_n)| \leq \phi.$$

Using Lebesgue theorem, we deduce that

$$a(t_n) \to a(t) \quad \text{in} \quad L^2.$$

This contradicts (22), and we conclude that $a \in \mathcal{C}([0,T]; L^2)$.

End of the proof of Theorem 4.4.
It is sufficient to check assumptions of Lemma 4.5. Obviously $w \in L^\infty([0,T], L^q(\mathbb{R}^2))$ for every $2 \leq q < \infty$. ∎

4.3 Global existence

Consider the solution u to (1) in the defocusing case $f(u) = -u(e^{u^2} - 1)$. With Proposition 3.6, (by applying to $\sqrt{p}u$ and using the inequality $e^{pu^2} - 1 \geq (e^{u^2} - 1)^p$), it follows that $u(e^{u^2} - 1) \in \mathcal{C}([0,T], L^p)$ for every $1 \leq p < \infty$. This implies that $u - e^{t\Delta}u_0 \in \mathcal{C}([0,T], L^\infty)$ so that $u \in \mathcal{C}([0,T), L^\infty)$. This means that u is a classical solution for positive time. The global existence is then a trivial consequence of the maximum principle and the next standard blow-up criterion (see for example [3]).

Lemma 4.6. *Let $u_0 \in H^1(\mathbb{R}^2)$ and $u \in \mathcal{C}([0,T^*); H^1(\mathbb{R}^2))$ solution to* (4). *Assume that $T^* < \infty$, then*

$$\limsup_{t \to T^*} \|u(t)\|_{L^\infty(\mathbb{R}^2)} = +\infty.$$

4.4 Blow-up solutions

Recall the energy

$$J(t) := J(u(t)) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \int_{\mathbb{R}^2} F(u(t))\, dx,$$

with $F(u) = \frac{1}{2}\left(e^{u^2} - 1 - u^2\right)$. We show that all solutions with non-positive energy have a finite lifespan time. More precisely

Proposition 4.7. *Let $u_0 \in H^1(\mathbb{R}^2)\backslash\{0\}$ such that $J(u_0) \leq 0$ and $u \in \mathcal{C}([0,T^*[; H^1(\mathbb{R}^2))$ be the maximal solution to* (4) *with data u_0. Then $T^* < \infty$.*

The proof is standard and follows for example [15] (see also [13] in the context of the Klein-Gordon equation). It consists in following the evolution in time of the function

$$y(t) := \frac{1}{2}\int_0^t \|u(s)\|_{L^2}^2\, ds.$$

Proof of Proposition 4.7. First, observe that since we have removed the quadratic term from the nonlinearity, then $f(u)$ enjoys the following property for a certain positive number ε

$$\Big(uf(u) - 2F(u)\Big) \geq \varepsilon F(u)\,. \tag{23}$$

Next, multiplying (4) by u, integrating in space we obtain

$$J'(t) = -\|\partial_t u(t)\|_{L^2}^2,$$

and by an integration in time

$$J(t) = J(0) - \int_0^t \int_{\mathbb{R}^2} (\partial_t u)^2(s,x)\, dx\, ds. \tag{24}$$

Finally, a straight calculation shows that

$$\begin{aligned} y''(t) &= -\|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^2} uf(u)\, dx \\ &\geq \frac{2+\varepsilon}{2}\Big(\int_{\mathbb{R}^2} 2F(u)\, dx - \|\nabla u\|_{L^2}^2\Big) \\ &\geq (2+\varepsilon)\Big(\int_0^t \int_{\mathbb{R}^2} \partial_t u^2\, dx\, ds - J(0)\Big), \end{aligned} \tag{25}$$

where we used property (23) in the second estimate and identity (24) in the last one. Now, the proof goes by contradiction assuming that $T^* = \infty$. We have
Claim 1: There exists $t_1 > 0$ such that $\int_0^{t_1} \|\partial_t u(s)\|_{L^2}^2\, ds > 0$.
Indeed, otherwise $u(t) = u_0$ almost everywhere and thus u solves the elliptic stationary equation $\Delta u = -f(u)$. Then $\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} uf(u)dx$, and therefore

$$0 \leq \varepsilon \int_{\mathbb{R}^2} F(u_0)\, dx \leq \int_{\mathbb{R}^2} \Big(u_0 f(u_0) - 2F(u_0)\Big)\, dx = 2J(0) \leq 0$$

giving $u_0 = 0$ which is an absurdity.
Claim 2: For any $0 < \alpha < 1$, there exists $t_\alpha > 0$ such that

$$(y'(t) - y'(0))^2 \geq \alpha y'(t)^2, \quad t \geq t_\alpha.$$

The claim immediately follows from the first one observing that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = +\infty.$$

Claim 3: One can choose $\alpha = \alpha(\varepsilon)$ such that

$$y(t)y''(t) \geq (1+\alpha)y'(t)^2, \quad t \geq t_\alpha. \tag{26}$$

Indeed, we have

$$\begin{aligned} y(t)y''(t) &\geq \frac{2+\varepsilon}{2}\Big(\int_0^t\int_{\mathbb{R}^2} u^2\,dxds\Big)\Big(\int_0^t\int_{\mathbb{R}^2}\partial_t u^2\,dxds\Big)\\ &\geq \frac{2+\varepsilon}{2}\Big(\int_0^t\int_{\mathbb{R}^2} u\partial_t u\,dxds\Big)^2\\ &\geq \frac{2+\varepsilon}{2}(y'(t)-y'(0))^2\\ &\geq \frac{(2+\varepsilon)\alpha}{2}(y'(t))^2, \end{aligned}$$

where we used (25) in the first estimate, Cauchy-Schwarz inequality in the second and Claim 2 in the last one. Now choose α such that $\frac{(2+\varepsilon)\alpha}{2} > 1$ then

$$y(t)y''(t) \geq \frac{(2+\varepsilon)\alpha}{2}(y'(t))^2.$$

The fact that this ordinary differential inequality blows up in finite time contradicts our assumption that the solution was global. ■

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