Applications of monotone operators to a class of semilinear elliptic BVPs in unbounded domain

Rasmita Kar

Abstract

We study the existence of a weak solution for a semilinear elliptic Dirichlet boundary-value problem

$$\begin{aligned} Lu(x) - \mu u g_1(x) + h(u) g_2(x) = f(x) &\quad \text{in } \Omega, \\ u(x) = 0 &\quad \text{on } \partial\Omega, \end{aligned}$$

in a suitable weighted Sobolev space, where $\Omega = \mathbb{R}^n \backslash K, n \geq 3$ is an unbounded domain, and where K is a closure of some bounded domain in $\mathbb{R}^n, n \geq 3$.

1 Introduction

Let $\Omega = \mathbb{R}^n \backslash K, n \geq 3$ be an unbounded domain with smooth boundary $\partial\Omega$, where K is a closure of some bounded domain in $\mathbb{R}^n$. Let L be an elliptic operator in the divergence form

$$Lu(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i u(x)) \quad \text{with } D_j = \frac{\partial}{\partial x_j}, \tag{1.1}$$

with coefficients $a_{ij} \in L^\infty(\Omega)$ and the matrix (a_{ij}) is symmetric and satisfy

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \text{a.e., } x \in \Omega, \tag{1.2}$$

Received by the editors in September 2012 - In revised form in September 2013.
Communicated by F. Bastin.
2010 *Mathematics Subject Classification* : 35J61; 47H05.
Key words and phrases : Monotone operators; Weighted Sobolev space, Semilinear elliptic equations, Unbounded domain.

for all $\xi \in \mathbb{R}^n$ $(\lambda > 0, \Lambda > 0)$. In this paper, we establish the existence results for a class of semilinear elliptic BVP

$$\begin{aligned} Lu - \mu g_1 u + g_2 h(u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $\mu \in \mathbb{R}$, $g_i (i = 1,2)$, f are elements of some weighted spaces and h is Lipschitzian and monotonic. The main tool used is a result by Browder [8] and Minty [9] on monotone hemi-continuous operators. The study is inspired by a problem in bounded domain given in the book by Zeidler [7]. Also, a degenerate elliptic BVP studied by Cavalheiro [2] in a bounded domain, say $U \subset \mathbb{R}^n$ with boundary ∂U. More precisely Cavalheiro [2] studied the following :

Suppose that $\tilde{\mathcal{L}}$ be an elliptic operator in the divergence form as in (1.1), where the coefficient matrix (a_{ij}) satisfies the degenerate ellipticity condition

$$\lambda|\xi|^2\omega(x) \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2\omega(x), \quad \text{a.e., } x \in U, \tag{1.4}$$

for all $\xi \in \mathbb{R}^n$, $(\lambda > 0, \Lambda > 0)$. Here, ω be an A_2-weight. For more details on A_p-weight $(1 \leq p < \infty)$, we refer to [3, 10, 11].

Consider the BVP

$$\begin{aligned} \tilde{\mathcal{L}}u - \mu u g_1 + h(u) g_2 = f \quad \text{in } U, \\ u = 0 \quad \text{on } \partial U. \end{aligned} \tag{1.5}$$

where $\mu \in \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Assume that $g_1/\omega \in L^\infty(U)$, $g_2/\omega \in L^2(U,\omega)$ and $f/\omega \in L^2(U,\omega)$. Under these hypotheses on the functions f, g_1, g_2, and h, the following proposition is due to Cavalheiro [2].

Proposition 1.1. *Suppose that $\mu > 0$ not be an eigenvalue of*

$$\tilde{\mathcal{L}}u - \mu u g_1 = 0 \quad \text{in } U, \qquad u = 0 \quad \text{on } \partial U.$$

Then, the BVP (1.5)*, has a weak solution $u \in W_0^{1,2}(U,\omega)$.*

In the Proposition 1.1 main tools used are compact embedding in weighted Sobolev space and a result introduced by Hess [14] in 1972 (also found in the book by Zeidler [7]). In the present paper, we study the elliptic BVP (1.3) in a class of unbounded domain. Elliptic BVPs in unbounded domains present specific difficulties, primarily due to lack of compactness. Another difficulty in the study of the elliptic BVPs is due to the non-availability of the Poincare-inequality in the Sobolev spaces $W_0^{1,p}(\Omega)$ for a general unbounded domain Ω. One of the classical technique employed is approximating a solution on unbounded domain say Ω by solutions on bounded subdomains of Ω under the assumption the suitable upper and lower solutions exist, as in Noussair and Swanson[5, 6]. Secondly, the use of weighted-norm Sobolev spaces which admit compact embeddings, as in Benci [16], Bongers, Heinz and Kupper [1]. In [12], Berger and Schechter have shown that a substitute for such embedding results can be obtained when Ω is unbounded, by introducing appropriate weighted L^p norms. As a consequence, to prove the existence of a weak solution to (1.3) we consider suitable weighted Sobolev space defined on a specific class of domains in $\mathbb{R}^n, n \geq 3$. Where as the restriction on the domain has yields us a required Hardy-type inequality. Also as in [2], we do not need the hypothesis of boundedness of the nonlinear function h and instead we have assumed h to be Lipschitzian and monotonic. The assumptions on h along with Hardy-type inequality helps us to establish the existence result without use of compact embedding theorems.

Section 2 deals with preliminaries. Section 3 concerns with the main result is about the existence of a weak solution of the BVP (1.3).

2 Preliminaries

Let $G \subset \mathbb{R}^n$ be a domain (not necessarily bounded) with a smooth boundary ∂G. For a weight function ω(*i.e.*, $\omega : G \to \mathbb{R}^+$ be a locally integrable function with $0 < \omega < \infty$ a.e.,) we define the weighted p-norm

$$\|v\|_{p,G,\omega} = \Big(\int_G |v(x)|^p \omega(x) dx \Big)^{1/p}, \; 1 \leq p < \infty, \tag{2.6}$$

and denote by $L^p(G, \omega)$ the space of all measurable functions v such that $\|v\|_{p,G,\omega}$ is finite. For weight functions ω_0 and ω_1, the weighted Sobolev space $W^{1,p}(G, \omega_0, \omega_1)$ is defined to be the space of all functions $v \in L^p(G, \omega_0)$ such that all weak derivatives $\partial v / \partial x_i$ belong to $L^p(G, \omega_1)$. In this space, the norm is defined by

$$\|v\|_{1,p,G,\omega_0,\omega_1} = \Big\{ \int_G (|v|^p \omega_0 + |\nabla v|^p \omega_1) \, dx \Big\}^{1/p}, \tag{2.7}$$

where $\nabla v = (D_1 v, D_2 v, \ldots, D_n v)$. It is known (cf.[4, 13]) that $W^{1,p}(G, \omega_0, \omega_1)$ is a uniformly convex Banach space, provided

$$p > 1, \; \omega_0^{-1}, \omega_1^{-1} \in L_{loc}^{\frac{1}{p-1}}(G), \tag{2.8}$$

and, moreover $C_0^\infty(G) \subset W^{1,p}(G, \omega_0, \omega_1)$, if and only if

$$\omega_0, \omega_1 \in L^1_{loc}(G). \tag{2.9}$$

Under the assumptions (2.8) and (2.9), let $W_0^{1,p}(G, \omega_0, \omega_1)$ be the closure of $C_0^\infty(G)$ with respect to the norm (2.7). We also note that $W^{1,2}(G, \omega_0, \omega_1)$ and $W_0^{1,2}(G, \omega_0, \omega_1)$, are Hilbert spaces. We denote the space $W_0^{1,p}(G, \omega_0, 1)$ by $W_0^{1,p}(G, \omega_0)$. More details on weighted Sobolev spaces are found in [3, 4, 10, 11, 13, 15].

We say domain G belongs to class $\mathcal{D}$ if there exists a compact set $K \subset \mathbb{R}^n$, such that $G = \mathbb{R}^n \backslash K$. For $G \in \mathcal{D}$, we set

$$a^* = \inf\{|x|; x \in G\}.$$

We denote by $\mathcal{D}^{0,1}$ the set of all $G \in \mathcal{D}$ such that

$$G = \mathbb{R}^n \backslash K \quad \text{with } K \in C^{0,1}.$$

More details on the classes $\mathcal{D}^{0,1}$, $C^{0,1}$ are given in the book by Opic and Kufner [4, p.269,p.289]. Another important and useful tool for the study of partial differential equations in unbounded domains is the Hardy-type inequality. Let $a^* > 0$ and further we assume $G \in \mathcal{D}[G \in \mathcal{D}^{0,1}]$ has the property that

$$x \in G \text{ implies } tx \in G, \quad \text{for every } t \geq 1,$$

and such a class of domains is denoted by $\mathcal{D}_*[\mathcal{D}_*^{0,1}]$. Moreover, the weight functions ω_0, ω_1 are assumed to be radial, i.e of the form $\omega_0(x) = \psi_0(|x|)$, $\omega_1(x) = \psi_1(|x|)$, where ψ_0, ψ_1 are defined on $(0,\infty)$ and bounded from below and above on each compact subinterval in $(0,\infty)$. Let $1 < p < \infty$ and p' be defined by $1/p + 1/p' = 1$. Then, the following Hardy-type inequality holds(Theorem 21.8,[4]):

Proposition 2.1. *(Hardy-type inequality) Assume that there are constants $k, t_0 > 0$ such that*

$$\psi_0(t) \geq k\,\psi_1(t)\,t^{-p} \quad \text{for a.e } t > t_0,$$
$$\mathfrak{B}_R(\psi_0(t)t^{n-1}, \psi_1(t)t^{n-1}, p) < \infty,$$

where

$$\mathfrak{B}_R(\psi_0(t)t^{n-1}, \psi_1(t)t^{n-1}, p) = \sup_{0<s<\infty} \|(\psi_0(t)t^{n-1})^{1/p}\|_{p,(0,s)} \|(\psi_1(t)t^{n-1})^{-1/p}\|_{p',(s,\infty)}.$$

Then, there is a constant $C > 0$ such that

$$\|v\|_{p,G,\omega_0} \leq C\||\nabla v|\|_{p,G,\omega_1} \tag{2.10}$$

holds for every $v \in W^{1,p}(G, \omega_0, \omega_1)$.

As a consequence of the Hardy-type inequality, we get the equivalence of the norms

$$\|v\|_{1,p,G,\omega_0,\omega_1} \quad \text{and} \quad |v|_{0,1,p,G,\omega_1} = \Big(\int_G |\nabla v(x)|^p \omega_1(x) dx \Big)^{1/p}$$

in $W^{1,p}(G; \omega_0, \omega_1)$.

Remark 2.2. Let $G \in \mathcal{D}_*$ with $a^* > 0$, $n \geq 3$, $\psi_0(t) = t^{-2}$, $\psi_1(t) = 1$, $p = 2$. We note that

$$\psi_0(t) \geq t^{-2}\psi_1(t), \text{ for } t > 0.$$

Also,

$$\begin{aligned}
\mathfrak{B}_R(\psi_0(t)t^{n-1}, \psi_1(t)t^{n-1}, 2) &= \sup_{0<s<\infty} \|(\psi_0(t)t^{n-1})^{1/2}\|_{2,(0,s)} \|(\psi_1(t)t^{n-1})^{-1/2}\|_{2,(s,\infty)} \\
&= \sup_{0<s<\infty} \|(t^{-2}t^{n-1})^{1/2}\|_{2,(0,s)} \|(t^{n-1})^{-1/2}\|_{2,(s,\infty)} \\
&= \sup_{0<s<\infty} \|t^{\frac{n-3}{2}}\|_{2,(0,s)} \|t^{\frac{1-n}{2}}\|_{2,(s,\infty)} \\
&= \sup_{0<s<\infty} \Big\{ \int_0^s t^{n-3} dt \times \int_s^\infty t^{-n+1} dt \Big\}^{1/2} < \infty,
\end{aligned}$$

and thus,

$$\mathfrak{B}_R(|t|^{-2}t^{n-1}, t^{n-1}, 2) < \infty.$$

From (2.10) we have for $p = 2$, $\omega_0(x) \equiv \psi_0(|x|) = |x|^{-2}$, and $\omega_1(x) \equiv \psi_1(|x|) = 1$

$$\|v\|_{2,G,|x|^{-2}} \leq C\||\nabla v|\|_{2,G}, \;\; v \in W^{1,2}(G, \omega). \tag{2.11}$$

Definition 2.3. Let $\Omega \subset \mathbb{R}^n$ be an open connected set. $u \in W_0^{1,2}(\Omega, \omega)$ is called a *weak solution* of (1.3) if

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j \phi(x) dx - \mu \int_\Omega u(x) g_1(x) \phi(x)\, dx + \int_\Omega h(u(x)) g_2(x) \phi(x) dx$$
$$= \int_\Omega f(x)\phi(x)\, dx, \;\; \text{for all } \phi \in W_0^{1,2}(\Omega, \omega).$$

From [7], we quote :

Definition 2.4. Let $B : X \to X^*$ be an operator on a real Banach space X.

(i) B is *monotone* iff

$$\big(Bu - Bv | u - v\big) \geq 0 \quad \text{for all } u, v \in X.$$

(ii) B is *uniformly monotone* iff

$$\big(Bu - Bv | u - v\big) \geq a(\|u - v\|)\|u - v\| \quad \text{for all } u, v \in X.$$

where the continuous function $a : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotone increasing with $a(0) = 0$ and $a(t) \to \infty$ as $t \to \infty$.

(iii) B is *coercive* iff

$$\lim_{\|u\| \to \infty} \frac{(Bu|u)}{\|u\|} = \infty.$$

(iv) B is *hemi-continuous* iff

$$t \mapsto \langle B(u + tv), w \rangle$$

is continuous on $[0, 1]$ for all $u, v, w \in X$.

We have the following implications:
B is uniformly monotone implies B is monotone. Furthermore, we note that B is uniformly monotone implies B is coercive.

In Definition 2.4(ii), we may choose the function $a(t) = c|t|^{p-1}$ with $p > 1$ and $c > 0$. In this case, we obtain

$$\big(Bu - Bv | u - v\big) \geq c\|u - v\|^p \quad \text{for all } u, v \in X.$$

In section 3, we use the following result. We consider the operator equation

$$Au = b, \quad u \in X. \tag{2.12}$$

Theorem 2.5. *(Browder-Minty(1963)) Assume that the operator $A : X \to X^*$ is monotone, hemi-continuous and coercive on the real, separable, reflexive Banach space X. Then, for each $b \in X^*$, the equation* (2.12) *has a solution.*

The proof of the Theorem 2.5 is found in [7, Theorem 26.A], Browder [8] and Minty [9].

3 The main result

Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be an unbounded domain of the type $\mathcal{D}_*^{0,1}$ with $a^* > 0$. The advantage of choosing such domains is the availability of Hardy-type inequality with suitable weights. In this section, we study the existence of a weak solution of the BVP (1.3).

We need the following hypotheses for further study.

(F_1) Suppose that $\omega = |x|^{-2}$, $g_1/\omega \in L^\infty(\Omega)$, $g_2/\omega \in L^\infty(\Omega)$, $g_2 \geq 0$ and $f/\omega \in L^2(\Omega, \omega)$;

(F_2) Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant A and $h(0) = 0$;

(F_3) Suppose that h satisfies, $(h(\xi) - h(\xi'))(\xi - \xi') \geq 0$, for all $\xi, \xi' \in \mathbb{R}$.

We define the operator $B_1 : W_0^{1,2}(\Omega, \omega) \times W_0^{1,2}(\Omega, \omega) \to \mathbb{R}$ by

$$\begin{aligned} B_1(u, \phi) = &\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) D_i(x) u D_j(x) \phi(x)\, dx - \mu \int_\Omega u(x) g_1(x) \phi(x)\, dx \\ &+ \int_\Omega h(u(x)) g_2(x) \phi(x)\, dx, \quad \text{for all } u, \phi \in W_0^{1,2}(\Omega, \omega) \end{aligned}$$

and we define $T : W_0^{1,2}(\Omega, \omega) \to \mathbb{R}$ by

$$T(\phi) = \int_\Omega f(x) \phi(x) dx.$$

A function $u \in W_0^{1,2}(\Omega, \omega)$ is a weak solution of (1.3) iff

$$B_1(u, \phi) = T(\phi), \quad \text{for all } \phi \in W_0^{1,2}(\Omega, \omega). \tag{3.13}$$

Below, we establish the existence of a weak solution of (1.3) under certain conditions.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be an unbounded domain of the type $\mathcal{D}_*^{0,1}$ with $a^* > 0$. Assume that the hypotheses (F_1)-(F_3) hold. Suppose that*

$$\mu C \|g_1/\omega\|_{\infty,\Omega} < \lambda, \quad \mu > 0$$

and C is the constant arising out of inequality (2.11). *Then, the problem* (1.3) *has a weak solution $u \in W_0^{1,2}(\Omega, \omega)$.*

Proof. Idea of proof is such. First we write the BVP (1.3) as operator equation

$$u \in W_0^{1,2}(\Omega, \omega) : Bu = T \text{ in } [W_0^{1,2}(\Omega, \omega)]^*, \tag{3.14}$$

where $T \in [W_0^{1,2}(\Omega, \omega)]^*$, $B : W_0^{1,2}(\Omega, \omega) \to [W_0^{1,2}(\Omega, \omega)]^*$ is monotone, hemicontinuous and coercive. Further, we put Proposition 2.5 to this operator equation. For convenience, we have divided the proof into five steps.

Step-1 : We note that, for some constant c, $|a_{ij}(x)| \leq c$. Since h is Lipschitzian and $h(0) = 0$, we get $|h(u)| \leq A|u|$. For all $u, \phi \in W_0^{1,2}(\Omega, \omega)$, we have

$$\begin{aligned}
|B_1(u,\phi)| \leq &\int_\Omega \sum_{i,j=1}^n |a_{ij}(x)||D_i u(x)||D_j \phi(x)|\,dx + \mu \int_\Omega |u(x)||\phi(x)||g_1(x)|\,dx \\
&+ \int_\Omega |g_2(x)||h(u(x))||\phi(x)|\,dx \\
&\leq c \int_\Omega \sum_{i,j=1}^n |D_i u(x)||D_j\phi(x)|\,dx + \mu \int_\Omega |u(x)||\phi(x)||\frac{g_1(x)}{\omega(x)}|\omega(x)\,dx \\
&+ A \int_\Omega |\frac{g_2(x)}{\omega(x)}||u(x)||\phi(x)|\omega(x)\,dx \\
&\leq c \Big(\int_\Omega \sum_{i=1}^n |D_i u(x)|^2\,dx \Big)^{1/2} \Big(\int_\Omega \sum_{j=1}^n |D_j\phi(x)|^2\,dx \Big)^{1/2} \\
&+ \mu \|\frac{g_1}{\omega}\|_{\infty,\Omega} \Big(\int_\Omega |u(x)|^2 \omega(x)\,dx \Big)^{1/2} \Big(\int_\Omega |\phi(x)|^2\omega(x)\,dx \Big)^{1/2} \\
&+ A \|\frac{g_2}{\omega}\|_{\infty,\Omega} \Big(\int_\Omega |u(x)|^2 \omega(x)\,dx \Big)^{1/2} \Big(\int_\Omega |\phi(x)|^2\omega(x)\,dx \Big)^{1/2} \\
&\leq c\,|u|_{0,1,2,\Omega}|\phi|_{0,1,2,\Omega} + \Big(\mu\|g_1/\omega\|_{\infty,\Omega} + A\|g_2/\omega\|_{\infty,\Omega} \Big) \|u\|_{2,\Omega,\omega}\|\phi\|_{2,\Omega,\omega} \\
&\leq \Big(c + C\mu\|g_1/\omega\|_{\infty,\Omega} + CA\|g_2/\omega\|_{\infty,\Omega} \Big) |u|_{0,1,2,\Omega}|\phi|_{0,1,2,\Omega}. \qquad (3.15)
\end{aligned}$$

where, C is a constant arising out of the Hardy-type inequality (2.11). Now, $B_1(u,.)$ is linear and bounded. Then, there exists an operator

$$B : W_0^{1,2}(\Omega,\omega) \to [W_0^{1,2}(\Omega,\omega)]^*,$$

defined by $(Bu|\,\phi) = B_1(u,\phi)$ for all $u, \phi \in W_0^{1,2}(\Omega,\omega)$. Also, we have

$$\begin{aligned}
|T(\phi)| \leq &\int_\Omega |f(x)||\phi(x)|\,dx \\
\leq &\ \|f/\omega\|_{2,\Omega,\omega}\|\phi\|_{2,\Omega,\omega} \\
\leq &\ C\,\|f/\omega\|_{2,\Omega,\omega}|\phi|_{0,1,2,\Omega}.
\end{aligned}$$

Then, problem (3.13) is equivalent to the operator equation

$$Bu = T,\ u \in W_0^{1,2}(\Omega,\omega). \qquad (3.16)$$

Step-2: We claim that B is hemi-continuous. That is,

$$t \mapsto (B(u+tv)|\phi)$$

is continuous on [0,1], for all $u, v, \phi \in W_0^{1,2}(\Omega,\omega)$. Let $F : [0,1] \to \mathbb{R}$ defined by

$$F(t) = (B(u+tv)|\phi) = B_1(u+tv,\phi),\ t \in [0,1].$$

Suppose $t_n \in [0,1]$ such that $t_n \to a$ in $[0,1]$ as $n \to \infty$. We show that $F(t_n) \to F(a)$ as $n \to \infty$.

We note that

$$
\begin{aligned}
|F(t_n)-F(a)| &= \big|(B(u+t_nv)|\phi)-(B(u+av)|\phi)\big| \\
&= \big|B_1(u+t_nv,\phi)-B_1(u+av,\phi)\big| \\
&= \Big|\int_\Omega \sum_{i,j=1}^n a_{ij}D_i(u+t_nv)D_j\phi dx-\mu\int_\Omega (u+t_nv)\phi g_1\,dx \\
&\quad+\int_\Omega h(u+t_nv)\phi g_2 dx-\int_\Omega \sum_{i,j=1}^n a_{ij}D_i(u+av)D_j\phi dx \\
&\quad+\mu\int_\Omega (u+av)\phi g_1\,dx-\int_\Omega h(u+av)\phi g_2\Big|\,dx \\
&\le \int_\Omega \Big|\sum_{i,j=1}^n a_{ij}\{(D_i(u+t_nv)-D_i(u+av)\}D_j\phi\Big|\,dx \\
&\quad+\mu\int_\Omega \big|(t_n-a)v\phi g_1\big|\,dx+\int_\Omega \big|(h(u+t_nv)-h(u+av))\phi g_2\,dx\big| \\
&\le \int_\Omega \Big|\sum_{i,j=1}^n a_{ij}D_i((t_n-a)v)D_j\phi\Big|\,dx+\mu||t_n-a|\int_\Omega |v\phi g_1|\,dx \\
&\quad+A\int_\Omega \big|(u+t_nv)-(u+av)\big|\big|\phi g_2\big|\,dx \\
&\le |t_n-a|\Big[c\int_\Omega \sum_{i,j=1}^n |D_iv||D_j\phi|dx+\mu\int_\Omega |v||\phi||g_1|\,dx \\
&\quad+A\int_\Omega |v||\phi||g_2|dx\Big] \\
&\le |t_n-a|\big[c\,|v|_{0,1,2,\Omega}|\phi|_{0,1,2,\Omega}+C\mu\|g_1/\omega\|_{\infty,\Omega}|v|_{0,1,2,\Omega}|\phi|_{0,1,2,\Omega} \\
&\quad+CA\|g_2/\omega\|_{\infty,\Omega}|v|_{0,1,2,\Omega}|\phi|_{0,1,2,\Omega}\big] \\
&= |t_n-a|\big[c+\mu C\|g_1/\omega\|_{\infty,\Omega}+CA\|g_2/\omega\|_{\infty,\Omega}\big]|v|_{0,1,2,\Omega}|\phi|_{0,1,2,\Omega}. \qquad (3.17)
\end{aligned}
$$

From (3.17), we note that F is continuous and as a consequence B turns out to be hemi-continuous.

Step-3 : It follows from the hypotheses (F_1) and (F_3) that

$$g_2(x)(h(u(x))-h(v(x)))(u(x)-v(x))\ge 0,\ \text{a.e., for all } u,v\in W_0^{1,2}(\Omega,\omega).$$

Since $\mu C\|g_1/\omega\|_{\infty,\Omega}<\lambda$, by (1.2), and the hypotheses (F_1) and (F_3), we obtain

$$
\begin{aligned}
B_1(u,u-v)-B_1(v,u-v) &= \int_\Omega \sum_{i,j=1}^n a_{ij}(D_iu-D_iv)D_j(u-v)dx \\
&\quad-\mu\int_\Omega (u-v)^2 g_1\,dx+\int_\Omega g_2(h(u)-h(v))(u-v)dx \\
&= \int_\Omega \sum_{i,j=1}^n a_{ij}D_i(u-v)D_j(u-v)dx-\mu\int_\Omega (u-v)^2 g_1\,dx \\
&\quad+\int_\Omega g_2(h(u)-h(v))(u-v)dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq \lambda \int_\Omega |\nabla(u-v)|^2 dx - \mu \|g_1/\omega\|_{\infty,\Omega} \int_\Omega (u-v)^2 \omega \, dx \\
&\geq \lambda \int_\Omega |\nabla(u-v)|^2 dx - \mu C \|g_1/\omega\|_{\infty,\Omega} \int_\Omega |\nabla(u-v)|^2 \omega \, dx \\
&\geq \left(\lambda - \mu C \|g_1/\omega\|_{\infty,\Omega}\right) |u-v|^2_{0,1,2,\Omega} \geq 0,
\end{aligned}
$$

for all $u, v \in W_0^{1,2}(\Omega,\omega)$. Consequently, B is monotone.

Step-4 : By (F_2), $h(0) = 0$ and by (F_3), $h(u)u \geq 0$, a.e., for all $u \in W_0^{1,2}(\Omega,\omega)$. Since $g_2 \geq 0$, we have $g_2 h(u)u \geq 0$, a.e., for all $u \in W_0^{1,2}(\Omega,\omega)$. By (1.2) and the hypotheses (F_1) and (F_3), we observe that

$$
\begin{aligned}
(Bu|u) = B_1(u,u) &= \int_\Omega \sum_{i,j=1}^{n} a_{ij} D_i u D_j u dx - \mu \int_\Omega u g_1 u \, dx + \int_\Omega g_2 h(u) u dx \\
&\geq \lambda \int_\Omega |Du|^2 dx - \mu \|g_1/\omega\|_{\infty,\Omega} \int_\Omega u^2 \omega \, dx + \int_\Omega g_2 h(u) u \, dx, \\
&\geq \left(\lambda - \mu C \|g_1/\omega\|_{\infty,\Omega}\right) |u|^2_{0,1,2,\Omega} \text{ for all } u, v \in W_0^{1,2}(\Omega,\omega).
\end{aligned}
$$

Since $\mu C \|g_1/\omega\|_{\infty,\Omega} < \lambda$, B is coercive.

Step-5 : We have B is hemi-continuous. Also, for $\mu C \|g_1/\omega\|_{\infty,\Omega} < \lambda$, B is monotone, and coercive. Hence, for $\mu C \|g_1/\omega\|_{\infty,\Omega} < \lambda$, by Theorem 2.5, BVP (1.3) has a weak solution, say $u \in W_0^{1,2}(\Omega,\omega)$. ∎

Remark 3.2. The Theorem 3.1 also holds true when h is monotonically decreasing with $g_2 \leq 0$.

In the following two results, we consider the cases $\mu < 0$, $\mu > 0$ and relax the hypothesis $\mu C \|g_1/\omega\|_{\infty,\Omega} < \lambda$ under the restriction g_1 does not change sign. The proof is similar to the Theorem 3.1; we restrict ourselves to sketch the deviations wherever needed.

Theorem 3.3. *Assume that the hypotheses (F_1)-(F_3) hold. Suppose that $g_1 \geq 0$, $\mu < 0$. Then, the BVP* (1.3) *has a weak solution in $W_0^{1,2}(\Omega,\omega)$.*

Proof. In Step-1 of Theorem 3.1, we note the following change :

$$
\begin{aligned}
|B_1(u,\phi)| \leq &\int_\Omega |a_{ij}(x)||D_i u(x)||D_j \phi(x)| \, dx + |\mu| \int_\Omega |u(x)||\phi(x)||g_1(x)| \, dx \\
&+ \int_\Omega |g_2(x)||h(u(x))||\phi(x)| \, dx \\
\leq &\left(c + C|\mu| \|g_1/\omega\|_{\infty,\Omega} + CA \|g_2/\omega\|_{\infty,\Omega}\right) |u|_{0,1,2,\Omega} |\phi|_{0,1,2,\Omega}.
\end{aligned}
$$

Also, by minor changes in Step-2 (in (3.17)) of Theorem 3.1, we note that the operator B is hemi-continuous. Since $g_1 \geq 0$, $\mu < 0$, we observe that

$$\begin{aligned}
B_1(u, u-v) - B_1(v, u-v) &= \int_\Omega a_{ij}(D_i u - D_i v) D_j(u-v) dx \\
&\quad - \mu \int_\Omega (u-v)^2 g_1 \, dx + \int_\Omega (h(u) - h(v))(u-v) g_2 \, dx \\
&= \int_\Omega a_{ij} D_i(u-v) D_j(u-v) dx + \int_\Omega (h(u) - h(v))(u-v) g_2 dx \\
&\geq \lambda \int_\Omega |\nabla(u-v)|^2 dx \ \text{ (By (1.2), and } (F_3)) \\
&\geq 0, \ \text{ for all } \ u, v \in W_0^{1,2}(\Omega, \omega).
\end{aligned}$$

Consequently, B is monotone. By the hypotheses $(F_1), (F_2)$ and (F_3), we obtain $g_2 h(u) u \geq 0$, a.e., for all $u \in W_0^{1,2}(\Omega, \omega)$. Since $g_1 \geq 0$, $\mu < 0$, by (1.2) as in Step-4 of Theorem 3.1 we note that

$$\begin{aligned}
(Bu|u) = B_1(u, u) &= \int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j u(x) dx \\
&\quad - \mu \int_\Omega u^2(x) g_1(x) \, dx + \int_\Omega g_2(x) h(u(x)) u(x) dx \\
&\geq \lambda \int_\Omega |\nabla u|^2 dx + \int_\Omega g_2 h(u) u \, dx \\
&\geq \lambda |u|^2_{0,1,2,\Omega} \ \text{ for all } u, v \in W_0^{1,2}(\Omega, \omega),
\end{aligned}$$

which shows that, B is coercive. Since B is monotone and hemi-continuous, and coercive, by Theorem 2.5, the BVP (1.3)(with $\mu < 0$ and $g_1 \geq 0$), has a weak solution in $W_0^{1,2}(\Omega, \omega)$. ∎

Similarly, we have the following result :

Theorem 3.4. *Assume the hypotheses* (F_1)-(F_3). *Suppose that* $g_1 \leq 0, \mu > 0$. *Then,* (1.3) *has a weak solution in* $W_0^{1,2}(\Omega, \omega)$

Acknowledgements. I thank the anonymous referee for the constructive comments and suggestions.

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The LNM Institute of Information Technology
Rupa ki Nangal, Post-Sumel, Via-Jamdoli
Jaipur-302031, (Rajasthan) INDIA
email:rasmitak6@gmail.com