Uniqueness of positive solutions for a singular nonlinear eigenvalue problem when a parameter is large

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Abstract

We study positive solutions to singular boundary value problems of the form

$$\begin{cases} -u''(t) = \lambda h(t)\frac{f(u(t))}{u(t)^\beta}; & (0,1) \\ u(0) = 0 = u(1), \end{cases}$$

where $\lambda > 0$ is a parameter, $\beta \in (0,1), f : [0,\infty) \to (0,\infty)$ is a C^1 function, $\frac{f(s)}{s^\beta}$ is decreasing for $s \gg 1, h : (0,1) \to (0,\infty)$ is a continuous function and there exist $C_i > 0, \alpha_i \in (0,1), i = 1,2$ such that $h(t) \le \frac{C_1}{t^{\alpha_1}};\ t \approx 0$ and $h(t) \le \frac{C_2}{(1-t)^{\alpha_2}};\ t \approx 1$. We establish the uniqueness of positive solutions for $\lambda \gg 1$, when $\alpha_i + \beta < 1, i = 1,2$.

1 Introduction

Study of positive solutions to reaction diffusion process and their analysis have great importance in understanding physical and biological phenomena. In particular, the steady states define the long term dynamics of these processes. Hence an analysis when a steady state reaction diffusion equation has a unique solution is a very significant question.

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In this paper, we consider $C^2(0,1)\cap C^1[0,1]$ positive solutions to steady state singular boundary value problems of the form

$$\begin{cases} -u''(t)=\lambda h(t)\dfrac{f(u(t))}{u(t)^{\beta}};\ (0,1) \\ u(0)=0=u(1), \end{cases} \tag{P}$$

where $\lambda>0,\beta\in(0,1)$ and $h:(0,1)\to(0,\infty)$ is a continuous function and there exist $C_i>0,\alpha_i\in(0,1),i=1,2$ such that $h(t)\le\frac{C_1}{t^{\alpha_1}};t\approx 0$ and $h(t)\le\frac{C_2}{(1-t)^{\alpha_2}};t\approx 1$. Here $f:[0,\infty)\to(0,\infty)$ is a C^1 function with $\inf\limits_{[0,\infty)} f(s)=:A>0$ and $\lim\limits_{s\to\infty}\dfrac{f(s)}{s^{\beta+1}}=0$. In [5], it has been established that (P) has a positive solution for each $\lambda>0$. Further under additional assumptions on the behavior of the function $\frac{s^{\beta+1}}{f(s)}$, a multiplicity result for a certain finite range of λ also has been established.

Such singular two point boundary value problems arise in the study of radial solutions to nonlinear eigenvalue problems on an exterior domain such as:

$$\begin{cases} -\Delta v(x)=\lambda K(|x|)\dfrac{f(v(x))}{v(x)^{\beta}} & \text{in } \Omega_E, \\ v(x)=0 & \text{if } |x|=r_0, \\ v(x)\to 0 & \text{if } |x|\to\infty, \end{cases} \tag{P_E}$$

where $\Omega_E:=\{x\in R^N:|x|>r_0,N>2\},\Delta v=div(\nabla v)$ is the Laplacian and $K:[r_0,\infty)\to(0,\infty)$ is a continuous function such that $\lim\limits_{s\to\infty}K(s)=0$. The transformation $r=|x|,t=(\frac{r}{r_0})^{2-N}$ and $v(r)=u(t)$ reduces (P_E) to (P) (see [5] for details) where $h(t)=\dfrac{r_0^2}{(N-2)^2}t^{-\frac{2(N-1)}{N-2}}K(r_0t^{\frac{1}{2-N}})$. Note that when $K(r)\le\frac{1}{r^{N+\rho}}$ for $r\gg 1$ with $0<\rho<N-2,h(t)$ is singular at 0 and there exists $C_1>0$ such that $h(t)\le\frac{C_1}{t^{\alpha_1}},t\approx 0$ with $\alpha_1=\frac{(N-2)-\rho}{N-2}$.

The main purpose of this paper is to establish the uniqueness of the positive solution for the problem (P) when $\lambda\gg 1$. In particular, we assume

(H) There exists $\sigma>0$ such that $\frac{f(s)}{s^{\beta}}$ is decreasing for $s>\sigma$.

We establish:

Theorem 1.1. *Assume (H) and $\alpha_i+\beta<1,i=1,2$. Then there exists $\lambda^*>0$ such that* (P) *has a unique positive solution for all $\lambda>\lambda^*$.*

See [1] where such a uniqueness result was recently discussed for the nonsingular eigenvalue problem

$$\begin{cases} -\Delta u(x)=\lambda\dfrac{f(u(x))}{u(x)^{\beta}} & \text{in } \Omega, \\ u(x)=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$ with smooth boundary. Our result here extends this result to the exterior domain problem (P_E). Also, in the case when $\beta = 0$, uniqueness results for the large value of a parameter have been established by many authors (see [2],[3] and [6]). A simple example of f satisfies all our hypothses is $f(s) = e^{\frac{\gamma s}{\gamma + s}}, \gamma > 0$ which arises in the theory of combustion.

We will establish some preliminaries in Section 2 and prove Theorem 1.1 in Section 3.

2 Preliminaries

Lemma 2.1. *If u_λ is a positive solution of* (P)*, then $u_\lambda \geq \delta_\lambda e$ in Ω, where $\delta_\lambda^{1+\beta} = \lambda \frac{A}{\|e\|_\infty^\beta}$ and e is the solution of $-e''(t) = h(t); (0,1)$ and $e(0) = 0 = e(1)$.*

Proof. Let u_λ be a positive solution of (P) for arbitrary fixed $\lambda > 0$. Assuming that $\Omega_\lambda = \{t \in (0,1) : u_\lambda(t) < \delta_\lambda e(t)\} \neq \emptyset$, we have

$$\begin{aligned}
-(u_\lambda(t) - \delta_\lambda e(t))'' &= \lambda h(t) \frac{f(u_\lambda(t))}{u_\lambda(t)^\beta} - \delta_\lambda h(t) \\
&> \lambda h(t) \frac{A}{\delta_\lambda^\beta e(x)^\beta} - \delta_\lambda h(t) \\
&\geq \lambda h(t) \frac{A}{\delta_\lambda^\beta \|e\|_\infty^\beta} - \delta_\lambda h(t) \\
&= 0
\end{aligned}$$

in Ω_λ and $u_\lambda - \delta_\lambda e = 0$ on $\partial\Omega_\lambda$, which contradicts the maximum principle. Hence, $\Omega_\lambda = \emptyset$, which proves Lemma 2.1.

3 Proof of Theorem 1.1

Let u_λ and v_λ be positive solutions of (P). Let $g(u) = \frac{f(u)}{u^\beta}$. Then we have

$$\int_0^1 -(v_\lambda - u_\lambda)''(v_\lambda - u_\lambda)dt = \lambda \int_0^1 h(t)(g(v_\lambda) - g(u_\lambda))(v_\lambda - u_\lambda)dt. \tag{3.1}$$

Next, since $u_\lambda, v_\lambda \in C^1[0,1]$ (see [4]), integrating by parts and using the boundary condition, we get

$$\int_0^1 -(v_\lambda - u_\lambda)''(v_\lambda - u_\lambda)dt = \int_0^1 |(v_\lambda - u_\lambda)'|^2 \, dt.$$

Further, by the fundamental theorem of calculus, we have

$$\begin{aligned}
&\lambda \int_0^1 h(t)(g(v_\lambda) - g(u_\lambda))(v_\lambda - u_\lambda)dt \\
&= \lambda \int_0^1 h(t) \left(\int_0^1 g'(u_\lambda + s(v_\lambda - u_\lambda))(v_\lambda - u_\lambda)ds \right) (v_\lambda - u_\lambda)dt \\
&= \lambda \int_0^1 h(t)\zeta(t)(v_\lambda - u_\lambda)^2 dt
\end{aligned}$$

where $\zeta(t) = \int_0^1 g'(u_\lambda + s(v_\lambda - u_\lambda))ds$. Hence from (3.1) we obtain

$$\int_0^1 |(v_\lambda - u_\lambda)'|^2 \, dt = \lambda \int_0^1 h(t)\zeta(t)(v_\lambda - u_\lambda)^2 dt. \tag{3.2}$$

Let $\Omega = [0,1]$ and $l > 0$ be such that $e(t) \geq l d(t, \partial\Omega)$ for all $t \in [0,1]$, where $d(t, \partial\Omega) = \min\{t, 1-t\}$. Now for $\lambda \gg 1$ with $\delta_\lambda > \frac{\sigma}{l}$, if $t \in [\frac{\sigma}{\delta_\lambda l}, 1 - \frac{\sigma}{\delta_\lambda l}] =: \Omega_2$, then by Lemma 2.1 $u_\lambda(t) \geq \delta_\lambda e(t) \geq \delta_\lambda l d(t, \partial\Omega) \geq \sigma$. Let $\Omega_1 := [0,1] - \Omega_2 = (0, a_\lambda) \cup (1 - a_\lambda, 1)$ where $a_\lambda = \frac{\sigma}{\delta_\lambda l} =: C\lambda^{-\frac{1}{1+\beta}}$ and $C = \frac{\sigma}{l}(\frac{\|e\|_\infty^\beta}{A})^{\frac{1}{1+\beta}}$. Now we rewrite (3.2) as:

$$\begin{aligned}
\int_0^1 |(v_\lambda - u_\lambda)'|^2 dt = \; & \lambda \int_{\Omega_1} h(t)\zeta(t)(v_\lambda - u_\lambda)^2 dt \\
+ \; & \lambda \int_{\Omega_2} h(t)\zeta(t)(v_\lambda - u_\lambda)^2 dt.
\end{aligned} \tag{3.3}$$

Since $u_\lambda + s(v_\lambda - u_\lambda) = (1-s)u_\lambda + s v_\lambda \geq (1-s)\delta_\lambda e + s\delta_\lambda e = \delta_\lambda e \geq \sigma$ for $s > 0$ in Ω_2, using (H), we know that $g'(u_\lambda + s(v_\lambda - u_\lambda)) < 0$ which gives $\zeta(t) = \int_0^1 g'(u_\lambda(t) + s(v_\lambda(t) - u_\lambda(t)))ds < 0$ in Ω_2. Since $\beta \in (0,1)$ and $f(0) > 0$, $\lim_{u \to 0^+} g'(u) = -\infty$. Also by (H), $g'(u) \leq 0$ for $u > \sigma$. Hence g' is bounded above; let then $M > 0$ be such that $g'(u) \leq M$ for all $u > 0$. Hence we obtain

$$\begin{aligned}
\int_0^1 |(v_\lambda - u_\lambda)'|^2 dt \;\; &\leq \;\; \lambda \int_{\Omega_1} h(t)\zeta(t)(v_\lambda - u_\lambda)^2 dt \\
&\leq \;\; \lambda M \int_{\Omega_1} h(t)(v_\lambda - u_\lambda)^2 dt.
\end{aligned} \tag{3.4}$$

Now for $t \in (0, a_\lambda)$ we have

$$\begin{aligned}
|v_\lambda(t) - u_\lambda(t)| \;\; &= \;\; \left| \int_0^t (v'_\lambda(s) - u'_\lambda(s))ds \right| \\
&\leq \;\; \left(\int_0^{a_\lambda} (v'_\lambda(s) - u'_\lambda(s))^2 ds \right)^{\frac{1}{2}} \left(\int_0^{a_\lambda} 1 \, ds \right)^{\frac{1}{2}} \\
&= \;\; C^{\frac{1}{2}} \lambda^{-\frac{1}{2(1+\beta)}} \left(\int_0^{a_\lambda} (v'_\lambda(s) - u'_\lambda(s))^2 ds \right)^{\frac{1}{2}} \\
&\leq \;\; C^{\frac{1}{2}} \lambda^{-\frac{1}{2(1+\beta)}} \left(\int_0^1 (v'_\lambda(s) - u'_\lambda(s))^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, if $t \in (1-a_\lambda, 1)$ then we obtain

$$\begin{aligned} |v_\lambda(t) - u_\lambda(t)| &\leq C^{\frac{1}{2}} \lambda^{-\frac{1}{2(1+\beta)}} \left(\int_{1-a_\lambda}^{1} (v'_\lambda(s) - u'_\lambda(s))^2 ds \right)^{\frac{1}{2}} \\ &\leq C^{\frac{1}{2}} \lambda^{-\frac{1}{2(1+\beta)}} \left(\int_{0}^{1} (v'_\lambda(s) - u'_\lambda(s))^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, it follows that

$$\int_{\Omega_1} h(t)(v_\lambda - u_\lambda)^2 dt \leq$$
$$C\lambda^{-\frac{1}{1+\beta}} \left(\int_0^1 [(v_\lambda - u_\lambda)']^2 ds \right) \left(\int_0^{a_\lambda} h(t)\, dt + \int_{1-a_\lambda}^{1} h(t)\, dt \right). \quad (3.5)$$

If $\int_0^1 [(v_\lambda - u_\lambda)']^2 ds \neq 0$, then from (3.4) and (3.5) we get

$$\begin{aligned} 1 &\leq \lambda M C \lambda^{-\frac{1}{1+\beta}} \left(\int_0^{a_\lambda} h(t)\, dt + \int_{1-a_\lambda}^{1} h(t)\, dt \right) \\ &= \tilde{C} \lambda^{\frac{\beta}{1+\beta} - \frac{1-\alpha_1}{1+\beta}} + \bar{C} \lambda^{\frac{\beta}{1+\beta} - \frac{1-\alpha_2}{1+\beta}}, \end{aligned}$$

which is a contradiction for $\lambda \gg 1$ since $\alpha_i + \beta < 1, i = 1, 2$. Hence $\int_0^1 [(v_\lambda - u_\lambda)']^2 ds = 0$. It follows that $v_\lambda - u_\lambda =$ constant. But $v_\lambda - u_\lambda = 0$ on $\partial\Omega$. Hence $v_\lambda = u_\lambda$ on $\bar{\Omega}$.

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