Coefficient bounds for bi-starlike analytic functions

A. K. Mishra M. M. Soren*

Abstract

In the present paper, we find new bounds on the modulii of the third and fourth Taylor-Maclaurin's coefficients of *bi-starlike functions of order* ρ and *strongly bi-starlike functions of order* β. Our estimates on the third coefficient improve upon earlier estimates found in [D.A. Brannan, T.S. Taha, On some classes of bi-univalent functions, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), Mathematical Analysis and its Applications, Kuwait; February 18-21, 1985, in: KFAS Proceedings Series, vol. 3, Pergamon Press, Elsevier Science Limited, Oxford, 1988, pp. 53-60].

1 Introduction and definitions

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and represented by the *normalized* series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

We denote by $\mathcal{S}$ the family of univalent functions in $\mathcal{A}$. (see, for details,[5, 15]).

*Corresponding author.

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For $f \in \mathcal{S}$ the inverse function f^{-1} is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}) \quad [5].$$

Further more,

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \quad (|w| < r_0(f)). \tag{1.2}$$

The function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if $(i) f \in \mathcal{S}$ and $(ii) f^{-1}(w)$ has an univalent *analytic continuation* to $|w| < 1$. Let σ denote the class of bi-univalent analytic functions in $\mathbb{U}$. Initial pioneering work on the class σ were done in [3, 9, 11]. Recently, Srivastava et al.[14] exhibited some interesting examples of functions in the class σ. We add that the family of functions defined by

$$\overline{\lambda}(e^{\lambda z} - 1) \quad (\lambda \in \mathbb{C}, |\lambda| = 1;\ z \in \mathbb{U})$$

are univalent in the larger disc $|z| < \pi$ and their inverse functions are univalent in $\mathbb{U}$. Therefore, these functions are also bi-univalent. For a brief history on the developments regarding the class σ see [7].

Earlier Brannan and Taha (cf [4], also see [16]) introduced two interesting subclasses of the function class σ, in analogy to the subclasses of *strongly starlike functions of order* β and *starlike functions of order* ρ of the class $\mathcal{S}$. We thus have the following definitions.

Definition 1.1. [4] The function $f(z)$, given by (1.1), is said to be in the class $\mathcal{S}_\sigma^{\star\beta}$ $(0 < \beta \leq 1)$, the class of *strongly bi-starlike functions of order* β, if each of the following conditions are satisfied:

$$f \in \sigma, \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2} \qquad (z \in \mathbb{U}) \tag{1.3}$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\beta\pi}{2} \qquad (w \in \mathbb{U}), \tag{1.4}$$

where the function g is the analytic continuation of $f^{-1}(w)$ to $\mathbb{U}$.

Definition 1.2. [4] The function $f(z)$, given by (1.1), is said to be in the class $\mathcal{S}_\sigma^\star(\rho)$, the class of *bi-starlike functions of order* ρ $(0 \leq \rho < 1)$ if each of the following conditions are satisfied:

$$f \in \sigma, \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \rho \qquad (z \in \mathbb{U}) \tag{1.5}$$

and

$$\Re\left(\frac{wg'(w)}{g(w)}\right) > \rho \qquad (w \in \mathbb{U}). \tag{1.6}$$

Example 1.3. The following considerations show that the family of functions defined by $f(z) = z + a_2 z^2 \quad (z \in \mathbb{U})$, are members of the class $\mathcal{S}_\sigma^\star(\rho)$ if $|a_2| \leq \frac{1-\rho}{4(2-\rho)}$. Direct verification shows that f is a univalent starlike function of order ρ. Moreover, we have

$$g^{-1}(w) = \frac{-1+\sqrt{1+4a_2 w}}{2a_2} = w + \sum_{n=2}^{\infty} A_n w^n \qquad (w \in \mathbb{U}), \tag{1.7}$$

where

$$A_n = \frac{1}{2}\binom{\frac{1}{2}}{n} 4^n a_2^{n-1} \qquad (n = 2, 3, \ldots).$$

Therefore,

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)|A_n| \\
&\leq \sum_{n=2}^{\infty}\frac{4^{n-1}}{1-\rho}\left(\frac{n-\rho}{n}\right)\left\{\frac{(n-1)-\frac{1}{2}}{n-1}\right\}\left\{\frac{(n-2)-\frac{1}{2}}{n-2}\right\}\cdots\left\{\frac{1-\frac{1}{2}}{1}\right\}|a_2|^{n-1} \\
&\leq \frac{1}{1-\rho}\sum_{n=2}^{\infty}4^{n-1}|a_2|^{n-1} \\
&\leq \frac{1}{1-\rho}\sum_{n=2}^{\infty}4^{n-1}\frac{(1-\rho)^{n-1}}{4^{n-1}(2-\rho)^{n-1}} \\
&\leq \frac{1}{2-\rho}\left(1+\sum_{n=1}^{\infty}\left(\frac{1-\rho}{2-\rho}\right)^n\right) = 1.
\end{aligned}$$

This shows that $g^{-1}$is a univalent starlike function of order ρ. Therefore, $f \in \mathcal{S}_\sigma^\star(\rho)$.

We shall also need the class $\mathcal{P}$ of analytic functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \qquad (z \in \mathbb{U})$$

and satisfying $\Re(p(z)) > 0 \quad (z \in \mathbb{U})$. The class $\mathcal{P}$ is popularly named after Carathéodory.

Brannan and Taha [4] found estimates for the second and third Taylor-Maclaurin's coefficients of the functions f in the classes $\mathcal{S}_\sigma^{\star\beta}$ and $\mathcal{S}_\sigma^\star(\rho)$. That is:

$$|a_2| \leq \frac{2\beta}{\sqrt{1+\beta}} \quad (f \in \mathcal{S}_\sigma^{\star\beta}) \text{ and } |a_2| \leq \sqrt{2(1-\rho)} \quad (f \in \mathcal{S}_\sigma^\star(\rho)). \tag{1.8}$$

Similarly,

$$|a_3| \leq 2\beta \quad (f \in \mathcal{S}_\sigma^{\star\beta}) \text{ and } |a_3| \leq 2(1-\rho) \quad (f \in \mathcal{S}_\sigma^\star(\rho)). \tag{1.9}$$

Srivastava et al. [14] introduced and investigated two novel subclasses of σ and found *non-sharp* bounds for functions in these classes. As a follow up of the work in [14] , at present there is renewed interest in the study of the class σ and its many new subclasses. For example see [1, 2, 6, 7, 8, 10, 12, 13, 17, 18].
In this note we improve upon the bound on $|a_3|$, $(f \in \mathcal{S}_\sigma^{\star\beta})$ of Brannan and Taha [4] given at (1.9). We also find estimates for $|a_4|$ when $f \in \mathcal{S}_\sigma^{\star\beta}$ and $\mathcal{S}_\sigma^\star(\rho)$.

2 Coefficient bounds for the function class $\mathcal{S}_\sigma^{\star\beta}$

We state and prove the following:

Theorem 2.1. *If the function $f(z)$ in $\mathcal{S}_\sigma^{\star\beta}$ is given by (1.1), then*

$$|a_3| \leq \begin{cases} \beta & (0 < \beta \leq \frac{1}{3}), \\ \frac{4\beta^2}{1+\beta} & (\frac{1}{3} \leq \beta \leq 1) \end{cases} \tag{2.1}$$

and

$$|a_4| \leq \begin{cases} \frac{2\beta}{3}\left(1 - \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\right) & (0 < \beta < \frac{3+\sqrt{73}}{32}), \\ \frac{2\beta}{3}\left(1 + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\right) & (\frac{3+\sqrt{73}}{32} \leq \beta < \frac{2}{5}), \\ \frac{2\beta}{3}\left(\frac{15\beta}{5\beta+4} + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\right) & (\frac{2}{5} \leq \beta \leq 1). \end{cases} \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{S}_\sigma^{\star\beta}$ $(0 < \beta \leq 1)$. Then by Definition 1.1, we have

$$\frac{zf'(z)}{f(z)} = [Q(z)]^\beta \tag{2.3}$$

and

$$\frac{wg'(w)}{g(w)} = [P(w)]^\beta, \tag{2.4}$$

respectively, where $Q(z)$ and $P(w)$ belong to the class $\mathcal{P}$ and have the forms:

$$Q(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \mathbb{U})$$

and

$$P(w) = 1 + l_1 w + l_2 w^2 + l_3 w^3 + \cdots \quad (w \in \mathbb{U}).$$

By equating the coefficients of $\frac{zf'(z)}{f(z)}$ with the coefficients of $[Q(z)]^\beta$, we get

$$a_2 = \beta c_1, \tag{2.5}$$

$$2a_3 - a_2^2 = \beta c_2 + \frac{\beta(\beta-1)}{2}c_1^2 \tag{2.6}$$

and

$$3a_4 - 3a_2a_3 + a_2^3 = \beta c_3 + \beta(\beta-1)c_1c_2 + \frac{\beta(\beta-1)(\beta-2)}{6}c_1^3. \tag{2.7}$$

Similarly, by equating the coefficients of $\frac{wg'(w)}{g(w)}$ and $[P(w)]^\beta$, we have

$$a_2 = -\beta l_1, \tag{2.8}$$

$$3a_2^2 - 2a_3 = \beta l_2 + \frac{\beta(\beta-1)}{2} l_1^2 \tag{2.9}$$

and

$$-(10a_2^3 - 12a_2a_3 + 3a_4) = \beta l_3 + \beta(\beta-1)l_1l_2 + \frac{\beta(\beta-1)(\beta-2)}{6} l_1^3. \tag{2.10}$$

The relations (2.5) and (2.8), together give

$$l_1 = -c_1. \tag{2.11}$$

We shall obtain a refined estimate on $|c_1|$ for use in the estimates of $|a_3|$ and $|a_4|$. For this purpose we first add (2.6) with (2.9); then use the relations (2.11) and get the following:

$$2a_2^2 = \beta(c_2 + l_2) + \beta(\beta-1)c_1^2.$$

Putting $a_2 = \beta c_1$ from (2.5), we have after simplification:

$$c_1^2 = \frac{c_2 + l_2}{1+\beta}. \tag{2.12}$$

By applying the familiar inequalities $|c_2| \leq 2$ and $|l_2| \leq 2$ we get:

$$|c_1| \leq \sqrt{\frac{4}{1+\beta}} = \frac{2}{\sqrt{1+\beta}}. \tag{2.13}$$

To find a bound on $|a_3|$ we wish express a_3 in terms of the coefficients of the functions $P(w)$ and $Q(z)$. For this we substract (2.9) from (2.6) and get

$$4a_3 = 4a_2^2 + \beta(c_2 - l_2) + \frac{\beta(\beta-1)}{2}(c_1^2 - l_1^2).$$

The relation $c_1^2 = l_1^2$ from (2.11), reduces the above expression to

$$4a_3 = 4a_2^2 + \beta(c_2 - l_2). \tag{2.14}$$

Next putting that $a_2 = \beta c_1$ and using (2.12), we obtain

$$\begin{aligned} 4a_3 &= 4\beta^2 c_1^2 + \beta(c_2 - l_2) \\ &= 4\beta^2 \left(\frac{c_2 + l_2}{1+\beta}\right) + \beta(c_2 - l_2) \\ &= \frac{\beta}{1+\beta}\left[(5\beta+1)c_2 + (3\beta-1)l_2\right]. \end{aligned}$$

Therefore, the inequalities $|c_2| \leq 2$ and $|l_2| \leq 2$ give the following:

$$4|a_3| \leq \begin{cases} \frac{2\beta}{1+\beta}(5\beta+1+1-3\beta) = 4\beta & (0 < \beta \leq \frac{1}{3}), \\ \frac{2\beta}{1+\beta}(5\beta+1+3\beta-1) = \frac{16\beta^2}{1+\beta} & (\frac{1}{3} \leq \beta \leq 1) \end{cases}$$

which simplifies to:

$$|a_3| \le \begin{cases} \beta & (0 < \beta \le \frac{1}{3}), \\ \frac{4\beta^2}{1+\beta} & (\frac{1}{3} \le \beta \le 1). \end{cases}$$

This is precisely the assertion of (2.1).

We shall next find an estimate on $|a_4|$. At first we shall derive a relation connecting c_1, c_2, c_3, l_2 and l_3. To this end, we first add the equations (2.7) and (2.10) and get

$$-9a_2^3 + 9a_2a_3 = \beta(c_3 + l_3) + \beta(\beta-1)(c_1c_2 + l_1l_2) + \frac{\beta(\beta-1)(\beta-2)}{6}(c_1^3 + l_1^3).$$

By putting $l_1 = -c_1$ the above expression reduces to the following:

$$-9a_2^3 + 9a_2a_3 = \beta(c_3 + l_3) + \beta(\beta-1)c_1(c_2 - l_2). \tag{2.15}$$

Substituting $a_3 = a_2^2 + \frac{\beta}{4}(c_2 - l_2)$ from (2.14) into (2.15) we get after simplification:

$$\frac{9\beta a_2}{4}(c_2 - l_2) = \beta(c_3 + l_3) + \beta(\beta-1)c_1(c_2 - l_2).$$

Since $a_2 = \beta c_1$, (see 2.5) we have

$$\frac{9\beta^2}{4}c_1(c_2 - l_2) = \beta(c_3 + l_3) + \beta(\beta-1)c_1(c_2 - l_2).$$

Or equivalently:

$$c_1(c_2 - l_2) = \frac{4(c_3 + l_3)}{5\beta + 4}. \tag{2.16}$$

We wish to express a_4 in terms of the first three coefficients of $P(w)$ and $Q(z)$. Now substracting (2.15) from (2.12), we get

$$6a_4 = -11a_2^3 + 15a_2a_3 + \beta(c_3 - l_3) + \beta(\beta-1)(c_1c_2 - l_1l_2) + \frac{\beta(\beta-1)(\beta-2)}{6}(c_1^3 - l_1^3).$$

Observing that $l_1 = -c_1$ we have $c_1^3 - l_1^3 = 2c_1^3$ and therefore

$$6a_4 = -9a_2^3 + 9a_2a_3 - 2a_2^3 + 6a_2a_3 + \beta(c_3 - l_3) + \beta(\beta-1)c_1(c_2 + l_2) + \frac{\beta(\beta-1)(\beta-2)}{3}c_1^3.$$

We replace $-9a_2^3 + 9a_2a_3$ by the right hand side of (2.15), put $a_3 = \beta^2c_1^2 + \frac{\beta}{4}(c_2 - l_2)$ (see (2.14)) and $a_2 = \beta c_1$. This gives

$$\begin{aligned} 6a_4 &= \beta(c_3 + l_3) + \beta(\beta-1)c_1(c_2 - l_2) - 2\beta^3c_1^3 + 6\beta c_1\left(\beta^2c_1^2 + \frac{\beta}{4}(c_2 - l_2)\right) \\ &\quad + \beta(c_3 - l_3) + \beta(\beta-1)c_1(c_2 + l_2) + \frac{\beta(\beta-1)(\beta-2)}{3}c_1^3 \\ &= 2\beta c_3 + \frac{\beta(5\beta-2)}{2}c_1(c_2 - l_2) + \beta(\beta-1)c_1(c_2 + l_2) + \frac{13\beta^3 - 3\beta^2 + 2\beta}{3}c_1^3. \end{aligned}$$

Next, replacing $c_1(c_2 - l_2)$ by the expression in the right hand side of (2.16) and c_1^2 by (2.12) we finally get

$$\begin{aligned}
6a_4 &= 2\beta c_3 + \frac{\beta(5\beta-2)}{2}\frac{4(c_3+l_3)}{5\beta+4} + \beta(\beta-1)c_1(c_2+l_2) + \\
&\qquad\qquad \frac{13\beta^3 - 3\beta^2 + 2\beta}{3} c_1 \frac{(c_2+l_2)}{1+\beta} \\
&= 2\beta c_3 + \frac{2\beta(5\beta-2)}{5\beta+4}(c_3+l_3) + \frac{16\beta^3 - 3\beta^2 - \beta}{3(1+\beta)} c_1(c_2+l_2) \\
&= \beta\left[\frac{4(5\beta+1)}{5\beta+4}c_3 + \frac{2(5\beta-2)}{5\beta+4}l_3 + \frac{16\beta^2-3\beta-1}{3(1+\beta)}c_1(c_2+l_2)\right].
\end{aligned}$$

This gives

$$|a_4| \le \frac{\beta}{6}\left\{\left|\frac{4(5\beta+1)}{5\beta+4}\right||c_3| + \left|\frac{2(5\beta-2)}{5\beta+4}\right||l_3| + \left|\frac{16\beta^2-3\beta-1}{3(1+\beta)}\right||c_1||(c_2+l_2)|\right\}.$$

We observe that $\beta_0 = \frac{3+\sqrt{73}}{32}$ and $\beta_1 = \frac{3-\sqrt{73}}{32}$ are the roots of the quadratic polynomial $16\beta^2 - 3\beta - 1$, out of which $\beta_1 < 0$. Therefore,

$$|a_4| \le \begin{cases}
\frac{\beta}{6}\left[\frac{4(5\beta+1)}{5\beta+4}|c_3| + \frac{2(2-5\beta)}{5\beta+4}|l_3| - \frac{16\beta^2-3\beta-1}{3(1+\beta)}|c_1||(c_2+l_2)|\right] & (0 < \beta < \frac{3+\sqrt{73}}{32}), \\
\frac{\beta}{6}\left[\frac{4(5\beta+1)}{5\beta+4}|c_3| + \frac{2(2-5\beta)}{5\beta+4}|l_3| + \frac{16\beta^2-3\beta-1}{3(1+\beta)}|c_1||(c_2+l_2)|\right] & (\frac{3+\sqrt{73}}{32} \le \beta < \frac{2}{5}), \\
\frac{\beta}{6}\left[\frac{4(5\beta+1)}{5\beta+4}|c_3| + \frac{2(5\beta-2)}{5\beta+4}|l_3| + \frac{16\beta^2-3\beta-1}{3(1+\beta)}|c_1||(c_2+l_2)|\right] & (\frac{2}{5} \le \beta \le 1).
\end{cases}$$

By applying the inequalities $|c_n| \le 2$, $|l_n| \le 2$ $(n = 2, 3)$ and the estimate (2.13) for $|c_1|$ we have:

$$|a_4| \le \begin{cases}
\frac{2\beta}{3}\left[1 - \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\right] & (0 < \beta < \frac{3+\sqrt{73}}{32}), \\
\frac{2\beta}{3}\left[1 + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\right] & (\frac{3+\sqrt{73}}{32} \le \beta < \frac{2}{5}), \\
\frac{2\beta}{3}\left[\frac{15\beta}{5\beta+4} + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\right] & (\frac{2}{5} \le \beta \le 1).
\end{cases}$$

We get the assertion (2.2). The proof of Theorem 2.1 is, thus, completed. ∎

We next find an estimate for $|a_4|$ for the function class $\mathcal{S}^\star_\sigma(\rho)$.

Theorem 2.2. *Let $f(z)$, given by (1.1), be in the class $\mathcal{S}^\star_\sigma(\rho)$. Then*

$$|a_4| \le \begin{cases}
\frac{2(1-\rho)}{3}\left[1 + 2\sqrt{2(1-\rho)}\right] & (0 \le \rho \le \frac{1}{2}) \\
\frac{2(1-\rho)}{3}\left[1 + 4(1-\rho)\right] & (\frac{1}{2} \le \rho < 1).
\end{cases} \tag{2.17}$$

Proof. Let $f(z) \in \mathcal{S}^{\star}{}_{\sigma}(\rho)$ $(0 \leq \rho < 1)$. Then by Definition 1.2, we get that

$$\frac{zf'(z)}{f(z)} = \rho + (1-\rho)Q_1(z) \tag{2.18}$$

and

$$\frac{wg'(w)}{g(w)} = \rho + (1-\rho)P_1(w) \tag{2.19}$$

respectively, where $\Re(Q_1(z)) > 0$,

$$Q_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U})$$

and $\Re(P_1(w)) > 0$,

$$P_1(w) = 1 + l_1 w + l_2 w^2 + \cdots \quad (w \in \mathbb{U}).$$

As in the proof of Theorem 2.1, by suitably comparing coefficients in (2.18) and (2.19) we get

$$a_2 = (1-\rho)c_1, \tag{2.20}$$

$$2a_3 - a_2^2 = (1-\rho)c_2, \tag{2.21}$$

$$3a_4 - 3a_2a_3 + a_2^3 = (1-\rho)c_3 \tag{2.22}$$

and

$$-a_2 = (1-\rho)l_1, \tag{2.23}$$

$$3a_2^2 - 2a_3 = (1-\rho)l_2, \tag{2.24}$$

$$-(10a_2^3 - 12a_2a_3 + 3a_4) = (1-\rho)l_3. \tag{2.25}$$

Addition of (2.21) with (2.24) yields:

$$2a_2^2 = (1-\rho)(c_2 + l_2). \tag{2.26}$$

Putting $a_2 = (1-\rho)c_1$ from (2.20) we have after simplification:

$$c_1^2 = \frac{c_2 + l_2}{2(1-\rho)}. \tag{2.27}$$

By applying the familiar inequalities $|c_2| \leq 2$ and $|l_2| \leq 2$ we get the first bound in the following and the second estimate is well known:

$$|c_1| \leq \begin{cases} \sqrt{\frac{2}{(1-\rho)}} & (0 \leq \rho \leq \frac{1}{2}) \\ 2 & (\frac{1}{2} \leq \rho < 1). \end{cases} \tag{2.28}$$

Next, we substract (2.24) from (2.21), add the equations (2.22) and (2.25) and get respectively:

$$4a_3 = 4a_2^2 + (1-\rho)(c_2 - l_2) \tag{2.29}$$

and

$$-9a_2^3 + 9a_2a_3 = (1-\rho)(c_3 + l_3). \tag{2.30}$$

We shall now find an estimate on $|a_4|$. We wish to express a_4 in terms of the first three coefficients of $P(w)$ and $Q(z)$. For this we substract (2.25) from (2.22), and get

$$\begin{aligned} 6a_4 &= -11a_2^3 + 15a_2a_3 + (1-\rho)(c_3 - l_3) \\ &= -9a_2^3 + 9a_2a_3 - 2a_2^3 + 6a_2a_3 + (1-\rho)(c_3 - l_3). \end{aligned}$$

We replace $-9a_2^3 + 9a_2a_3$ by the right hand side of (2.30), put $a_3 = (1-\rho)^2c_1^2 + \frac{(1-\rho)}{4}(c_2 - l_2)$ (see (2.29)) and $a_2 = (1-\rho)c_1$. Thus, we have:

$$\begin{aligned} 6a_4 = (1-\rho)(c_3 + l_3) - 2(1-\rho)^3c_1^3 + 6(1-\rho)c_1\left((1-\rho)^2c_1^2 + \frac{(1-\rho)}{4}(c_2 - l_2)\right) \\ + (1-\rho)(c_3 - l_3) \\ = 2(1-\rho)c_3 + 4(1-\rho)^3c_1^3 + \frac{6(1-\rho)^2}{4}c_1(c_2 - l_2). \end{aligned}$$

Next replacing c_1^2 by (2.27) we finally get

$$\begin{aligned} 6a_4 &= 2(1-\rho)c_3 + 4(1-\rho)^3c_1\frac{c_2 + l_2}{2(1-\rho)} + \frac{6(1-\rho)^2}{4}c_1(c_2 - l_2) \\ &= 2(1-\rho)c_3 + 2(1-\rho)^2c_1(c_2 + l_2) + \frac{3(1-\rho)^2}{2}c_1(c_2 - l_2) \\ &= 2(1-\rho)c_3 + \frac{7(1-\rho)^2}{2}c_1c_2 + \frac{(1-\rho)^2}{2}c_1l_2. \end{aligned}$$

By applying the inequalities $|c_3| \le 2$, $|c_2| \le 2$ and $|l_2| \le 2$, the estimate for $|c_1|$ from (2.28) we have

$$\begin{aligned} 6|a_4| &\le 2(1-\rho)|c_3| + \frac{7(1-\rho)^2}{2}|c_1||c_2| + \frac{(1-\rho)^2}{2}|c_1||l_2| \\ &\le \begin{cases} 4(1-\rho) + 8\sqrt{2(1-\rho)} & (0 \le \rho \le \frac{1}{2}) \\ 4(1-\rho) + 16(1-\rho)^2 & (\frac{1}{2} \le \rho < 1). \end{cases} \end{aligned}$$

Or equivalently:

$$|a_4| \le \begin{cases} \frac{2(1-\rho)}{3}[1 + 2\sqrt{2(1-\rho)}] & (0 \le \rho \le \frac{1}{2}) \\ \frac{2(1-\rho)}{3}[1 + 4(1-\rho)] & (\frac{1}{2} \le \rho < 1). \end{cases}$$

We get the assertion (2.17). This completes the proof of the Theorem 2.2. ∎

3 Concluding Remarks

By definition every bi-starlike analytic function $f(z)$ in $\mathbb{U}$ is associated with a function $Q(z)$ in the Carathéodory class $\mathcal{P}$ and its inverse function $g(w)$ is associated with another function $P(w) \in \mathcal{P}$. In this paper suitable relationships between the first and second coefficients of the two functions $P(w)$ and $Q(z)$ are

obtained. Using these relationships, the third Taylor-Maclaurin's coefficient of a bi-starlike function $f(z)$ is expressed in terms of the first and second coefficients of $P(w)$ and $Q(z)$. Similarly the fourth coefficient of $f(z)$ is expressed in terms of the first three coefficients of $P(w)$ and $Q(z)$. A refined estimate for the first coefficient of the function $Q(z)$ is also derived. These relationships and the refined estimate yield coefficient bounds for the third and fourth coefficients of the functions in the classes $\mathcal{S}_{\sigma}^{\star\beta}$ and $\mathcal{S}_{\sigma}^{\star}(\rho)$.
By comparing our result (2.1) with (1.9) we observe that our estimate on $|a_3|$ improves upon the earlier bound of Brannan and Taha [4] for the class $\mathcal{S}_{\sigma}^{\star\beta}$.

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Department of Mathematics
Berhampur University
Bhanja Bihar 760007
Ganjam, Odisha
India
email:akshayam2001@yahoo.co.in

Department of Mathematics
Berhampur University
Bhanja Bihar 760007
Ganjam, Odisha
India
email:soren85@rediffmail.com