Starlikeness and convexity of polyharmonic mappings*

J. Chen A. Rasila† X. Wang

Abstract

In this paper, we first find an estimate for the range of polyharmonic mappings in the class HC_p^0. Then, we obtain two characterizations in terms of the convolution for polyharmonic mappings to be starlike of order α, and convex of order β, respectively. Finally, we study the radii of starlikeness and convexity for polyharmonic mappings, under certain coefficient conditions.

1 Introduction

Let $F = u + iv$ be a $2p$ times continuously differentiable complex-valued mapping, where $p \geq 1$, defined in a domain $D \subset \mathbb{C}$. The mapping F is called *polyharmonic* (or *p-harmonic*), if it satisfies the polyharmonic equation $\Delta^p F = \Delta(\Delta^{p-1}F) = 0$, where $\Delta^1 := \Delta$ is the usual complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For a simply connected domain D, it is well known (see [8, 22]) that a mapping F is polyharmonic if and only if F has the representation

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_k(z),$$

*The research was partly supported by NSF of China and Hunan Provincial Innovation Foundation for Postgraduates (No. 125000-4242).

†Corresponding author

Received by the editors in February 2013.
Communicated by H. De Schepper.
2010 *Mathematics Subject Classification* : Primary 30C45; Secondary 30C20, 30C65.
Key words and phrases : polyharmonic mapping, starlike, convex.

where G_k are complex-valued harmonic mappings in D for all $k \in \{1, \cdots, p\}$. The mappings G_k can be presentented (see [12, 13]) in the form

$$G_k = h_k + \overline{g_k},$$

where all h_k and g_k are analytic in D for all $k \in \{1, \cdots, p\}$. Clearly, if $p = 1$, then we have the usual class of harmonic mappings and, for $p = 2$, we obtain the class of biharmonic mappings, as special cases.

The biharmonic equation is related to numerous modeling problems in science and engineering. For example, it arises from certain problems in solid mechanics, and also from the theory of steady Stokes flow (i.e., speed ≈ 0) of viscous fluids, where it is the equation satisfed by the stream function (see e.g. [16, 18, 19]). In the geometric function theory, the class biharmonic mappings can be understood as a natural generalization of the harmonic mappings, but it has only recently been studied from this point of view (see [1, 2, 3, 6, 7, 11]). The reader is referred to [8, 9, 10, 22] for the properties of polyharmonic mappings, and [12, 13] for basic results on harmonic mappings.

For $r > 0$, write $\mathbb{U}_r = \{z : |z| < r\}$, and let $\mathbb{U} := \mathbb{U}_1$, i.e., the unit disk. Let S_H denote the set of all univalent harmonic mappings f in $\mathbb{U}$, where

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{j=2}^{\infty} a_j z^j + \sum_{j=1}^{\infty} \overline{b_j z^j} \tag{1.1}$$

with $|b_1| < 1$. We denote by S_H^0 the set of all mappings in S_H with $b_1 = 0$. Let S_H^* and $S_H^{*,0}$ denote the respective subclasses of S_H and S_H^0, where the images of $f(\mathbb{U})$ are starlike. Let C_H and C_H^0 denote the respective subclasses of S_H and S_H^0, where the images of $f(\mathbb{U})$ are convex.

In [5], Avci and Złotkiewicz introduced the class HS of univalent harmonic mappings F with the series expansion (1.1) such that

$$\sum_{j=2}^{\infty} j(|a_j| + |b_j|) \leq 1 - |b_1|, \quad (0 \leq |b_1| < 1),$$

and the subclass HC of HS, where

$$\sum_{j=2}^{\infty} j^2(|a_j| + |b_j|) \leq 1 - |b_1|, \quad (0 \leq |b_1| < 1).$$

The corresponding subclasses of HS and HC with $b_1 = 0$ are denoted by HS^0 and HC^0, respectively. These two classes constitute a harmonic counterpart of classes introduced by Goodman [15]. They are useful in studying questions of so-called δ-neighborhoods (Ruscheweyh [25], see also [22]) and in constructing explicit k-quasiconformal extensions (Fait et al. [14]).

Our aim is to generalize the following result, due to Duren [13], to the mappings of the class HC_p^0.

Theorem A. ([13, Theorem 1, p. 50]) *Each function $f \in C_H^0$ contains the full disk $|w| < 1/2$ in its range $f(\mathbb{U})$.*

A well-known coefficient conjecture of Clunie and Sheil-Small [12], is that if $f = h + \overline{g} \in S_H^0$, then the Taylor coefficients of the series of h and g satisfy the inequality

$$|a_j| \leq \frac{1}{6}(2j+1)(j+1) \quad \text{and} \quad |b_j| \leq \frac{1}{6}(2j-1)(j-1) \tag{1.2}$$

for all $j \geq 1$. Although, this coefficient conjecture remains an open problem for the full class S_H^0, this statement has been verified for certain subclasses, namely, the class T_H (see [13, Section 6.6]) of harmonic univalent typically real mappings, the class of harmonic convex mappings in one direction, harmonic starlike mappings in S_H^0 (see [13, Section 6.7]), and the class of harmonic close-to-convex mappings (see [24]). Equality occurs in (1.2) for the harmonic Koebe mapping

$$K(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}}, \tag{1.3}$$

which is constructed by shearing the Koebe function $k(z) = z/(1-z)^2$ horizontally with the dilatation $w(z) = z$. Note that K maps the unit disk $\mathbb{U}$ onto the slit-plane $\mathbb{C} \setminus (-\infty, -1/6]$.

This paper is organized as follows. In Section 3, we generalize Theorem A to the class HC_p^0 of polyharmonic mappings. The main result of this section is Theorem 1. In Section 4, we obtain two convolution characterizations for polyharmonic mappings to be starlike of order α and convex of order β, respectively. Our results are Theorems 2 and 3, where Theorem 2 extends [4, Theorems 2.6], and Theorem 3 is a generalization of [4, Theorem 2.8]. In Section 5, we find the radii of convexity and starlikeness for polyharmonic mappings, under certain coefficient conditions. The results in this section are Theorems 4 $\sim$ 7, which are the generalizations of [21, Theorems 3.1 and 3.3] and [17, Theorem 1.11], respectively.

2 Preliminaries

In this paper, we consider the polyharmonic mappings in $\mathbb{U}$. We use H_p to denote the set of all polyharmonic mappings F in $\mathbb{U}$ with a series expansion of the following form:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(h_k(z) + \overline{g_k(z)} \big) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} (a_{k,j} z^j + \overline{b_{k,j} z^j}), \tag{2.1}$$

with $a_{1,1} = 1$, $|b_{1,1}| < 1$. Let H_p^0 denote the subclass of H_p for $b_{1,1} = 0$ and $a_{k,1} = b_{k,1} = 0$ for $k \in \{2, \cdots, p\}$.

Definition 1. ([22]) We say that a univalent polyharmonic mapping F with $F(0) = 0$ is starlike with respect to the origin if the curve $F(re^{i\theta})$ is starlike with respect to the origin for each $r \in (0,1)$.

Proposition 1. ([23]) *If F is univalent, $F(0) = 0$ and $\frac{\partial}{\partial\theta}\big(\arg F(re^{i\theta})\big) > 0$ for $z = re^{i\theta} \neq 0$, then F is starlike with respect to the origin.*

Definition 2. ([22]) A univalent polyharmonic mapping F with $F(0) = 0$ and $\frac{\partial}{\partial\theta}F(re^{i\theta}) \neq 0$ whenever $r \in (0,1)$, is said to be convex if the curve $F(re^{i\theta})$ is convex for each $r \in (0,1)$.

Proposition 2. ([23]) *If F is univalent, $F(0) = 0$, $\frac{\partial}{\partial\theta}F(re^{i\theta}) \neq 0$ whenever $r \in (0,1)$, and $\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right] > 0$ for $z = re^{i\theta} \neq 0$, then F is convex.*

In [22], J. Qiao and X. Wang introduced the subclass of H_p^0 denoted by HS_p^0 of polyharmonic mappings F of the form (2.1) satisfying the condition

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left(2(k-1)+j\right)\left(|a_{k,j}|+|b_{k,j}|\right) \leq 2, \tag{2.2}$$

and the subclass HC_p^0 of HS_p^0, where

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left(2(k-1)+j^2\right)\left(|a_{k,j}|+|b_{k,j}|\right) \leq 2. \tag{2.3}$$

Their main result is the following:

Theorem B. ([22, Theorems 3.1, 3.2 **and** 3.3**])** *Suppose $F \in HS_p^0$. Then F is univalent, sense preserving, starlike in $\mathbb{U}$. In particularly, for each member of HC_p^0, F maps $\mathbb{U}$ onto a convex domain.*

Obviously, if $p = 1$, then the classes HS_p^0 and HC_p^0 reduce to HS^0 and HC^0, respectively.

3 Coefficient estimates

Now, we will generalize the Theorem A [13] from the class C_H^0 to the class HC_p^0 of polyharmonic mappings.

Theorem 1. *Let $F \in HC_p^0$ of the form* (2.1)*. Then the range $F(\mathbb{U})$ contains the full disk $|w| < 1/2$.*

Proof. Let $F \in HC_p^0$, and let $r \in (0,1)$. Write

$$F_r(z) = z + \sum_{j=2}^{\infty}\left(\sum_{k=1}^{p} a_{k,j}r^{2(k-1)}\right)z^j + \sum_{j=2}^{\infty}\left(\sum_{k=1}^{p}\overline{b_{k,j}}r^{2(k-1)}\right)\overline{z^j},\ z \in \mathbb{U}.$$

Then F_r is harmonic. By the hypothesis and (2.3), $F \in HC_p^0$, which implies

$$\sum_{j=2}^{\infty} j^2\left|\sum_{k=1}^{p} r^{2(k-1)}a_{k,j}\right| + \sum_{j=2}^{\infty} j^2\left|\sum_{k=1}^{p} r^{2(k-1)}b_{k,j}\right| \leq 1,$$

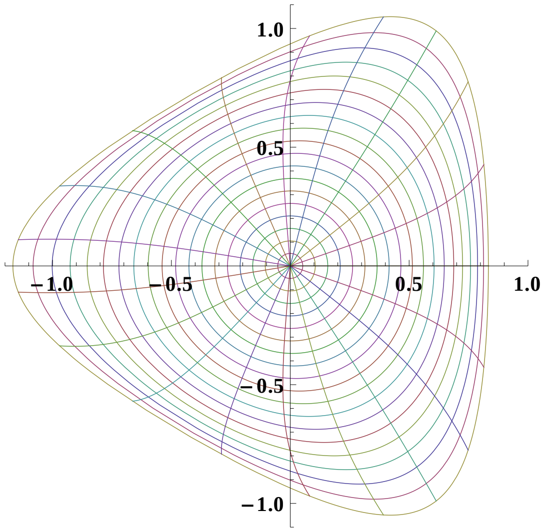

Figure 1: The image of $\mathbb{U}$ under the mapping $F(z) = z - \frac{1}{6}\overline{z^2}|z|^2$.

i.e., $F_r \in C_H^0$. As in the proof of Theorem A, we see that the range $F_r(\mathbb{U})$ is convex. Thus, if $w \notin F_r(\mathbb{U})$, a suitable rotation gives

$$\mathrm{Re}\left\{e^{i\theta}\left(F_r(z) - w\right)\right\} > 0,$$

for all $z \in \mathbb{U}$. But if $F_r(z) = \sum_{k=1}^{p} r^{2(k-1)}\left(h_k(z) + \overline{g_k(z)}\right)$, it follows that $\mathrm{Re}\{\varphi(z)\} > 0$ for

$$\begin{aligned}\varphi(z) =& \mathrm{Re}\left\{e^{i\theta}\left(\sum_{k=1}^{p} r^{2(k-1)} h_k(z) - w\right) + e^{-i\theta}\sum_{k=1}^{p} r^{2(k-1)} g_k(z)\right\}\\ =& \mathrm{Re}\{c_0 + c_1 z + \cdots\},\end{aligned}$$

where $c_0 = -e^{i\theta}w$ and $c_1 = e^{i\theta}$. Following the proof of Theorem A, we get

$$1 = |e^{i\theta}| = |c_1| \le 2|c_0| = 2|-e^{i\theta}w| = 2|w|,$$

or $|w| \ge 1/2$. This proves the result. ■

Example 1. *Let* $F(z) = z - \frac{1}{6}\overline{z^2}|z|^2 \in HC_2^0$. *Then* $F(\mathbb{U})$ *contains the full disk* $|w| < 1/2$. *See Figure 1.*

4 Convolution characterization

In this section, we obtain two convolution characterizations concerning polyharmonic mappings which are starlike of order α, and convex of order β, respectively.

Definition 3. ([20]) We say that a univalent polyharmonic mapping F with $F(0) = 0$ is starlike of order $\alpha \in [0,1)$ with respect to the origin if

$$\frac{\partial}{\partial\theta}\big(\arg F(re^{i\theta})\big) = \mathrm{Re}\left\{\frac{zF_z(z) - \overline{z}F_{\overline{z}}(z)}{F(z)}\right\} > \alpha \tag{4.1}$$

for all $z = re^{i\theta} \neq 0$.

Definition 4. ([20]) A univalent polyharmonic mapping F with $F(0) = 0$ and $\frac{\partial}{\partial\theta}F(re^{i\theta}) \neq 0$ whenever $r \in (0,1)$, is said to be convex of order $\beta \in [0,1)$ if

$$\begin{aligned}&\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right]\\ =&\mathrm{Re}\left\{\frac{zF_z(z) + z^2F_{z^2}(z) - 2|z|^2F_{z\overline{z}}(z) + \overline{z}F_{\overline{z}}(z) + \overline{z}^2F_{\overline{z}^2}(z)}{zF_z(z) - \overline{z}F_{\overline{z}}(z)}\right\} > \beta\end{aligned} \tag{4.2}$$

for all $z = re^{i\theta} \neq 0$.

Theorem 2. *Let $F = \sum_{k=1}^{p}|z|^{2(k-1)}(h_k(z) + \overline{g_k(z)}) \in H_p^0$ be univalent. Then F is starlike of order α if and only if*

$$\begin{aligned}&\sum_{k=1}^{p}|z|^{2(k-1)}\left\{h_k(z) * \left[\frac{z + ((\alpha\xi + \alpha + \xi - 1)/(2 - \alpha - \alpha\xi))z^2}{(1-z)^2}\right] - \overline{g_k(z)} *\right.\\ &\left.\left[\frac{(2\xi + \alpha + \alpha\xi)/(2 - \alpha - \alpha\xi)\overline{z} - ((\alpha\xi + \alpha + \xi - 1)/(2 - \alpha - \alpha\xi))\overline{z}^2}{(1-\overline{z})^2}\right]\right\} \neq 0,\end{aligned} \tag{4.3}$$

for all $z \neq 0$ in $\mathbb{U}$ and all $\xi \in \mathbb{C}$ with $|\xi| = 1$.

Proof. Let $F \in H_p^0$ be univalent. Since

$$\frac{zF_z(z) - \overline{z}F_{\overline{z}}(z)}{F(z)} = 1$$

at $z = 0$, the condition (4.1) is equivalent to the condition

$$\frac{zF_z(z) - \overline{z}F_{\overline{z}}(z)}{F(z)} - \alpha \neq \frac{\xi - 1}{\xi + 1}, \tag{4.4}$$

for all $z \neq 0$ in $\mathbb{U}$ and all $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\xi \neq -1$. By the hypothesis that F is univalent in $\mathbb{U}$, we get that $F(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$. Then, (4.4) holds if and only if

$$(\xi+1)\left(\sum_{k=1}^{p}|z|^{2(k-1)}\left(zh_k'(z)-\overline{z}\overline{g_k'(z)}-\alpha h_k(z)-\alpha\overline{g_k(z)}\right)\right)$$
$$\neq(\xi-1)\left(\sum_{k=1}^{p}|z|^{2(k-1)}\left(h_k(z)+\overline{g_k(z)}\right)\right)$$

for all $z\neq 0$ in $\mathbb{U}$ and all $\xi\in\mathbb{C}$ with $|\xi|=1$. Straightforward computations show that

$$\begin{aligned}
&(\xi+1)\left(\sum_{k=1}^{p}|z|^{2(k-1)}\left(zh_k'(z)-\overline{z}\overline{g_k'(z)}-\alpha h_k(z)-\alpha\overline{g_k(z)}\right)\right)\\
&\quad-(\xi-1)\left(\sum_{k=1}^{p}|z|^{2(k-1)}\left(h_k(z)+\overline{g_k(z)}\right)\right)\\
&=\sum_{k=1}^{p}|z|^{2(k-1)}\left\{h_k(z)*\left[\frac{(\xi+1)z}{(1-z)^2}-\frac{(\alpha\xi+\alpha+\xi-1)z}{1-z}\right]\right.\\
&\qquad\left.-\overline{g_k(z)}*\left[\frac{(\xi+1)\overline{z}}{(1-\overline{z})^2}+\frac{(\alpha\xi+\alpha+\xi-1)\overline{z}}{1-\overline{z}}\right]\right\}\\
&=\sum_{k=1}^{p}|z|^{2(k-1)}\left\{h_k(z)*\left[\frac{(2-\alpha-\alpha\xi)z+(\alpha\xi+\alpha+\xi-1)z^2}{(1-z)^2}\right]\right.\\
&\qquad\left.-\overline{g_k(z)}*\left[\frac{(2\xi+\alpha+\alpha\xi)\overline{z}-(\alpha\xi+\alpha+\xi-1)\overline{z}^2}{(1-\overline{z})^2}\right]\right\},
\end{aligned}$$

from which we see that (4.4) is true if and only if so is (4.3). The proof is complete. ■

Remark 1. The above result gives a sufficient condition for mappings in H_p^0 to be starlike in terms of their coefficients. Let $F\in H_p^0$ be of the form (2.1). If

$$\sum_{k=1}^{p}\sum_{j=2}^{\infty}\frac{2(k-1)+j-\alpha}{1-\alpha}|a_{k,j}|+\sum_{k=1}^{p}\sum_{j=2}^{\infty}\frac{2(k-1)+j+\alpha}{1-\alpha}|b_{k,j}|\leq 1,$$

then F is sense-preserving, univalent and starlike of order α. The result follows from Theorem 2 and Lemma B by a straightforward calculation. In fact, this case is already covered by Theorem B.

Theorem 3. *Let* $F=\sum_{k=1}^{p}|z|^{2(k-1)}\left(h_k(z)+\overline{g_k(z)}\right)\in H_p^0$ *be univalent such that* $\frac{\partial}{\partial\theta}F(re^{i\theta})\neq 0$ *for all* $r\in(0,1)$*. Then* F *is convex of order* β *if and only if*

$$\begin{aligned}
&\sum_{k=1}^{p}|z|^{2(k-1)}\left\{h_k(z)*\left[\frac{(2-\beta\xi-\beta)z+(2\xi+\beta\xi+\beta)z^2}{(1-z)^3}\right]\right.\\
&\qquad\left.+\overline{g_k(z)}*\left[\frac{(2\xi+\beta\xi+\beta)\overline{z}+(2-\beta\xi-\beta)\overline{z}^2}{(1-\overline{z})^3}\right]\right\}\neq 0,
\end{aligned}\tag{4.5}$$

for all $z \neq 0$ in $\mathbb{U}$ and all $\xi \in \mathbb{C}$ with $|\xi| = 1$.

Proof. Let $F \in H_p^0$ be univalent. Since

$$\frac{zF_z(z) + z^2F_{z^2}(z) - 2|z|^2F_{z\overline{z}}(z) + \overline{z}F_{\overline{z}}(z) + \overline{z}^2F_{\overline{z}^2}(z)}{zF_z(z) - \overline{z}F_{\overline{z}}(z)} = 1$$

at $z = 0$, the required condition (4.2) is equivalent to

$$\frac{zF_z(z) + z^2F_{z^2}(z) - 2|z|^2F_{z\overline{z}}(z) + \overline{z}F_{\overline{z}}(z) + \overline{z}^2F_{\overline{z}^2}(z)}{zF_z(z) - \overline{z}F_{\overline{z}}(z)} - \beta \neq \frac{\xi - 1}{\xi + 1}, \tag{4.6}$$

for all $z \neq 0$ in $\mathbb{U}$ and all $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\xi \neq -1$. Note that $\frac{\partial}{\partial\theta}F(re^{i\theta}) \neq 0$ for all $r \in (0,1)$. Then, (4.6) holds if and only if

$$\begin{aligned}
&(\xi + 1)\sum_{k=1}^{p} |z|^{2(k-1)}\left((1-\beta)zh_k'(z) + z^2h_k''(z) + (1+\beta)\overline{z}\overline{g_k'(z)} + \overline{z}^2\overline{g_k''(z)}\right) \\
&- (\xi - 1)\sum_{k=1}^{p} |z|^{2(k-1)}\left(zh_k'(z) - \overline{z}\overline{g_k'(z)}\right) \neq 0
\end{aligned}$$

for all $z \neq 0$ in $\mathbb{U}$ and all $\xi \in \mathbb{C}$ with $|\xi| = 1$. Straightforward computations show that

$$\begin{aligned}
&(\xi + 1)\sum_{k=1}^{p} |z|^{2(k-1)}\left((1-\beta)zh_k'(z) + z^2h_k''(z) + (1+\beta)\overline{z}\overline{g_k'(z)} + \overline{z}^2\overline{g_k''(z)}\right) \\
&\qquad - (\xi - 1)\sum_{k=1}^{p} |z|^{2(k-1)}\left(zh_k'(z) - \overline{z}\overline{g_k'(z)}\right) \\
&= \sum_{k=1}^{p} |z|^{2(k-1)}\left\{h_k(z) * \left[\frac{z(2 - \beta\xi - \beta)}{(1-z)^2} + \frac{2z^2(\xi+1)}{(1-z)^3}\right]\right. \\
&\qquad \left. + \overline{g_k(z)} * \left[\frac{\overline{z}(2\xi + \beta\xi + \beta)}{(1-\overline{z})^2} + \frac{2\overline{z}^2(\xi+1)}{(1-\overline{z})^3}\right]\right\} \\
&= \sum_{k=1}^{p} |z|^{2(k-1)}\left\{h_k(z) * \left[\frac{(2 - \beta\xi - \beta)z + (2\xi + \beta\xi + \beta)z^2}{(1-z)^3}\right]\right. \\
&\qquad \left. + \overline{g_k(z)} * \left[\frac{(2\xi + \beta\xi + \beta)\overline{z} + (2 - \beta\xi - \beta)\overline{z}^2}{(1-\overline{z})^3}\right]\right\},
\end{aligned}$$

from which we see that (4.6) is true if and only if (4.5) is. The proof is complete. ∎

Remark 2. By a straightforward calculation, we obtain from Theorem 3 and Lemma B a sufficient coefficient bound for polyharmonic mappings which are convex of order β. Let $F \in H_p^0$ be of the form (2.1). If

$$\sum_{k=1}^{p}\sum_{j=2}^{\infty} \frac{2(k-1) + j^2 - \beta}{1-\beta}|a_{k,j}| + \sum_{k=1}^{p}\sum_{j=2}^{\infty} \frac{2(k-1) + j^2 + \beta}{1-\beta}|b_{k,j}| \leq 1,$$

then F is convex of order β. In fact, this case is already covered by Theorem B.

5 Radii for starlikeness and convexity

In this section, we will first generalize the results [21, Theorems 3.1 and 3.3] to the polyharmonic mappings. The following identities, where $r \in (0,1)$, are used in the proofs of our results:

$$\begin{aligned}
&\sum_{j=1}^{\infty} r^{j-1} = \frac{1}{1-r}, \quad \sum_{j=1}^{\infty} j r^{j-1} = \frac{1}{(1-r)^2}, \quad \sum_{j=1}^{\infty} j^2 r^{j-1} = \frac{1+r}{(1-r)^3}, \\
&\sum_{j=1}^{\infty} j^3 r^{j-1} = \frac{1+4r+r^2}{(1-r)^4} \quad \text{and} \quad \sum_{j=1}^{\infty} j^4 r^{j-1} = \frac{(1+r)(1+10r+r^2)}{(1-r)^5}.
\end{aligned} \tag{5.1}$$

Theorem 4. *Let $F \in H_p^0$ of the form* (2.1) *and the coefficients of the series satisfy the conditions*

$$|a_{k,j}| \le \tfrac{1}{6}(2j+1)(j+1) \text{ and } |b_{k,j}| \le \tfrac{1}{6}(2j-1)(j-1).$$

Then F is univalent and starlike of order α in $|z| < r_0(\alpha)$, where $r_0(\alpha)$ is the smallest positive root of the equation

$$6(1-\alpha)(1-r)^4 - \sum_{k=1}^{p} r^{2(k-1)} \left(3(r+1)^2 - 3\alpha(1-r)^2 + 2(k-1)(r^2+3)(1-r)\right) = 0, \tag{5.2}$$

in the interval $(0,1)$. The result is sharp.

Proof. Let $F_r(z) := r^{-1}F(rz)$, where $F \in H_p^0$ is of the form (2.1), and fix $r \in (0,1)$. Then

$$F_r(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left(a_{k,j} r^{2k+j-3} z^j + \overline{b_{k,j}} r^{2k+j-3} \overline{z^j} \right), \; z \in \mathbb{U}.$$

By the hypotheses, $|a_{k,j}| \le \frac{1}{6}(2j+1)(j+1)$ and $|b_{k,j}| \le \frac{1}{6}(2j-1)(j-1)$. By using these coefficient estimates and (5.1), we obtain

$$
\begin{aligned}
S_0 :=& \sum_{k=1}^{p}\sum_{j=1}^{\infty}\left(\frac{2(k-1)+j-\alpha}{1-\alpha}|a_{k,j}|r^{2k+j-3}+\frac{2(k-1)+j+\alpha}{1-\alpha}|b_{k,j}|r^{2k+j-3}\right)\\
\leq& \frac{1}{6}\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left(\frac{2(k-1)+j-\alpha}{1-\alpha}(2j+1)(j+1)\right.\\
&\left.+\frac{2(k-1)+j+\alpha}{1-\alpha}(2j-1)(j-1)\right)r^{2k+j-3}\\
=& \sum_{k=1}^{p}\sum_{j=1}^{\infty}\frac{1}{3(1-\alpha)}\left(2j^3+4j^2(k-1)+(1-3\alpha)j+2(k-1)\right)r^{2k+j-3}\\
=& \frac{1}{3(1-\alpha)}\sum_{k=1}^{p}\left(2r^{2k-3}\sum_{j=1}^{\infty}j^3r^j+4(k-1)r^{2k-3}\sum_{j=1}^{\infty}j^2r^j\right.\\
&\left.+(1-3\alpha)r^{2k-3}\sum_{j=1}^{\infty}jr^j+2(k-1)r^{2k-3}\sum_{j=1}^{\infty}r^j\right)\\
=& \frac{1}{3(1-\alpha)}\sum_{k=1}^{p}r^{2k-2}\left(\frac{3(r+1)^2}{(1-r)^4}-\frac{3\alpha}{(1-r)^2}+\frac{2(k-1)(r^2+3)}{(1-r)^3}\right).
\end{aligned}
$$

According to Remark 1, it suffices to show that $S_0 \leq 2$. By the last inequality, $S_0 \leq 2$ if r satisfies the inequality

$$
\begin{aligned}
s_0(r) = 6(1-\alpha)(1-r)^4 - \sum_{k=1}^{p} r^{2(k-1)}\big(3(r+1)^2 - 3\alpha(1-r)^2&\\
+2(k-1)(r^2+3)(1-r)\big) \geq 0.&
\end{aligned}
$$

Since $s_0(0) = 3-3\alpha > 0$ and $s_0(1) < 0$, then exists a smallest positive root $r_0(\alpha)$ of the equation $s_0(r) = 0$ in the interval (0,1). In particular, F is sense-preserving, univalent and starlike of order α in $|z| < r_0(\alpha)$.

As in [21, Theorem 3.1], the mapping

$$
f_0(z) = 2z - \frac{z-\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3}} \tag{5.3}
$$

shows that the bound given by $r_0(\alpha)$ is the best possible. See Figure 2. ■

Theorem 5. *Under the hypothesis of Theorem 4, $F \in H_p^0$ is univalent and convex of order β in the disk $|z| < r_1(\beta)$, where $r_1(\beta)$ is the smallest positive root of the equation*

$$
\begin{aligned}
0 =& 6(1-\beta)(1-r)^5 - \sum_{k=1}^{p} r^{2(k-1)}\big((8k-6-6\beta)(1+r)(1-r)^2\\
&+4(k-1)(1-r)^4+4(1+r)(1+10r+r^2)-6(2k-1-\beta)(1-r)^5\big)
\end{aligned} \tag{5.4}
$$

in the interval $(0,1)$. The result is sharp.

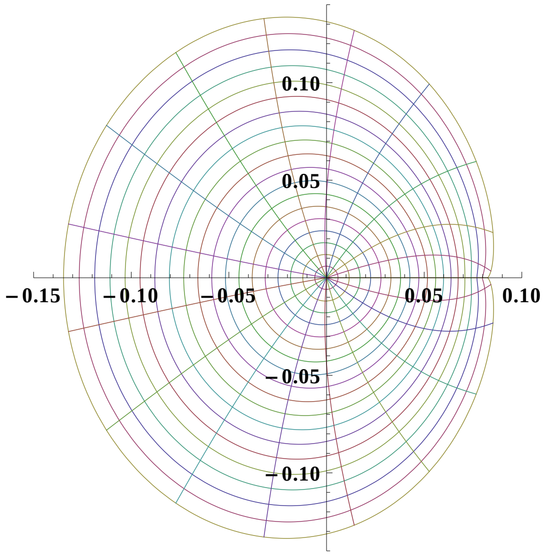

Figure 2: The image of $\mathbb{U}(r_0(\alpha))$, where $\alpha = 0$ and $r_0 \approx 0.11290$ is given by (5.2), under the mapping f_0 of (5.3).

Proof. The proof of this result is similar to Theorem 4, where Remark 2 is used instead of Remark 1, and we omit it. The bound $r_1(\beta)$ given by (5.4) is again sharp, which can be seen by considering the mapping

$$f_1(z) = 2z - \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} - \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}}, \tag{5.5}$$

see Figure 3. Then, by [21, Theorem 3.3], we see that the bound given by $r_1(\beta)$ is the best possible. ∎

In [17, Theorem 1.11], D. Kalaj et al. gave the radii of close-to-convexity of harmonic mappings under certain coefficients conditions. Now, we will study the radii of starlikeness and convexity of mappings in H_p^0 under the same coefficients condition. Our results are the following:

Theorem 6. *Let $F \in H_p^0$ of the form* (2.1) *and the coefficients of the series satisfy the conditions*

$$|a_{k,j}| + |b_{k,j}| \leq C \text{ for all } j \geq 2.$$

Then F is univalent and starlike of order α in $|z| < r_2(\alpha)$, where $r_2(\alpha)$ is the smallest positive root of the equation

$$(1-\alpha)(1-r)^2 - \sum_{k=1}^{p} Cr^{2(k-1)}\left((2k-2+\alpha)(1-r) + 1 - (2k+\alpha-1)(1-r)^2\right) = 0 \tag{5.6}$$

in the interval (0,1). The result is sharp.

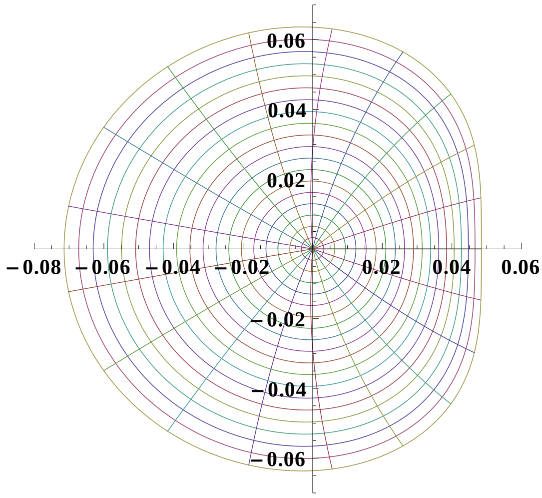

Figure 3: The image of $\mathbb{U}(r_1(\beta))$, where $\beta = 0$ and $r_1 \approx 0.06143$ is given by (5.4), under the mapping f_1 of (5.5).

Proof. Let $F_r(z) := r^{-1}F(rz)$, where $F \in H_p^0$ is of the form (2.1), and fix $r \in (0,1)$. Then

$$F_r(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left(a_{k,j} r^{2k+j-3} z^j + \overline{b_{k,j}} r^{2k+j-3} \overline{z^j} \right), \ z \in \mathbb{U}.$$

As in the proof of Theorem 4, under the hypothesis that $|a_{k,j}| + |b_{k,j}| \leq C$ and (5.1), we get

$$\begin{aligned} S_1 &:= \sum_{k=1}^{p} \sum_{j=2}^{\infty} \left(\frac{2(k-1)+j-\alpha}{1-\alpha} |a_{k,j}| + \frac{2(k-1)+j+\alpha}{1-\alpha} |b_{k,j}| \right) r^{2k+j-3} \\ &\leq \sum_{k=1}^{p} \sum_{j=2}^{\infty} \frac{2(k-1)+j+\alpha}{1-\alpha} C r^{2k+j-3} \\ &= \sum_{k=1}^{p} \left(\frac{(2k-2+\alpha)(1-r)+1}{(1-r)^2} - 2k+1-\alpha \right) \frac{Cr^{2k-2}}{1-\alpha}. \end{aligned}$$

According to Remark 1, it suffices to show that $S_1 \leq 1$. By the last inequality, $S_1 \leq 1$ if r satisfies the following inequality:

$$\begin{aligned} s_1(r) = (1-\alpha)(1-r)^2 - \\ \sum_{k=1}^{p} Cr^{2(k-1)} \left((2k-2+\alpha)(1-r) + 1 - (2k+\alpha-1)(1-r)^2 \right) \geq 0. \end{aligned}$$

Since $s_1(0) = 1-\alpha > 0$ and $s_1(1) < 0$, then exists a smallest positive root $r_2(\alpha)$ of the equation $s_1(r) = 0$ in the interval $(0,1)$. In particular, F is sense-preserving, univalent and starlike of order α in $|z| < r_2(\alpha)$.

To prove the sharpness part of the statement, one may consider the mapping

$$f_2(z) = z - \frac{Cz^2}{2(1-z)} - \overline{\frac{Cz^2}{2(1-z)}}, \tag{5.7}$$

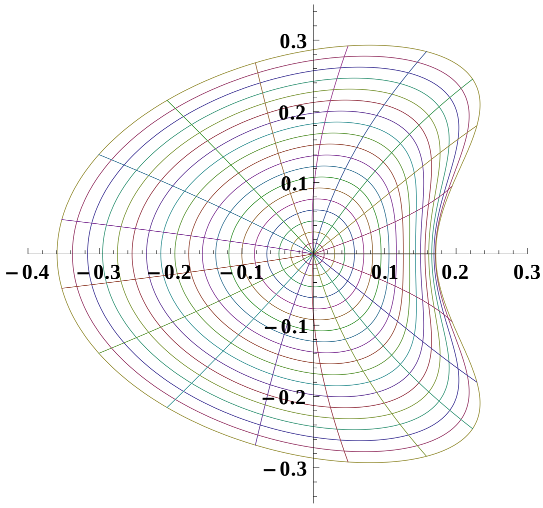

Figure 4: The image of $\mathbb{U}(r_2(\alpha))$, where $\alpha = 0$, $C = 1$ and $r_2 \approx 0.29289$ is given by (5.6), under the mapping f_2 of (5.7).

see Figure 4. Then by [17, Theorem 1.11], we see that the bound given by $r_2(\alpha)$ is the best possible. The proof of the theorem is complete. ∎

Theorem 7. *Under the hypothesis of Theorem 6, $F \in H_p^0$ is univalent and convex of order β in the disk $|z| < r_3(\beta)$, where $r_3(\beta)$ is the smallest positive real root of the equation*

$$0 = (1-\beta)(1-r)^3 - \sum_{k=1}^{p} Cr^{2k-2}\big((2k-2)(1-r)^2 + 1 + r + \beta - \beta r - (2k+\beta-1)(1-r)^3\big) \tag{5.8}$$

in the interval $(0,1)$. The result is sharp.

Proof. The proof of this result is similar to that of Theorem 6, where Remark 2 is used instead of Remark 1, and we omit it. The bound $r_3(\beta)$ given by (5.8) is sharp by considering the mapping

$$f_3(z) = z - \overline{\frac{Cz^2}{1-z}}, \tag{5.9}$$

see Figure 5. Note that the root of the equation (5.8) in $(0,1)$ is decreasing as a function of $\beta \in [0,1)$. As f_3 has real coefficients, we obtain

$$\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta} f_3(re^{i\theta})\right)\right]\bigg|_{\theta=0, r=r_3(\beta)} = \frac{(1+C)-(C+Cr)/(1-r)^3}{1-C+C/(1-r)^2}\bigg|_{r=r_3(\beta)} = \beta.$$

Therefore, the mapping f_3 will not be convex of order β in the disk $|z| < r$, where $r > r_3(\beta)$. ∎

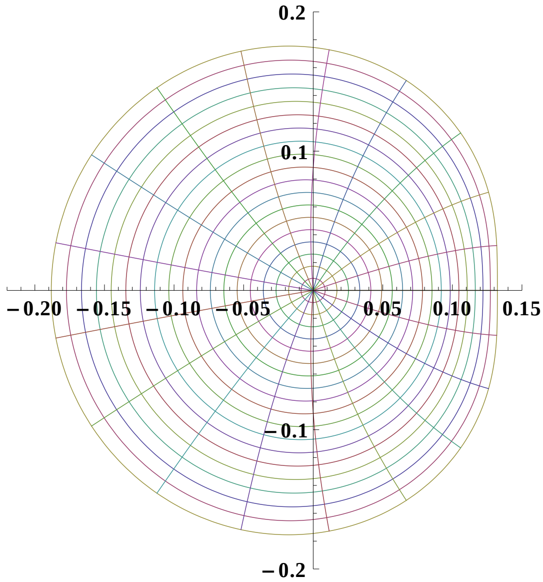

Figure 5: The image of $\mathbb{U}(r_3(\beta))$, where $\beta = 0$, $C = 1$ and $r_3 \approx 0.16488$ is given by (5.8), under the mapping f_3 of (5.9).

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Department of Mathematics, Hunan Normal University,
Changsha, Hunan 410081,
People's Republic of China.
email:jiaolongchen@sina.com

Department of Mathematics, Hunan Normal University,
Changsha, Hunan 410081,
People's Republic of China,
and
Department of Mathematics and Systems Analysis,
Aalto University, P.O. Box 11100, FI-00076 Aalto, Finland.
email:antti.rasila@iki.fi

Department of Mathematics, Hunan Normal University,
Changsha, Hunan 410081,
People's Republic of China.
email:xtwang@hunnu.edu.cn