Nonlinear Sturm-Liouville dynamic equation with a measure of noncompactness in Banach spaces

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Abstract

This paper is devoted to prove the existence of solutions of the nonlinear Sturm-Liouville boundary value problem on time scales in Banach spaces. We obtain the sufficient conditions for the existence of solutions in terms of Kuratowski measure of noncompactness. Mönch's fixed point theorem is used to prove the main result. By the unification property of time scales, our result is valid for Sturm-Liouville differential equations and difference equations, but more interestingly by the extension property, it is also valid for Sturm-Liouville q-difference equation.

1 Introduction

The measure of noncompactness, initiated by fundamental papers of Kuratowski [28] and Darbo [17], developed by Banaś and Goebel [9] and many authors in the literature, plays an important role in the theory of nonlinear analysis which has been improved fast recently because of its extensive practical applications in many fields such as engineering, economics, optimal control and optimization. The measure of noncompactness has been successfully applied to the theory of differential equations (see [3, 14, 31, 35] and references therein), difference equations [1, 21], integral equations [5] and differential inclusions, *i.e.*, multi-valued

Received by the editors in March 2012.
Communicated by J. Mawhin.
2000 *Mathematics Subject Classification* : 34G20, 34A40, 34N05, 39A13, 46B50.
Key words and phrases : Sturm-Liouville problem; Banach space; measure of noncompactness, time scale.

functional differential equations [13, 22, 32].
After Hilger [24] initiated the concept of time scale and Aulbach and Hilger [25] published the first article on time scales, the differential equations, difference equations and quantum equations [26] (h-difference and q -difference equations are based on h-calculus and q-calculus, respectively) were unified and extended as *the dynamic equations*. The most important advantage of a time scale is that it provides not only a unified approach to study the discrete intervals with uniform step size (the lattice $h\mathbb{Z}$), continuous intervals and discrete intervals with non-uniform (variable) step size (for instance q-numbers), but also, more interestingly, it gives an opportunity to extend the approach to study the combination of continuous and discrete intervals. Therefore, the concept of time scales can build bridges between the continuous, discrete and q-discrete analysis.
Sturm Liouville equation has been extensively studied in both continuous and discrete cases [7, 20, 29, 33, 36]. After the theory of time scale is created and it has been shown to be applicable to any field with discrete or continuous models (or combination of these), the study on Sturm-Liouville equation turned out to be Sturm-Liouville dynamic equation for which the existence of the solutions has been presented [8, 18, 19, 34].
However the theory of dynamic equations in Banach spaces on an arbitrary time scale is *still* a new research area. In this article, we focus on this gap. By unifying both the discrete and continuous cases, as well as extending to the q-discrete case, we derive the existence of the solutions of the dynamic Sturm-Liouville problem in Banach spaces. These kind of dynamic equations have the same advantages as in a real-valued case and an increasing number of possible applications. For the dynamic equations in Banach spaces, the first articles are written by Cichoń *et.al.* [15, 16]. Authors prove the existence of the classical, Carathéodory and the weak solutions of the first order Cauchy dynamic problem via measure of (weak) noncompactness.
The more general form of Sturm-Liouville equation with mixed derivatives in finite interval is introduced by Atici and Guseinov [8]. Then Topal *et.al* [34] study the existence of positive solutions for the the Sturm-Liouville boundary value problem with real valued nonlinear term f. In this paper we prove the existence of the solutions of boundary value problem

$$(p(t)x^{\Delta}(t))^{\nabla} + f(t, x(t), x^{\Delta}(t), x^{\nabla}(t)) = 0, \quad t \in [0, \infty), \tag{1}$$

$$\begin{aligned} \alpha_1 x(0) - \beta_1 \lim_{t\to 0^+} p(t)x^{\Delta}(t) = 0, \\ \alpha_2 \lim_{t\to\infty} x(t) + \beta_2 \lim_{t\to\infty} p(t)x^{\Delta}(t) = 0, \end{aligned} \tag{2}$$

in Banach spaces. For this purpose, we first express the boundary value problem (1)-(2) as an integral equation by means of Green's function. Then by imposing some conditions on the nonlinear term f in terms of Kuratowski measure of noncompactness, we obtain a fixed point of the operator associated to this integral equation using Mönch's fixed point theorem.

2 Preliminaries

We refer to the books [10, 11] for the general theory of dynamic equations on time scales. To understand the so-called dynamic equation, we present some preliminary definitions and notations of time scale which are very common in the literature [2, 10, 11, 24, 25, 27].
Let $(E, ||\cdot||)$ be a Banach space and $\mathbb{T}$ denote a time scale (nonempty closed subset of real numbers $\mathbb{R}$). Throughout this paper, by an interval $[a, b]$ we mean $\{t \in \mathbb{T} : a \leq t \leq b\}$. In particular, the time scale half line is denoted by $J = [0, \infty) = \{t \in \mathbb{T} : 0 \leq t < \infty\}$. Other types of intervals are assumed in a similar manner. By a subinterval I of J, we mean the time scale subinterval.

Definition 2.1. *The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ respectively. We put $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).*
The jump operators σ and ρ allow the classification of points in time scale in the following way: t is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$ respectively.
Moreover $\mathbb{T}^k$ denotes Hilger's above truncated set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point. Similarly, $\mathbb{T}_k$ denotes the below truncated set obtained from $\mathbb{T}$ by deleting a possible right-scattered minimal point.

Next we define the so - called $\Delta(\nabla)$-derivative and $\Delta(\nabla)$-integral for Banach valued functions similar as $\Delta(\nabla)$-derivative and $\Delta(\nabla)$-integral on time scales [10, 11]. The basic properties of $\Delta(\nabla)$-derivative and integral for Banach valued functions are analogue to $\Delta(\nabla)$-derivative and integral for the real case.

Definition 2.2. *Let $f : \mathbb{T} \to E$. For $t \in \mathbb{T}^k$, we define $f^\Delta(t)$ by*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s},$$

and for $t \in \mathbb{T}_k$, and $f^\nabla(t)$ by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\rho(t)) - f(s)}{\rho(t) - s}.$$

For the most common time scales, the Δ and the ∇ derivatives are stated below:

(a) If $\mathbb{T} = \mathbb{R}$, $f^\Delta = f^\nabla = f'$,

(b) If $\mathbb{T} = \mathbb{Z}$, $f^\Delta = \Delta f$, *i.e.*, the usual forward difference operator and $f^\nabla = \nabla f$, *i.e.*, the usual backward difference operator,

(c) If $\mathbb{T} = \mathbb{K}_q = q^{\mathbb{Z}} \cup \{0\} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q \neq 1$ is a fixed real number, $f^\Delta(t) = \Delta_q f(t) = \dfrac{f(qt) - f(x)}{(q-1)t}$ and $f^\nabla(t) = \nabla_q f(t) = \dfrac{f(x) - f(t/q)}{t(1 - 1/q)}$.

Hence the theory time scale allows the unification and the extension of the differential and the difference equations.

Definition 2.3. *We say that $f : \mathbb{T} \to E$ is right dense continuous (rd- continuous) if f is continuous at every right dense point $t \in \mathbb{T}$ and $\lim_{s \to t^-} f(s)$ exists and is finite at every left dense point $t \in \mathbb{T}$.*

Similarly, we say that $f : \mathbb{T} \to E$ is left dense continuous (ld- continuous) if f is continuous at every left dense point $t \in \mathbb{T}$ and $\lim_{s \to t^+} f(s)$ exists and is finite at every right dense point $t \in \mathbb{T}$.

Definition 2.4. *A function $F : \mathbb{T} \to E$ is called a Δ-antiderivative of the function $f : \mathbb{T} \to E$ if $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the Δ-integral is defined by*

$$\int_a^t f(\tau)\Delta\tau = F(t) - F(a),$$

and similarly a function $F : \mathbb{T} \to E$ is called a ∇-antiderivative of $f : \mathbb{T} \to E$ if $F^{\nabla}(t) = f(t)$ holds for all $t \in \mathbb{T}_k$. Then the ∇-integral is defined by

$$\int_a^t f(\tau)\nabla\tau = F(t) - F(a).$$

Remark 2.5. [10] (Existence of antiderivatives)

(i) *Every rd-continuous function f has a Δ antiderivative. In particular if $t_0 \in \mathbb{T}$ then F defined by*

$$F(t) := \int_{t_0}^t f(\tau)\Delta\tau, \quad t \in \mathbb{T}$$

is an antiderivative of f.

(ii) *Every ld-continuous function f has a ∇ antiderivative. In particular if $t_0 \in \mathbb{T}$ then F defined by*

$$F(t) := \int_{t_0}^t f(\tau)\nabla\tau, \quad t \in \mathbb{T}$$

is a ∇ antiderivative of f.

Theorem 2.6. (The mean value theorem for ∇-integrals) *If $f : J \to E$ is ∇- integrable then*

$$\int_I f(t)\nabla t \in \mu_{\nabla}(I) \cdot \overline{\text{conv}} f(I),$$

where I is an arbitrary subinterval of J, $\mu_{\nabla}(I)$ is the Lebesgue ∇-measure of I and $\overline{\text{conv}} f(I)$ is the closure of the convex extension of $f(I)$.

Proof. Let I_{RS} be the set of right scattered points of I. By the properties of ∇-integral on Banach spaces (∇-analogue Theorem 5.2 of [12]), we obtain

$$\int_I f(t)\nabla t = \int_{I \setminus I_{RS}} f(t)dt + \sum_{t_i \in I_{RS}} f(t_i)\mu_{\nabla}(t_i).$$

Theorem 5.2 and Lemma 3.1 of [12] lead us to have

$$\begin{aligned}\int_I f(t)\nabla t &\in \operatorname{mes}(I \setminus I_{RS}) \cdot \overline{\operatorname{conv}} f(I) + \sum_{t_i \in I_{RS}} f(t_i)\mu_\nabla(t_i) \\ &\subset \operatorname{mes}(I \setminus I_{RS}) \cdot \overline{\operatorname{conv}} f(I) + f(I) \cdot \sum_{t_i \in I_{RS}} \mu_\nabla(t_i) \\ &\subset (\operatorname{mes}(I \setminus I_{RS}) + \sum_{t_i \in I_{RS}} \mu_\nabla(t_i)) \cdot \overline{\operatorname{conv}} f(I) \\ &= \mu_\nabla(I) \cdot \overline{\operatorname{conv}} f(I).\end{aligned}$$

Here $\operatorname{mes}(I)$ denotes the measure of the interval I. ∎

Theorem 2.7. (Mean value theorem for Δ-integrals) *If $f : J \to E$ is Δ- integrable then*

$$\int_I f(t)\Delta t \in \mu_\Delta(I) \cdot \overline{conv} f(I),$$

where I is an arbitrary subinterval of J and $\mu_\Delta(I)$ is the Lebesgue Δ-measure of I.

Proof. The proof is Δ-analogue of Theorem 2.6. ∎

See [6, 23] for the detailed research about the theory of Lebesgue $\Delta(\nabla)$- measure and Lebesgue $\Delta(\nabla)$-integral. Authors developed the theory by using the measure-theoretical approach of Hilger [25].

Our fundamental tool is the Kuratowski measure of noncompactness [9]. For any bounded subset A of E, the Kuratowski measure of noncompactness of A, denoted by $\alpha(A)$, is the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter smaller than ε. For convenience we present the properties of the measure of noncompactness α:

1. If $A \subset B$ then $\alpha(A) \leq \alpha(B)$,
2. $\alpha(A) = \alpha(\bar{A})$, where $\bar{A}$ denotes the closure of A,
3. $\alpha(A) = 0$ if and only if A is relatively compact,
4. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,
5. $\alpha(\lambda A) = |\lambda|\alpha(A)$ $(\lambda \in \mathbb{R})$,
6. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$,
7. $\alpha(\operatorname{conv} A) = \alpha(A)$, where $\operatorname{conv}(A)$ denotes the convex extension of A.

For the proof of the main result, we require the adaptation of Ambrosetti's result [4] for the space $\mathcal{C}_1^{\Delta\nabla}(\mathbb{T}, E)$, *i.e*, the space of Banach-valued functions defined on a time scale $\mathbb{T}$ having continuous Δ and ∇ derivatives.

Lemma 2.8. *Assume that $A \subset \mathcal{C}_1^{\Delta\nabla}(\mathbb{T}, E)$ is bounded. Also assume that A^Δ and A^∇ defined by $\{g^\Delta : g \in A\}$ and $\{g^\nabla : g \in A\}$ respectively, are bounded and equicontinuous. Then the measure of noncompactness in $\mathcal{C}_1^{\Delta\nabla}(\mathbb{T}, E)$ is given by*

$$\alpha(A) = \max\left\{\sup_{t \in \mathbb{T}} \alpha(A(t)), \sup_{t \in \mathbb{T}} \alpha(A^\Delta(t)), \sup_{t \in \mathbb{T}} \alpha(A^\nabla(t))\right\}.$$

Proof. First we show that

$$\alpha(A) \geq \max\left\{\sup_{t\in\mathbb{T}} \alpha(A(t)), \sup_{t\in\mathbb{T}} \alpha(A^{\Delta}(t)), \sup_{t\in\mathbb{T}} \alpha(A^{\nabla}(t))\right\}.$$

Let $d = \alpha(A)$ and $\epsilon > 0$. By definition of $\alpha(A)$ there exists sets $T_1, T_2, \ldots T_k \in \mathcal{C}_1^{\Delta\nabla}(\mathbb{T}, E)$ such that $A \subset \bigcup_{i=1}^{k} T_i$ and diam $T_i < d + \varepsilon$, $\forall i = 1, 2, \ldots, k$.
For $t_0 \in \mathbb{T}$, observe that

$$A(t_0) \subset \bigcup_{i=1}^{k} T_i(t_0), \quad A^{\Delta}(t_0) \subset \bigcup_{i=1}^{k} T_i^{\Delta}(t_0), \quad A^{\nabla}(t_0) \subset \bigcup_{i=1}^{k} T_i^{\nabla}(t_0)$$

and

$$\text{diam } T_i(t_0) < d + \varepsilon, \quad \text{diam } T_i^{\Delta}(t_0) < d + \varepsilon, \quad \text{diam} T_i^{\nabla}(t_0) < d + \varepsilon.$$

Thus we have

$$\alpha(A(t_0)) < d + \varepsilon, \quad \alpha(A^{\Delta}(t_0)) < d + \varepsilon, \quad \alpha(A^{\nabla}(t_0)) < d + \varepsilon.$$

Since ε and t_0 are arbitrary, we obtain

$$\alpha(A) \geq \max\left\{\sup_{t\in\mathbb{T}} \alpha(A(t)), \sup_{t\in\mathbb{T}} \alpha(A^{\Delta}(t)), \sup_{t\in\mathbb{T}} \alpha(A^{\nabla}(t))\right\}. \tag{3}$$

Next we show

$$\alpha(A) \leq \max\left\{\sup_{t\in\mathbb{T}} \alpha(A(t)), \sup_{t\in\mathbb{T}} \alpha(A^{\Delta}(t)), \sup_{t\in\mathbb{T}} \alpha(A^{\nabla}(t))\right\}.$$

By the hypothesis, A^{Δ} and A^{∇} are equicontinuous. By the continuity of A and mean value theorems for Δ and ∇ integrals (Theorems 2.6 and 2.7), we obtain the equicontinuity of A. Since A, A^{Δ} and A^{∇} are equicontinuous, for $\varepsilon > 0$ there exist $\delta > 0$ and the set of points $t_0, t_1, \ldots, t_n \in \mathbb{T}$ such that if $t \in \mathbb{T}$ and $g \in A$ then there exists $t_i \in \mathbb{T}$ such that $|t - t_i| < \delta$ implies

$$||g(t) - g(t_i)|| \leq \varepsilon, \quad ||g^{\Delta}(t) - g^{\Delta}(t_i)|| \leq \varepsilon, \quad ||g^{\nabla}(t) - g^{\nabla}(t_i)|| \leq \varepsilon.$$

Let $\overline{d} = \alpha(\bigcup_{i=0}^{n}[A(t_i) \bigcup A^{\Delta}(t_i) \bigcup A^{\nabla}(t_i)])$. Then there exist sets $A_1, A_2, \ldots, A_m$ such that

$$\bigcup_{i=0}^{n}[A(t_i) \bigcup A^{\Delta}(t_i) \bigcup A^{\nabla}(t_i)] \subset \bigcup_{i=1}^{m} A_i$$

and diam $A_i < \overline{d} + \varepsilon$ for $i = 1, 2, \ldots, m$. Define

$$B_{ijkl} := \{g \in A : g(t_i) \in A_j, \quad g^{\Delta}(t_i) \in A_k^{\Delta}, \quad g^{\nabla}(t_i) \in A_l^{\nabla}\}$$

for $i = 1, 2, \ldots, n$ and $j, k, l = 1, 2, \ldots, m$. Note that $A \subset \bigcup_{i,j,k,l} B_{ijkl}$. If $g, h \in B_{ijkl}$ and $|t - t_i| < \delta$, then we obtain the followings:

$$\begin{aligned} ||g(t) - h(t)|| &\leq ||g(t_i) - g(t)|| + ||h(t_i) - h(t)|| + ||h(t_i) - g(t_i)|| \\ &\leq \varepsilon + \varepsilon + \varepsilon + \overline{d} = \overline{d} + 3\varepsilon, \end{aligned} \tag{4}$$

$$\begin{aligned} ||g^\Delta(t) - h^\Delta(t)|| &\leq ||g^\Delta(t_i) - g^\Delta(t)|| + ||h^\Delta(t_i) - h^\Delta(t)|| + ||h^\Delta(t_i) - g^\Delta(t_i)|| \\ &\leq \varepsilon + \varepsilon + \varepsilon + \overline{d} = \overline{d} + 3\varepsilon, \end{aligned} \tag{5}$$

$$\begin{aligned} ||g^\nabla(t) - h^\nabla(t)|| &\leq ||g^\nabla(t_i) - g^\nabla(t)|| + ||h^\nabla(t_i) - h^\nabla(t)|| + ||h^\nabla(t_i) - g^\nabla(t_i)|| \\ &\leq \varepsilon + \varepsilon + \varepsilon + \overline{d} = \overline{d} + 3\varepsilon. \end{aligned} \tag{6}$$

Therefore diam $B_{ijkl} < \overline{d} + 3\varepsilon$, which implies

$$\begin{aligned} \alpha(A) &\leq \overline{d} + 3\varepsilon = \alpha\Big(\bigcup_{i=0}^{n} [A(t_i) \bigcup A^\Delta(t_i) \bigcup A^\nabla(t_i)] + 3\varepsilon \\ &\leq \max_{1 \leq i \leq n} \{\alpha(A(t_i)), \quad \alpha(A^\Delta(t_i)), \quad \alpha(A^\nabla(t_i))\} + 3\varepsilon \\ &\leq \max_{1 \leq i \leq n} \{\sup_{t_i \in \mathbb{T}} \alpha(A(t_i)), \quad \sup_{t_i \in \mathbb{T}} \alpha(A^\Delta(t_i)), \quad \sup_{t_i \in \mathbb{T}} \alpha(A^\nabla(t_i))\} + 3\varepsilon. \end{aligned}$$

Since ε is arbitrary,

$$\alpha(A) \leq \max\left\{\sup_{t \in \mathbb{T}} \alpha(A(t)), \sup_{t \in \mathbb{T}} \alpha(A^\Delta(t)), \sup_{t \in \mathbb{T}} \alpha(A^\nabla(t))\right\}. \tag{7}$$

Inequalities (3) and (7) complete the proof. ∎

In the proof of the main theorem, we apply the following fixed point theorem.

Theorem 2.9. (Mönch Fixed Point Theorem)[30] *Let D be a closed convex subset of E, and let F be a continuous map from D into itself. If for some $x \in D$ the implication*

$$\bar{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \quad \Rightarrow \quad V \textit{ is relatively compact}$$

holds for every countable subset V of D, then F has a fixed point.

3 Main Results

Definition 3.1. *A function $x : \mathbb{T} \to E$ is said to be a solution of* (1)-(2) *provided that x is Δ-differentiable on $\mathbb{T}^k$, $x^\Delta : \mathbb{T} \to E$ is ∇-differentiable on $\mathbb{T}^* = \mathbb{T}^k \bigcap \mathbb{T}_k$, $x^{\Delta\nabla} : \mathbb{T}^* \to E$ is continuous and* (1)-(2) *hold for all $t \in \mathbb{T}^*$.*

Let $p, f, \alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy the following conditions:

(C1) $p : \mathbb{T} \to \mathbb{R}$ is a ∇-differentiable function on $\mathbb{T}_k = [0, \infty)_k$, $p^{\nabla} : \mathbb{T}_k \to \mathbb{R}$ is continuous, $p(t) \neq 0$, for all $t \in \mathbb{T}$, and $\int_0^{\infty} \frac{\Delta s}{p(s)} < \infty$,

(C3) $f : \mathbb{T} \times E^3 \to E$ is a Banach valued function,

(C4) $|\alpha_1| + |\beta_1| \neq 0, \quad |\alpha_2| + |\beta_2| \neq 0.$

To state the boundary value problem (1)-(2) as an equivalent integral equation, we consider the homogenous equation

$$-(p(t)x^{\Delta}(t))^{\nabla} = 0. \tag{8}$$

Let $u_1(t)$ and $u_2(t)$ be the solutions of (8) the satisfying the boundary conditions

$$u_1(0) = \beta_1, \quad \lim_{t \to 0^+} u_1^{[\Delta]}(t) = \alpha_1, \tag{9}$$

$$\lim_{t \to \infty} u_2(t) = \beta_2, \quad \lim_{t \to \infty} u_2^{[\Delta]}(t) = -\alpha_2, \tag{10}$$

respectively where $x^{[\Delta]}(t) = p(t)x^{\Delta}(t)$. Clearly

$$u_1(t) = \beta_1 + \alpha_1 \int_0^t \frac{\Delta s}{p(s)} \text{ and } u_2(t) = \beta_2 + \alpha_2 \int_t^{\infty} \frac{\Delta s}{p(s)}.$$

Moreover $u_1(t)$ and $u_2(t)$ satisfy (2). Let us set

$$D = -W_t(u_1, u_2) = u_1^{[\Delta]}(t)u_2(t) - u_1(t)u_2^{[\Delta]}(t) \tag{11}$$

Since the Wronskian of any two solution of (8) is independent of t, while $t \to \infty$ in (11) we also get

$$D = \alpha_2 \lim_{t \to \infty} u_1(t) + \beta_2 \lim_{t \to \infty} u_1^{[\Delta]}(t). \tag{12}$$

Theorem 3.2. *Under the condition $D \neq 0$ and $\int_0^{\infty} \frac{\Delta s}{p(s)} < \infty$ the dynamic equation*

$$(p(t)x^{\Delta}(t))^{\nabla} + h(t) = 0, \tag{13}$$

with the boundary conditions (2) *has a unique solution*

$$x(t) = \int_0^{\infty} G(t,s)h(s)\nabla s, \tag{14}$$

where $h : \mathbb{T} \to E$ is any ∇-integrable function and

$$G(t,s) = \frac{1}{D} \begin{cases} u_1(t)u_2(s), & 0 \leq t < s < \infty, \\ u_1(s)u_2(t), & 0 \leq s < t < \infty. \end{cases} \tag{15}$$

Proof. If $D \neq 0$ then the solutions $u_1(t)$ and $u_2(t)$ of (8) are linearly independent and therefore the general solution of the equation (13) has of the form

$$x(t) = c_1 u_1(t) + c_2 u_2(t) + \frac{1}{D} \int_0^t (u_1(s) u_2(t) - u_1(t) u_2(s)) h(s) \nabla s. \tag{16}$$

Now we try to chose the constants c_1 and c_2 so that the function defined by (16) satisfies (13)-(2). From (16) we obtain the following:

$$\begin{aligned} x^{[\Delta]}(t) &= c_1 u_1^{[\Delta]}(t) + c_2 u_2^{[\Delta]}(t) + \frac{1}{D} \int_0^t (u_1(s) u_2^{[\Delta]}(t) - u_1^{[\Delta]}(t) u_2(s)) h(s) \nabla s, \\ x(0) &= c_1 u_1(0) + c_2 u_2(0) = c_1 \beta_1 + c_2 u_2(0), \\ x^{[\Delta]}(0) &= c_1 u_1^{[\Delta]}(0) + c_2 u_2^{[\Delta]}(0) = c_1 \alpha_1 + c_2 u_2^{[\Delta]}(0). \end{aligned}$$

Using the first boundary condition of (2), we have

$$\alpha_1 [c_1 \beta_1 + c_2 u_2(0)] - \beta_1 [c_1 \alpha_1 + c_2 u_2^{[\Delta]}(0)] = c_2 [\alpha_1 u_2(0) - \beta_1 u_2^{[\Delta]}(0)] = 0.$$

Since $D \neq 0$, we obtain $c_2 = 0$. Therefore

$$\begin{aligned} x(t) &= c_1 u_1(t) + \frac{1}{D} \int_0^t (u_1(s) u_2(t) - u_1(t) u_2(s)) h(s) \nabla s, \\ x^{[\Delta]}(t) &= c_1 u_1^{[\Delta]}(t) + \frac{1}{D} \int_0^t (u_1(s) u_2^{[\Delta]}(t) - u_1^{[\Delta]}(t) u_2(s)) h(s) \nabla s. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{t \to \infty} x(t) &= c_1 \lim_{x \to \infty} u_1(t) + \frac{1}{D} \int_0^\infty (u_1(s) \beta_2 - \lim_{x \to \infty} u_1(t) u_2(s)) h(s) \nabla s, \\ \lim_{t \to \infty} x^{[\Delta]}(t) &= c_1 \lim_{x \to \infty} u_1^{[\Delta]}(t) + \frac{1}{D} \int_0^\infty (-\alpha_2 u_1(s) - \lim_{x \to \infty} u_1^{[\Delta]}(t) u_2(s)) h(s) \nabla s. \end{aligned}$$

Using the second boundary condition of (2) and taking (12) into account, it can be verified that

$$c_1 = \frac{1}{D} \int_0^\infty u_2(s) h(s) \nabla s.$$

Replacing c_1 and c_2 in (16) we obtain that (14) is the solution of the boundary value problem (13)-(2). ∎

By the virtue of Theorem 3.2, finding a solution $x(t)$ of boundary value problem (1)-(2) is equivalent to finding a continuous solution $x(t)$ of the integral equation

$$x(t) = \int_0^\infty G(t,s) f(s, x(s), x^\Delta(s), x^\nabla(s)) \nabla s. \tag{17}$$

For the existence of the solution of the integral equation (17), we propose the corresponding to integral operator:

$$F(x)(t) = \int_0^\infty G(t,s) f(s, x(s), x^\Delta(s), x^\nabla(s)) \nabla s. \tag{18}$$

By our considerations presented above, it follows that the fixed points of the operator F is the solution of (1)-(2). For $x \in C_1^{\Delta\nabla}(\mathbb{T}, E)$, we define the norm of x by $||x|| = \sup\{||x(t)|| : t \in \mathbb{T}\}$.

To fulfill the conditions on the integral operator, we define the followings:

$$M_1 = \frac{1}{D}\left(\beta_1 + \alpha_1 \int_0^\infty \frac{\Delta s}{p(s)}\right) \cdot \left(\beta_2 + \alpha_2 \int_0^\infty \frac{\Delta s}{p(s)}\right), \tag{19}$$

$$M_0 = M_1 \int_0^\infty m(s)\nabla s, \tag{20}$$

where $m : [0, \infty) \to \mathbb{R}^+$ is $L - \nabla$-integrable function,

$$M_2 = \sup_{t\in\mathbb{T}} \left|\frac{1}{p(t)}\right|, \qquad M_3 = \sup_{t\in\mathbb{T}} \left|\frac{1}{p(\rho(t))}\right|, \tag{21}$$

$$\bar{G}(t,\tau) = \int_0^\infty |G(t,s) - G(\tau,s)| \cdot m(s)\nabla s, \quad t, \tau \in J \tag{22}$$

$$\begin{aligned}\tilde{B} = \{x \in C(J,E) : ||x|| \le M_0, ||x(t) - x(\tau)|| \le \bar{G}(t,\tau),\\ ||x^\Delta(t) - x^\Delta(\tau)|| \le \left|\frac{1}{p(\tau)} - \frac{1}{p(t)}\right| \int_0^\infty m(s)\nabla s + \left|\frac{1}{p(\tau)}\right| \int_t^\tau m(s)\nabla s,\\ ||x^\nabla(t) - x^\nabla(\tau)|| \le \left|\frac{1}{p(\rho(\tau))} - \frac{1}{p(\rho(t))}\right| \int_0^\infty m(s)\nabla s + \left|\frac{1}{p(\rho(\tau))}\right| \int_t^\tau m(s)\nabla s\}.\end{aligned}$$

It is easy to verify that $\tilde{B}$ is a closed, bounded and convex subset of E.

Theorem 3.3. *Let f be a bounded and continuous function. Assume that there exist L-∇-integrable functions $m, k : [0, \infty) \to \mathbb{R}^+$ satisfying*

$$||f(t, x, x_1, x_2)|| \le m(t) \quad \textit{for} \quad t \in J, \quad x, x_1, x_2 \in E, \tag{23}$$

$$\alpha(f(I \times A \times B \times C)) \le k(t) \max\{\alpha(A), \alpha(B), \alpha(C)\} \tag{24}$$

and

$$0 < M_i \int_0^\infty k(s)\nabla s < 1 \quad \textit{for } i = 1, 2, 3 \tag{25}$$

for $I \subset J$ and for any bounded subsets A, B, C of E. Then the boundary value problem (1)-(2) *has a solution.*

Proof. Let F be defined as in (18). In order to fulfill the conditions of Mönch fixed point theorem (Theorem 2.9) we first show that the operator F maps $\tilde{B}$ into $\tilde{B}$.

$$\begin{aligned}||F(x)(t)|| &= ||\int_0^\infty G(t,s) f(s, x(s), x^\Delta(s), x^\nabla(s))\nabla s||\\ &\le \int_0^\infty |G(t,s)| \cdot ||f(s, x(s), x^\Delta(s), x^\nabla(s))||\nabla s.\end{aligned}$$

As

$$G(t,s) \le \frac{1}{D} u_1(\infty) u_2(0) = \frac{1}{D}\left(\beta_1 + \alpha_1 \int_0^\infty \frac{\Delta s}{p(s)}\right)\left(\beta_2 + \alpha_2 \int_0^\infty \frac{\Delta s}{p(s)}\right) = M_1,$$

we have

$$||F(x)(t)|| \leq \int_0^\infty M_1||f(s,x(s),x^\Delta(s),x^\nabla(s))||\nabla s \leq M_1 \int_0^\infty m(s)\nabla s = M_0.$$

Consequently we show that the image sets $F(\tilde{B}), F^\Delta(\tilde{B}), F^\nabla(\tilde{B})$ are equicontinuous.

$$\begin{aligned} ||F(x)(t) - F(x)(\tau)|| &\leq \int_0^\infty |G(t,s) - G(\tau,s)|\, ||f(s,x(s),x^\Delta(s),x^\nabla(s))||\nabla s \\ &\leq \int_0^\infty |G(t,s) - G(\tau,s)| m(s)\nabla s = \bar{G}(t,\tau) \end{aligned}$$

Since $\bar{G}(t,\tau) \to 0$ as $t \to \tau$, $F(\tilde{B})$ is equicontinuous.
For the equicontinuity of $F^\Delta(\tilde{B})$, we show that $||F(x)^\Delta(t) - F(x)^\Delta(\tau)|| \to 0$ as $t \to \tau$. Using the Green's function (15), we obtain

$$\begin{aligned} F(x)^\Delta(t) &= \left(\int_0^\infty G(t,s)f(s,x(s),x^\Delta(s),x^\nabla(s))\nabla s\right)^\Delta \\ &= \frac{1}{D}\int_0^t \frac{-\alpha_2}{p(t)} u_1(s)f(s,x(s),x^\Delta(s),x^\nabla(s))\nabla s \\ &\qquad + \frac{1}{D}\int_t^\infty \frac{\alpha_1}{p(t)} u_2(s)f(s,x(s),x^\Delta(s),x^\nabla(s))\nabla s. \end{aligned}$$

Therefore $\forall t,\tau \in J$, we have

$$\begin{aligned} &||F(x)^\Delta(t) - F(x)^\Delta(\tau)|| \\ &\quad\leq \frac{1}{D}\left|\frac{1}{p(\tau)} - \frac{1}{p(t)}\right| \left(\int_0^t \alpha_2||u_1||\cdot||f||\nabla s + \int_t^\infty \alpha_1||u_2||\cdot||f||\nabla s\right) \\ &\qquad + \frac{1}{D}\left|\frac{1}{p(\tau)}\right| \int_t^\tau (\alpha_2||u_1|| + \alpha_1||u_2||)\, ||f||\nabla s \\ &\quad\leq \left|\frac{1}{p(\tau)} - \frac{1}{p(t)}\right| \int_0^\infty m(s)\nabla s + \left|\frac{1}{p(\tau)}\right| \int_t^\tau m(s)\nabla s, \end{aligned}$$

which leads $||F(x)^\Delta(t) - F(x)^\Delta(\tau)|| \to 0$ as $t \to \tau$.

The equicontinuity of $F^\nabla(\tilde{B})$ can be obtained in a similar way.

The continuity of F is the direct result of the continuity of f. Indeed, let $x_n \to x$ in $\tilde{B}$.

$$\begin{aligned} ||F(x_n)(t) - F(x)(t)|| &= ||\int_0^\infty G(t,s)[f(s,x_n,x_n^\Delta,x_n^\nabla) - f(s,x,x^\Delta,x^\nabla)]\nabla s|| \\ &\leq \int_0^\infty |G(t,s)|\cdot||f(s,x_n,x_n^\Delta,x_n^\nabla) - f(s,x,x^\Delta,x^\nabla)||\nabla s, \end{aligned}$$

which implies the continuity of F.

The existence of a fixed point of operator F is guaranteed via Mönch fixed point theorem. For this purpose, we let V be a countable subset of $\tilde{B}$ satisfying the condition

$$\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \text{ for some } x \in \tilde{B}$$

and $V(t) = \{v(t) \in E : v \in V,\ t \in J\}$. Since V is equicontinuous then by Lemma 2.8 the function $t \mapsto v(t) = \alpha(V(t))$ is continuous on J. For any $\varepsilon > 0$ we separate the time scale interval $J = [0, K] \cup [K, \infty)$ into two subintervals in such way that

$$\int_K^\infty G(t,s) f(s, V(s), V^\Delta(s), V^\nabla(s)) \nabla s < \varepsilon.$$

We split the subinterval $[0, K]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = K$ and denote $T_i = [t_i, t_{i+1}],\ \ i = 0, 1, 2, \ldots, m-1$. The mean value theorem for ∇-integrals (Theorem 2.6) implies the following embedding:

$$\begin{aligned}\int_0^K G(t,s) f(s, V, V^\Delta, V^\nabla) \nabla s &= \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} G(t,s) f(s, V, V^\Delta, V^\nabla) \nabla s \\ &\subset \sum_{i=0}^{m-1} \mu_\nabla(T_i) \overline{\text{conv}} \left(G(t, T_i) f(T_i, V(T_i), V^\Delta(T_i), V^\nabla(T_i)) \right).\end{aligned}$$

The definition of operator F, properties of Kuratowski measure of noncompactness, Lebesgue Δ-measure, Lemma 2.8 and the assumption (19), we lead

$$\begin{aligned}\alpha(F(V)(t)) &\le \sum_{i=0}^{m-1} \mu_\nabla(T_i) \alpha \left(G(t, T_i) f(T_i, V(T_i), V^\Delta(T_i), V^\nabla(T_i)) \right) \nabla s \\ &\le \sum_{i=0}^{m-1} \mu_\nabla(T_i) \sup_{s \in T_i} G(t,s) \alpha \left(f(T_i, V(T_i), V^\Delta(T_i), V^\nabla(T_i)) \right) \nabla s \\ &\le M_1 \sum_{i=0}^{m-1} \mu_\nabla(T_i) \sup_{s \in T_i} k(s) \max\{\alpha(V(T_i)), \alpha(V^\Delta(T_i)), \alpha(V^\nabla(T_i))\} \nabla s \\ &\le M_1 \alpha(V(t)) \sum_{i=0}^{m-1} \mu_\nabla(T_i) k(q_i) \nabla s\end{aligned}$$

for all $t \in I_K = [0, K]$ and for some $q_i \in T_i$. Hence as

$$F(V)(t) = \int_0^K G(t,s) f(s, V, V^\Delta, V^\nabla) \nabla s + \int_K^\infty G(t,s) f(s, V, V^\Delta, V^\nabla) \nabla s,$$

we acquire $\alpha(F(V)(t)) \le M_1 \alpha(V(I_K)) \int_0^\infty k(s) \nabla s + \varepsilon$. Since ε is arbitrary

$$\alpha(F(V)(t)) \le M_1 \alpha(V(I_K)) \int_0^\infty k(s) \nabla s. \tag{26}$$

Analogously by the use of (21)

$$\begin{aligned}&\alpha(F(V)^\Delta(t)) \\ &\quad = \alpha \left(\int_0^K G^\Delta(t,s) f(s, V(s), V^\Delta(s), V^\nabla(s)) \nabla s \right) \\ &\quad \le \sum_{i=0}^{m-1} \mu_\nabla(T_i) \alpha \left(G^\Delta(t, T_i) f(T_i, V(T_i), V^\Delta(T_i), V^\nabla(T_i)) \right) \nabla s \\ &\quad \le M_2 \sum_{i=0}^{m-1} \mu_\nabla(T_i) \sup_{s \in T_i} k(s) \cdot \max\{\alpha(V(T_i)), \alpha(V^\Delta(T_i)), \alpha(V^\nabla(T_i))\} \nabla s \\ &\quad \le M_2 \cdot \alpha(V(t)) \sum_{i=0}^{m-1} \mu_\nabla(T_i) k(q_i) \nabla s\end{aligned}$$

for all $t \in I_K = [0, K]$ and for some $q_i \in T_i$. Hence

$$\alpha(F(V)^{\Delta}(t)) \leq M_2\alpha(V(I_K)) \int_0^{\infty} k(s)\nabla s. \tag{27}$$

Moreover

$$\alpha(F(V)^{\nabla}(t)) \leq M_3\alpha(V(I_K)) \int_0^{\infty} k(s)\nabla s \tag{28}$$

can be obtained in a similar way. Therefore Lemma 2.8, equations (26), (27), (28) and the condition (25) enable us to have

$$\begin{aligned}
\alpha(F(V)) &\leq \max\{M_1\alpha(V)\int_0^{\infty} k(s)\nabla s, M_2\alpha(V)\int_0^{\infty} k(s)\nabla s, M_3\alpha(V)\int_0^{\infty} k(s)\nabla s\} \\
&= \max\{M_1\int_0^{\infty} k(s)\nabla s, M_2\int_0^{\infty} k(s)\nabla s, M_3\int_0^{\infty} k(s)\nabla s\} \cdot \alpha(V) \\
&< \alpha(V). \qquad (29)
\end{aligned}$$

Since $\overline{V} = \overline{\mathrm{conv}}(\{x\} \cup F(V))$, then $\alpha(V) \leq \alpha(F(V))$. Therefore $\alpha(V) = 0$, *i.e.*, V is relatively compact.
As a result all the assumption hypothesis of Mönch fixed point theorem are fulfilled and we conclude the fact that F has a fixed point which is the solution of problem (1)-(2). ■

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