Multiplicity of solutions for a biharmonic equation with subcritical or critical growth

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Abstract

We consider the fourth-order problem

$$\begin{cases} \epsilon^4 \Delta^2 u + V(x)u = f(u) + \gamma |u|^{2_{**}-2}u & \text{in } \mathbb{R}^N \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $\epsilon > 0$, $N \geq 5$, V is a positive continuous potential, f is a function with subcritical growth and $\gamma \in \{0,1\}$. We relate the number of solutions with the topology of the set where V attain its minimum values. We consider the subcritical case $\gamma = 0$ and the critical case $\gamma = 1$. In the proofs we apply Ljusternik-Schnirelmann theory.

1 Introduction

In this paper we are concerned with the following class of elliptic Schrödinger biharmonic equation

$$(P) \qquad \begin{cases} \epsilon^4 \Delta^2 u + V(x)u = f(u) + \gamma |u|^{2_{**}-2}u & \text{in } \mathbb{R}^N \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $\epsilon > 0$, $N \geq 5$, V is a positive continuous potential, f is a function with subcritical growth and $\gamma \in \{0,1\}$. To the related second order problems involving either the Laplacian or the p-Laplacian operator, there are so many works

*Supported by CNPq/PQ 300705/2008-5
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Received by the editors in May 2012.
Communicated by J. Mahwin.
2010 *Mathematics Subject Classification* : Primary 35J30; Secondary 35J35.
Key words and phrases : variational methods, biharmonic equations, nontrivial solutions.

dealing with questions like existence, concentration and multiplicity of nontrivial solutions. Among them we could cite [13, 16, 19, 20, 8], in which they deal with existence and concentration of solutions and [4, 5, 7] in which they get multiplicity results by studying some topological information of $V^{-1}(0)$. On this last issue, we can also mention the works of Alves and Figueiredo [2, 3] in which they study the multiplicity and concentration of positive solutions of

$$\begin{cases} -\epsilon^p \Delta_p u + V(x)|u|^{p-2}u = f(u) \ \text{ in } \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N), \ 1 < p < N, \end{cases}$$

where V satisfies a global assumption in the first work and a local condition in the second one. They succeed in use similar arguments to Cingolani and Lazzo in [7] in order to get multiplicity of solutions, taking advantage of the richness of the set $V^{-1}(0)$ in a topological way.

Recently, Pimenta and Soares in [17, 18] studied the existence and the concentration of solutions of (P) assuming that the potential V satisfies a global and a local condition, respectively. In the first work, the authors have been inspired by Rabinowitz in [19] and Wang in [20], in which they use some alternative methods in order to overcome the lack of a maximum principle to the biharmonic operator. Later, they use the penalization method developed by del Pino and Felmer in [8] to prove the same kind of results, but now considering a local type condition in V. The works just described have induced us to wonder if would be possible to prove some similar multiplicity results to [3], but now to the biharmonic problem (P), considering even nonlinearities with critical growth. In this sense, the intend of this work is to give an affirmative answer to this question.

In the first part of this paper we are concerned with the existence of multiple solutions for the fourth-order problem

$$(P_\epsilon) \qquad \begin{cases} \epsilon^4 \Delta^2 u + V(x)u = f(u) \ \text{ in } \mathbb{R}^N \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $\epsilon > 0$, $N \geq 5$ and

$$\Delta^2 u = \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} u + \sum_{i \neq j}^{N} \frac{\partial^4}{\partial x_i^2 x_j^2} u.$$

is the bi-Laplacian operator. In order to make precise assumptions on the continuous potential V we define

$$V_0 := \inf_{x \in \mathbb{R}^N} V(x),$$

$$V_\infty := \liminf_{|x| \to +\infty} V(x).$$

and suppose that V satisfy

(V_0) $V_0 > 0$ and the set

$$M := \{x \in \mathbb{R}^N : V(x) = V_0\}$$

is nonempty.

(V_1) $V_0 < V_\infty$.

Here we consider two cases: $V_\infty < \infty$ or $V_\infty = \infty$. We can state our hypothesis on $f \in C^1(\mathbb{R}, \mathbb{R})$ in the following way.

(f_1) $f(0) = f'(0) = 0$,

(f_2) There exist constants $c_1, c_2 > 0$ and $q \in (2, 2_{**})$, such that $|f(s)| \leq c_1|s| + c_2|s|^{q-1}$, for all $s \in \mathbb{R}$, where $2_{**} = \frac{2N}{N-4}$,

(f_3) There exists $\mu > 2$ such that $0 < \mu F(s) \leq sf(s)$, for all $s \neq 0$,

(f_4) $\dfrac{f(s)}{s}$ is increasing for $s > 0$ and decreasing for $s < 0$,

(f_5) $f'(s)s^2 - f(s)s \geq C|s|^{\bar{q}}$ for $C > 0$, $\bar{q} \in (2, 2_{**})$ and $s \neq 0$.

We recall that, if Y is a closed set of a topological space X, $\mathrm{cat}_X(Y)$ is the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y. We denote by

$$M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \leq \delta\}$$

the closed δ-neighborhood of M and we shall prove the following multiplicity result.

Theorem 1.1. *Suppose that $(V_0) - (V_1)$ and $(f_1) - (f_5)$ hold. Then, for any $\delta > 0$, there exists $\epsilon_\delta > 0$ such that, for any $\epsilon \in (0, \epsilon_\delta)$, the problem (P_ϵ) has at least $\mathrm{cat}_{M_\delta}(M)$ solutions.*

Note that the problem (P_ϵ) has a variational structure and therefore the solutions can be found as critical points of a functional I_ϵ defined on an appropriated subspace of $H^2(\mathbb{R}^N)$. In order to obtain such critical points we use a technique introduced by Benci and Cerami [4], which consists in making precise comparisons between the category of some sublevel sets of I_ϵ and the category of the set M. This kind of argument for a scalar Schrödinger equation has appeared in [7]. Since we are intending to apply Ljusternik-Schnirelmann theory, we need to prove some compactness property for the functional I_ϵ. Following the ideas of [17], [19] and [7], we prove that the levels of compactness are strongly related with the behavior of the potential V at infinity.

In the second part of the paper we deal with a critical version of (P_ϵ), namely the problem

$$(CP_\epsilon), \qquad \begin{cases} \epsilon^4 \Delta^2 u + V(x)u = f(u) + |u|^{2_{**}-2}u \ \text{ in } \mathbb{R}^N \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

In order to deal with the critical growth of the nonlinearity we assume the same technical condition of [15], namely

(f_6) $f(s) \geq \lambda s^{q_1 - 1}$ for all $s \in \mathbb{R}$, with $q_1 \in (2, 2_{**})$ and λ satisfying

- $(f_6 a)$ $\lambda > 0$ if $\max\{\frac{N}{N-4}, \frac{8}{N-4}\} < q_1 < 2_{**}$,
- $(f_6 b)$ λ is sufficiently large if $2 < q_1 \leq \max\{\frac{N}{N-4}, \frac{8}{N-4}\}$.

The critical version of Theorem 1.1 can be stated as follows.

Theorem 1.2. *Suppose that* $(V_0)-(V_1)$ *and* $(f_1)-(f_6)$ *hold. Then , for any* $\delta > 0$ *given, there exists* $\epsilon_\delta > 0$ *such that, for any* $\epsilon \in (0, \epsilon_\delta)$*, the problem* (CP_ϵ) *has at least* $\mathrm{cat}_{M_\delta}(M)$ *solutions.*

The proof of Theorem 1.2 follows the same lines of the subcritical case. However, this new problem has an extra difficulty when compared with the subcritical one. This occurs because the level of non-compactness is affected by the critical growth of the nonlinearity. This problem is overcame by using the ideas of Brezis and Nirenberg [6] with some adaptations of the calculations performed in [21]. We will prove that the number

$$S := \inf_{u \in H^2(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta u|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2_{**}} \mathrm{d}x\right)^{2/2_{**}}}$$

plays an important role when dealing with critical problems as in (CP_ϵ).

2 Variational framework

In order to simplify the notation, we write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$.

Hereafter, we will work with the following problem equivalent to (P_ϵ), which is obtained under the change of variables $z \mapsto \epsilon x$

$$(\widehat{P}_\epsilon) \qquad \begin{cases} \Delta^2 u + V(\epsilon x)u = f(u) \ \text{ in } \mathbb{R}^N \\ u \in H^2(\mathbb{R}^N). \end{cases}$$

For any $\epsilon > 0$, we consider the Sobolev space

$$X_\epsilon := \left\{ u \in H^2(\mathbb{R}^N) : \int V(\epsilon x)|u|^2 < \infty \right\}$$

endowed with the norm

$$\|u\|_\epsilon := \left(\int |\Delta u|^2 + \int V(\epsilon x)|u|^2 \right)^{1/2}.$$

The growth condition (f_2) implies that, for some constant $C > 0$,

$$|F(s)| \le C(|s|^2 + |s|^q) \ \text{ for all } \ s \in \mathbb{R}. \tag{2.1}$$

Hence, the weak solutions of the problem $(\widehat{P}_\epsilon)$ are related with the critical points of the functional $I_\epsilon : X_\epsilon \to \mathbb{R}$ given by

$$I_\epsilon(u) := \frac{1}{2}\|u\|_\epsilon^2 - \int F(u).$$

We introduce the Nehari manifold associated to I_ϵ by setting

$$\mathcal{N}_\epsilon := \{ u \in X_\epsilon \setminus \{0\} : \langle I'_\epsilon(u), u \rangle = 0 \}$$

and define the minimax level c_ϵ as being

$$c_\epsilon := \inf_{u \in \mathcal{N}_\epsilon} I_\epsilon(u).$$

In what follows, we present some properties of c_ϵ and $\mathcal{N}_\epsilon$ whose proofs can be carried out as in [17]. First of all, we note that there exists $r > 0$, independent of ϵ, such that

$$\|u\|_\epsilon \geq r > 0 \quad \text{for any } \epsilon > 0,\ u \in \mathcal{N}_\epsilon. \tag{2.2}$$

Since I_ϵ satisfies the Mountain Pass geometry, from [17], c_ϵ can be alternatively characterized by

$$c_\epsilon = \inf_{\gamma \in \Gamma_\epsilon} \max_{t \in [0,1]} I_\epsilon(\gamma(t)) = \inf_{u \in X_\epsilon \setminus \{0\}} \max_{t \geq 0} I_\epsilon(tu) > 0, \tag{2.3}$$

where $\Gamma_\epsilon := \{\gamma \in C([0,1], X_\epsilon) : \gamma(0) = 0,\ I_\epsilon(\gamma(1)) < 0\}$. Moreover, for any $u \neq 0$, there exists a unique $\bar{t} > 0$ such that $\bar{t}u \in \mathcal{N}_\epsilon$, which has the property that the maximum of $t \mapsto I_\epsilon(tu)$ for $t \geq 0$ is achieved at $t = \bar{t}$.

2.1 The Palais-Smale condition

We start this subsection by recalling the definition of the Palais-Smale condition. Let E be a Banach space, $\mathcal{V}$ be a C^1-manifold of E and $I : E \to \mathbb{R}$ a C^1-functional. We say that $I|_\mathcal{V}$ satisfies the Palais-Smale condition at level d ($(\mathrm{PS})_d$ for short) if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to d$ and $\|I'(u_n)\|_* \to 0$ contains a strongly convergent subsequence. Here, we are denoting by $\|I'(u)\|_*$ the norm of the derivative of I restricted to $\mathcal{V}$ at the point u.

If $V_\infty < \infty$, let us set $X_\infty = \big(H^2(\mathbb{R}^N), \langle \cdot, \cdot \rangle_\infty\big)$, where

$$\langle u, v \rangle_\infty = \int (\Delta u \Delta v + V_\infty uv),$$

is an inner product which gives rise to the norm

$$\|u\|_\infty = \left(\int \left(|\Delta u|^2 + V_\infty |u|^2 \right) \right)^{\frac{1}{2}}.$$

Let us consider the limit functional $I_\infty : X_\infty \to \mathbb{R}$, given by

$$I_\infty(u) := \frac{1}{2} \int \left(|\Delta u|^2 + V_\infty |u|^2 \right) - \int F(u),$$

and denote by c_∞ the ground state level of I_∞, namely

$$c_\infty := \inf_{u \in \mathcal{N}_\infty} I_\infty(u) = \inf_{u \in X_\infty \setminus \{0\}} \max_{t \geq 0} I_\infty(tu) > 0,$$

where $\mathcal{N}_\infty := \{u \in X_\infty \setminus \{0\} : \langle I'_\infty(u), u \rangle = 0\}$. If $V_\infty = \infty$, we set $c_\infty := \infty$.

Now we state an important result which can be found in [17].

Proposition 2.1. *The functional I_ϵ satisfies the $(\mathrm{PS})_d$ condition at any $d < c_\infty$.*

We state below our compactness result for I_ϵ constrained to $\mathcal{N}_\epsilon$.

Proposition 2.2. *The functional I_ϵ constrained to $\mathcal{N}_\epsilon$ satisfies the $(\mathrm{PS})_d$ condition at any level $d < c_\infty$.*

Proof. Let $(u_n) \subset \mathcal{N}_\epsilon$ be such that

$$I_\epsilon(u_n) \to d \text{ and } \|I'_\epsilon(u_n)\|_* \to 0.$$

Then there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_\epsilon(u_n) = \lambda_n J'_\epsilon(u_n) + o_n(1) \quad \text{in } X^*_\epsilon,$$

where $J_\epsilon(u) = \|u\|^2_\epsilon - \int f(u)u$. Hence

$$\begin{aligned} 0 = \langle I'_\epsilon(u_n), (u_n)\rangle &= \lambda_n \langle J'_\epsilon(u_n), (u_n)\rangle + o_n(1) \\ &= \lambda_n \left(\int f(u_n)u_n - \int f'(u_n)u_n^2 \right) + o_n(1). \end{aligned}$$

This expression, (f_5) and (2.2) imply that $\lambda_n \to 0$, and therefore $I'_\epsilon(u_n) \to 0$ in the dual space X^*_ϵ.

Therefore, (u_n) is a $(PS)_d$ sequence for I_ϵ and the result follows by Proposition 2.1. ∎

Corollary 2.3. *The critical points of the functional I_ϵ constrained to $\mathcal{N}_\epsilon$ are critical points of I_ϵ in X_ϵ*

Proof. It suffices to argue as in the second part of the above proof. We omit the details. ∎

3 Proof of Theorem 1.1

As we will see, in order to relate the Ljusternik-Schnirelmann category of the sub-levels of I_ϵ and of the subset M_δ, an important role will be played by the ground-state solution of an autonomous problem. More precisely, let us consider

$$(A) \qquad \begin{cases} \Delta^2 u + V_0 u = f(u) \text{ in } \mathbb{R}^N \\ u \in H^2(\mathbb{R}^N). \end{cases}$$

We will denote by X_0 the space $H^2(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_0 := \left(\int \left(|\Delta u|^2 + V_0 |u|^2 \right) \right)^{1/2}$$

and let us consider $I_0 : X_0 \to \mathbb{R}$ the functional given by

$$I_0(u) := \frac{1}{2}\|u\|^2_0 - \int F(u).$$

Associated with I_0 we have the minimax level

$$c_0 := \inf_{u \in \mathcal{N}_0} I_0(u) = \inf_{u \in X_0 \setminus \{0\}} \max_{t \geq 0} I_0(tu) > 0,$$

where $\mathcal{N}_0 := \{u \in X_0 \setminus \{0\} : \langle I_0'(u), u \rangle = 0\}$ is the Nehari manifold associated to I_0. As one can see, in [17] the authors succeeded in proving that this minimax level is achieved.

Let us state a technical result.

Lemma 3.1. *Let $\epsilon_n \to 0^+$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $I_{\epsilon_n}(u_n) \to c_0$. Then there exists $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that the translated sequence*

$$(\widetilde{u}_n(x)) := (u_n(x + \widetilde{y}_n))$$

has a strongly convergent subsequence in X_0. Moreover, up to a subsequence, $(y_n) := (\epsilon_n \widetilde{y}_n)$ is such that $y_n \to y \in M$.

Proof. Since $\langle I_{\epsilon_n}'(u_n), (u_n) \rangle = 0$ and $I_{\epsilon_n}(u_n) \to c_0$, the sequence (u_n) is easily shown to be bounded in X_ϵ. Moreover, since $c_0 > 0$, we cannot have $u_n \to 0$. Hence, using Lemma I.1 of [14], we obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \text{ in } H^2(\mathbb{R}^N),$$

where $(\widetilde{u}_n(x)) := (u_n(x + \widetilde{y}_n))$ and $\widetilde{u} \neq 0$.

Let $(t_n) \subset (0, +\infty)$ be such that $\widehat{u}_n := t_n \widetilde{u}_n \in \mathcal{N}_0$. If we set $y_n := \epsilon_n \widetilde{y}_n$ we can use the change of variables $z \mapsto x + \widetilde{y}_n$ to get

$$\begin{aligned} I_0(\widehat{u}_n) &\leq \frac{t_n^2}{2} \int |\Delta \widetilde{u}_n|^2 - \int F(t_n \widetilde{u}_n) \\ &\quad + \frac{t_n^2}{2} \int V(\epsilon_n (x + \widetilde{y}_n)) |\widetilde{u}_n|^2 \\ &= I_{\epsilon_n}(t_n u_n) \leq I_{\epsilon_n}(u_n) = c_0 + o_n(1). \end{aligned}$$

Since $c_0 \leq I_0(\widehat{u}_n)$ we conclude that $I_0(\widehat{u}_n) \to c_0$.

Since $(\widetilde{u}_n)$ and $(\widehat{u}_n)$ are bounded and $\widetilde{u}_n \not\to 0$, the sequence (t_n) is bounded. Thus, up to a subsequence, $t_n \to t_0 \geq 0$. If $t_0 = 0$ were true, by using the boundedness of $(\widetilde{u}_n)$ we would obtain that $(\widehat{u}_n) = t_n \widetilde{u}_n \to 0$, which contradicts the fact that $I_0(\widehat{u}_n) \to c_0 > 0$. Thus, $t_0 > 0$. We notice that, up to a subsequence, $\widehat{u}_n \rightharpoonup t_0 \widetilde{u} = \widehat{u}$ weakly in X_0. Since $t_0 > 0$ and $\widetilde{u} \neq 0$, we have concluded that

$$I_0(\widehat{u}_n) \to c_0 \quad \text{and} \quad \widehat{u}_n \rightharpoonup \widehat{u} \neq 0 \text{ weakly in } X_0.$$

We can now use the same calculations performed in [1, Theorem 3.1] to conclude that $\widehat{u}_n \to \widehat{u}$ in X_0, which implies that $\widetilde{u}_n \to \widetilde{u}$ in X_0.

It remains to show that (y_n) has a subsequence such that $y_n \to y \in M$. We start by proving that (y_n) is bounded. Indeed, suppose by contradiction that there exists a subsequence, still denoted by (y_n), such that $|y_n| \to +\infty$. A contradiction will be obtained in each of the following cases:

Case 1: $V_\infty = \infty$.

Since $(u_n) \in \mathcal{N}_{\epsilon_n}$

$$\int f(u_n(x+\widetilde{y}_n))u_n(x+\widetilde{y}_n) \quad \geq \quad \int V(\epsilon_n x + y_n)|u_n(x+\widetilde{y}_n)|^2.$$

Applying Fatou's lemma we obtain

$$\liminf_{n\to\infty} \int f(u_n(x+\widetilde{y}_n))u_n(x+\widetilde{y}_n) \geq \infty.$$

On the other hand, the boundedness of (u_n) and (f_2) imply that the left hand side in the above expression is bounded. Thus, we obtain a contradiction.

Case 2: $V_\infty < \infty$.

In this case, since $\widehat{u}_n \to \widehat{u}$ strongly in X_0 and $V_0 < V_\infty$ we have

$$\begin{aligned} c_0 &= I_0(\widehat{u}) < I_\infty(\widehat{u}) \\ &\leq \liminf_{n\to\infty}\left[\frac{1}{2}\int |\Delta \widehat{u}_n|^2 - \int F(\widehat{u}_n) \right. \\ &\quad \left. + \frac{1}{2}\int V(\epsilon_n x + y_n)|\widehat{u}_n|^2\right] \\ &= \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\epsilon_n}(u_n) = c_0, \end{aligned} \tag{3.1}$$

which does not make sense.

Then we conclude that (y_n) is bounded and therefore, up to a subsequence, $y_n \to y$. If $y \notin M$ then $V_0 < V(y)$ and we have that

$$c_0 < \frac{1}{2}\int \left(|\Delta \widehat{u}|^2 + V(y)|\widehat{u}|^2\right) - \int F(\widehat{u}).$$

This inequality and the same kind of calculations performed in (3.1) provide a contradiction. Thus, $y \in M$ and we succeeded in proving the lemma. ∎

Fix $\delta > 0$ and choose a cut-off function $\eta \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\eta(s) = 1$ if $0 \leq s \leq \delta/2$ and $\eta(s) = 0$ if $s \geq \delta$. Let $\omega \in X_0$ be the solution of (A). For each $y \in M$ we define

$$\Psi_{\epsilon,y}(x) := \eta(|\epsilon x - y|)\omega\left(\frac{\epsilon x - y}{\epsilon}\right).$$

If t_ϵ denotes the unique positive number satisfying

$$\max_{t\geq 0} I_\epsilon(t\Psi_{\epsilon,y}) = I_\epsilon(t_\epsilon \Psi_{\epsilon,y}),$$

we introduce the map $\Phi_\epsilon : M \to \mathcal{N}_\epsilon$ by setting

$$\Phi_\epsilon(y) := (t_\epsilon \Psi_{\epsilon,y}).$$

Since $I_0(\omega) = c_0$, by using Lebesgue's theorem and the compactness of M, we can proceed as in [12] to check that

$$\lim_{\epsilon\to 0^+} I_\epsilon(\Phi_\epsilon(y)) = c_0, \text{ uniformly for } y \in M. \tag{3.2}$$

We take now $\rho = \rho_\delta > 0$ such that $M_\delta \subset B_\rho(0)$ and consider $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ defined as $\Upsilon(x) := x$ for $|x| < \rho$ and $\Upsilon(x) := \rho x/|x|$ for $|x| \geq \rho$. We define the barycenter map $\beta_\epsilon : \mathcal{N}_\epsilon \to \mathbb{R}^N$ as being

$$\beta_\epsilon(u) := \frac{\displaystyle\int \Upsilon(\epsilon x)|u(x)|^2}{\displaystyle\int |u(x)|^2}.$$

Lemma 3.2. *The function Φ_ϵ satisfies*

$$\lim_{\epsilon\to 0^+} \beta_\epsilon(\Phi_\epsilon(y)) = y \;\; \text{uniformly for } y \in M. \tag{3.3}$$

Proof. Suppose the assertion of the lemma is false. Then, there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| \geq \delta_0.$$

By using the change of variables $z := (\epsilon_n x - y_n)/\epsilon_n$, we can write

$$\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = y_n + \frac{\displaystyle\int_{\mathbb{R}^N} (\Upsilon(\epsilon_n z + y_n) - y_n)\, |\eta(|\epsilon_n z|)|^2 |\omega(z)|^2 \mathrm{d}z}{\displaystyle\int_{\mathbb{R}^N} |\eta(|\epsilon_n z|)|^2 |\omega(z)|^2 \mathrm{d}z}.$$

Since $M \subset B_\rho(0)$ and $\Upsilon|_{B_\rho(0)} \equiv \mathrm{Id}$, we can use the above expression and Lebesgue's theorem to conclude that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts our assumption and hence proves the lemma. ■

Following [7], we take a function $h : [0,\infty) \to [0,\infty)$ such that $h(\epsilon) \to 0$ as $\epsilon \to 0^+$ and set

$$\Sigma_\epsilon := \{u \in \mathcal{N}_\epsilon : I_\epsilon(u) \leq c_0 + h(\epsilon)\}.$$

Given $y \in M$, we can use (3.2) to conclude that $h(\epsilon) = |I_\epsilon(\Phi_\epsilon(y)) - c_0|$ is such that $h(\epsilon) \to 0^+$ as $\epsilon \to 0$. Thus, $\Phi_\epsilon(y) \in \Sigma_\epsilon$ and we have that $\Sigma_\epsilon \neq \varnothing$ for any $\epsilon > 0$. Moreover, the following holds

Lemma 3.3. *For any $\delta > 0$ we have that*

$$\lim_{\epsilon\to 0^+} \sup_{u\in\Sigma_\epsilon} \mathrm{dist}(\beta_\epsilon(u), M_\delta) = 0.$$

Proof. Let $(\epsilon_n) \subset \mathbb{R}$ be such that $\epsilon_n \to 0^+$. By definition, there exists $(u_n) \subset \Sigma_{\epsilon_n}$ such that

$$\mathrm{dist}(\beta_{\epsilon_n}(u_n), M_\delta) = \sup_{u\in\Sigma_{\epsilon_n}} \mathrm{dist}(\beta_{\epsilon_n}(u), M_\delta) + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\epsilon_n}(u_n) - y_n| = o_n(1). \tag{3.4}$$

Since $(u_n) \subset \Sigma_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$, we have that

$$c_0 \leq c_{\epsilon_n} \leq I_{\epsilon_n}(u_n) \leq c_0 + h(\epsilon_n),$$

and therefore $I_{\epsilon_n}(u_n) \to c_0$. We may now invoke Lemma 3.1 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) := (\epsilon_n \widetilde{y}_n) \subset M_\delta$. We set

$$(\widetilde{u}_n(x)) := (u_n(\epsilon_n x + \widetilde{y}_n))$$

and observe that, since $\widetilde{u}_n \to u$ in X_0 and $\epsilon_n x + y_n \to y \in M$, a direct calculation shows that $\beta_{\epsilon_n}(u_n) = y_n + o_n(1)$. Hence, the lemma is proved. ∎

We are now ready to present the proof of the multiplicity result in the subcritical case.

Proof of Theorem 1.1. Given $\delta > 0$ we can use (3.2), (3.3), Lemma 3.3, and argue as in [7, Section 6] to obtain $\epsilon_\delta > 0$ such that, for any $\epsilon \in (0, \epsilon_\delta)$, the diagram

$$M \xrightarrow{\Phi_\epsilon} \Sigma_\epsilon \xrightarrow{\beta_\epsilon} M_\delta$$

is well defined and $\beta_\epsilon \circ \Phi_\epsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Using the definition of Σ_ϵ and taking ϵ_δ small if necessary, we may suppose that I_ϵ satisfies the Palais-Smale condition in Σ_ϵ. Standard Ljusternik-Schnirelmann theory provides at least $\mathrm{cat}_{\Sigma_\epsilon}(\Sigma_\epsilon)$ critical points u_i of I_ϵ restricted to $\mathcal{N}_\epsilon$. The same ideas contained in the proof of [5, Lemma 4.3] show that $\mathrm{cat}_{\Sigma_\epsilon}(\Sigma_\epsilon) \geq \mathrm{cat}_{M_\delta}(M)$. By using Corollary 2.3 we conclude that u_i is a solution of $(\widehat{P}_\epsilon)$ and this proves the theorem. ∎

4 The critical case

In this section, in order to avoid repetition we just describe the differences between the critical and subcritical cases, since the calculations are almost the same.

We first consider the critical version of problem (A), namely

$$(CA) \qquad \begin{cases} \Delta^2 u + V_0 u = f(u) + |u|^{2_{**}-2}u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N) \end{cases}$$

whose solutions are related with the critical points of $J_0 : X_0 \to \mathbb{R}$ defined as

$$J_0(u) := \frac{1}{2}\|u\|_0^2 - \int F(u) - \frac{1}{2_{**}} \int |u|^{2_{**}}.$$

We denote by m_0 the ground state level of J_0, that is,

$$m_0 := \inf_{u \in X_0 \setminus \{0\}} \max_{t \geq 0} J_0(tu) > 0.$$

As usual, we denote by S the best constant of the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{2_{**}}(\mathbb{R}^N)$. By Gazzola and Berchio [10],

$$S := \inf_{u \in H^2(\mathbb{R}^N) \setminus \{0\}} \frac{\int |\Delta u|^2}{\left(\int |u|^{2_{**}} \right)^{2/2_{**}}}.$$

Lemma 4.1. *Let $(u_n) \subset X_0$ be a* $(\mathrm{PS})_d$ *sequence for the functional J_0 with $d < \frac{2}{N} S^{N/4}$. Then we have either*

(i) $\|u_n\|_0 \to 0$, *or*

(ii) *there exists a sequence $(y_n) \subset \mathbb{R}^N$ and constants R, $\gamma > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \geq \gamma > 0.$$

Proof. Suppose that (ii) does not hold. Using Lemma I.1 in [14], we can prove that $\int f(u_n)u_n = o_n(1)$ and $\int F(u_n) = o_n(1)$. Since (u_n) is bounded, we get $\langle J_0'(u_n), (u_n)\rangle \to 0$. Taking a subsequence, we obtain $l \geq 0$ such that

$$\|u_n\|_0^2 \to l \quad \text{and} \quad \int |u_n|^{2_{**}} \to l. \tag{4.1}$$

Since $J_0(u_n) \to d$, we can use (4.1) to conclude that $l = \frac{N}{2} d$. Recalling the definition of S we get

$$\|u_n\|_0^2 \geq S\left(\int |u_n|^{2_{**}}\right)^{2/2_{**}}.$$

Taking the limit we conclude that $l \geq S l^{2/2_{**}}$. If $l > 0$ we obtain

$$\frac{N}{2} d = l \geq S^{N/4},$$

which does not make sense. Hence $l = 0$ and therefore (i) holds. ■

Proposition 4.2. *The problem (CA) has a nontrivial weak solution.*

Proof. Since J_0 has the Mountain Pass geometry, there exists $(u_n) \subset X_0$ such that

$$J_0(u_n) \to m_0 \quad \text{and} \quad J_0'(u_n) \to 0.$$

We claim that the number m_0 satisfies

$$m_0 < \frac{2}{N} S^{N/4}.$$

Assuming for a moment that this is true, we can use Lemma 4.1 and argue as in the proof of Theorem 4.23 in [19] to obtain the desired solution.

What is left is to show that $m_0 < \frac{2}{N} S^{\frac{N}{4}}$. In view of the definition of m_0 it suffices to obtain $u \in X_0$ such that

$$\max_{t \geq 0} J_0(tu) < \frac{2}{N} S^{N/4}.$$

We proceed as in [11, Lemma 3] and firstly recall (see [9]) that, for any $\delta > 0$, the instanton

$$w_\delta(x) := C_N \delta^{(N-4)/2} \left(\delta^2 + |x|^2\right)^{(4-N)/2},$$

satisfies the problem

$$\begin{cases} \Delta^2 w = |w|^{2_{**}-2} w \ \text{ in } \mathbb{R}^N, \\ w \in D^{2,2}(\mathbb{R}^N), \quad w(x) > 0 \ \text{ for all } x \in \mathbb{R}^N, \end{cases}$$

and

$$\int |\Delta w_\delta|^2 = S \quad \text{and} \quad \int |w_\delta|^{2_{**}} = 1.$$

Let $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta \equiv 1$ on $B_R(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_{2R}(0)$. Setting

$$\psi_\delta(x) := \frac{\eta(x) w_\delta(x)}{\|\eta w_\delta\|_{2_{**}}},$$

we can use the definition of ψ_δ and (f_6) to get

$$J_0(t\psi_\delta) \leq \frac{t^2}{2} D_\delta - \frac{t^{2_{**}}}{2_{**}} - \lambda t^{q_1} \int_{B_{2R}(0)} |\psi_\delta|^{q_1},$$

where $q_1 \in (2, 2_{**})$ is given by condition (f_6) and

$$D_\delta = \int |\Delta \psi_\delta|^2 + V_0 |\psi_\delta|^2.$$

Let $h_\delta(t)$ be the t-function on the right hand side of the above expression and denote by t_δ the maximum point of h_δ on $(0, \infty)$. Since $h'_\delta(t_\delta) = 0$ we have that

$$\bar{t}_\delta := D_\delta^{1/(2_{**}-2)} \geq t_\delta > 0.$$

Since the function $t \mapsto t^2 D_\delta / 2 - t^{2_{**}} / 2_{**}$ is increasing in $(0, \bar{t}_\delta)$, we can use the definition of h_δ to get

$$h_\delta(t_\delta) \leq \frac{2}{N} D_\delta^{N/4} - C \lambda t^{q_1} \int_{B_{2R}(0)} |\psi_\delta|^{q_1}. \tag{4.2}$$

If $a, b \geq 0$ and $s \geq 1$, then $(a+b)^s \leq a^s + s(a+b)^{s-1} b$. Therefore, there exists $C_1 > 0$ such that

$$D_\delta^{N/4} \leq S^{N/4} + O(\delta^{(N-4)}) + C_1 \int_{B_2(0)} |\psi_\delta|^2.$$

Moreover, we can obtain $\rho > 0$ such that $t_\delta > \rho$ for any δ small. Hence, it follows from the above inequality and (4.2) that

$$h_\delta(t_\delta) \leq \frac{2}{N} S^{N/4} + \delta^{(N-4)} \left[C_2 + \frac{C_3}{\delta^{(N-4)}} \left(\int_{B_2(0)} |\psi_\delta|^2 - \lambda C_4 |\psi_\delta|^{q_1} \right) \right],$$

for positive constants C_2, C_3 and C_4. In view of the hypotheses on $\lambda > 0$ given in (f_6), we can argue as in the proof of [15, Claim 2] to check that, if δ is sufficiently small, the second term in the right hand side above is negative. Thus,

$$\max_{t \geq 0} J_0(t\psi_\delta) \leq \max_{t \geq 0} h_\delta(t) = h_\delta(t_\delta) < \frac{2}{N} S^{N/4}$$

and the proposition is proved. ∎

In order to obtain solutions for (CP_ϵ) we will consider the problem

$$\begin{cases} \Delta^2 u + V(\epsilon x)u = f(u) + |u|^{2_{**}-2}u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N) \end{cases} \qquad (\widehat{CP}_\epsilon)$$

and look for critical points of the functional $J_\epsilon : X_\epsilon \to \mathbb{R}$ given by

$$J_\epsilon(u) := \frac{1}{2}\|u\|_\epsilon^2 - \int F(u) - \int |u|^{2_{**}},$$

where X_ϵ as in Section 2.

The critical points of J_ϵ belong to the Nehari manifold

$$\mathcal{M}_\epsilon := \{u \in X_\epsilon \setminus \{0\} : \langle J'_\epsilon(u)u\rangle = 0\}$$

and the ground state level is given by

$$m_\epsilon := \inf_{u\in\mathcal{M}_\epsilon} J_\epsilon(u) = \inf_{u\in X_\epsilon\setminus\{0\}} \max_{t\geq 0} J_\epsilon(tu) > 0.$$

As before, the Palais-Smale condition for the functional J_ϵ is related with V_∞. When this quantity is finite we define the limit functional $J_\infty : X_\infty \to \mathbb{R}$ as being

$$J_\infty(u) := \frac{1}{2}\int \left(|\Delta u|^2 + V_\infty |u|^2\right) - \int F(u) - \frac{1}{2_{**}}\int |u|^{2_{**}},$$

and its ground state level

$$m_\infty := \inf_{u\in X_\infty\setminus\{0\}} \max_{t\geq 0} J_\infty(tu) > 0.$$

If $V_\infty = \infty$, we set $m_\infty := \infty$.

Since the function $u \mapsto \int |u|^{2_{**}}$ is 2_{**}-homogeneous, we can argue as in Subsection 2.1 to get a compactness result for the functional J_ϵ. We only notice that, in some arguments, we need to use an analogous of Lemma 4.1 to J_ϵ, rather than Lions Lemma. Hence, the following result holds.

Proposition 4.3. *The functional J_ϵ constrained to $\mathcal{M}_\epsilon$ satisfies the* $(\mathrm{PS})_d$ *condition at any level $d < \min\{m_\infty, 2S^{N/4}/N\}$. Moreover, critical points of J_ϵ constrained to $\mathcal{M}_\epsilon$ are critical points of J_ϵ in X_ϵ.*

We are now ready to prove our second multiplicity result.

Proof of Theorem 1.2. Since the proof is very similar to that presented for Theorem 1.1, we only sketch it. Fix $\delta > 0$ and choose $\eta \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\eta(s) = 1$ if $0 \leq s \leq \delta/2$ and $\eta(s) = 0$ if $s \geq \delta$. Let $\widetilde{\omega} \in X_0$ be the ground-state solution of (CA) given by Proposition 4.2 and define, for each $y \in M$,

$$\widetilde{\Psi}_{\epsilon,y}(x) := \eta(|\epsilon x - y|)\widetilde{\omega}\left(\frac{\epsilon x - y}{\epsilon}\right).$$

We introduce the map $\widetilde{\Phi}_\epsilon : M \to \mathcal{M}_\epsilon$ by setting

$$\widetilde{\Phi}_\epsilon(y) := \widetilde{t}_\epsilon \widetilde{\Psi}_{\epsilon,y},$$

where $\widetilde{t}_\epsilon$ is the unique positive number satisfying

$$\max_{t\geq 0} J_\epsilon(t\widetilde{\Psi}_{\epsilon,y}) = J_\epsilon(\widetilde{t}_\epsilon\widetilde{\Psi}_{\epsilon,y}).$$

The following holds

$$\lim_{\epsilon\to 0^+} J_\epsilon(\widetilde{\Phi}_\epsilon(y)) = m_0 \text{ uniformly for } y \in M.$$

Let $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ be the function defined in Section 3 and consider the barycenter map $\widetilde{\beta}_\epsilon : \mathcal{M}_\epsilon \to \mathbb{R}^N$ given by

$$\widetilde{\beta}_\epsilon(u) := \frac{\displaystyle\int \Upsilon(\epsilon x)|u(x)|^2 \mathrm{d}x}{\displaystyle\int |u(x)|^2 \mathrm{d}x}.$$

As before we can check that

$$\lim_{\epsilon\to 0^+} \widetilde{\beta}_\epsilon(\widetilde{\Phi}_\epsilon(y)) = y \ \text{ uniformly for } y \in M$$

and

$$\lim_{\epsilon\to 0^+} \sup_{u\in\widetilde{\Sigma}_\epsilon} \operatorname{dist}(\widetilde{\beta}_\epsilon(u), M_\delta) = 0,$$

where

$$\widetilde{\Sigma}_\epsilon := \{u \in \mathcal{M}_\epsilon : J_\epsilon(u) \leq m_0 + \widetilde{h}(\epsilon)\}$$

and $\widetilde{h} : [0,\infty) \to [0,\infty)$ satisfies $\widetilde{h}(\epsilon) \to 0$ as $\epsilon \to 0^+$.

The above equations provide $\epsilon_\delta > 0$ such that, for any $\epsilon \in (0, \epsilon_\delta)$, the diagram

$$M \xrightarrow{\widetilde{\Phi}_\epsilon} \widetilde{\Sigma}_\epsilon \xrightarrow{\widetilde{\beta}_\epsilon} M_\delta$$

is well defined and $\widetilde{\beta}_\epsilon \circ \widetilde{\Phi}_\epsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Hence we conclude that $\operatorname{cat}_{\widetilde{\Sigma}_\epsilon}(\widetilde{\Sigma}_\epsilon) \geq \operatorname{cat}_{M_\delta}(M)$. In view of Proposition 4.3 and recalling that

$$m_0 < \frac{2}{N} S^{N/4},$$

we may suppose that ϵ_δ is small in such a way that J_ϵ satisfies the Palais-Smale condition in $\widetilde{\Sigma}_\epsilon$. The proof now follows from Ljusternik-Schnirelmann theory and the same arguments used in the subcritical case. ∎

Acknowledgment. The authors are grateful to Prof. Sérgio H. M. Soares to valuable discussions about this subject. Marcos T.O. Pimenta was supported by FAPESP (2012/20160-0) and Giovany J.M. Figueiredo was supported by PROCAD/CASADINHO (552101/2011-7), CNPq/PQ (301242/2011-9) and CNPQ/CSF (200237/2012-8).

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