Existence of Solutions for Some Boundary Value Problems of Fractional p-Laplacian Equation at Resonance *

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Abstract

This paper considers the existence of solutions for two boundary value problems of fractional p-Laplacian equation at resonance. Under certain nonlinear growth condition of the nonlinearity, two new existence results are obtained by using the coincidence degree theory. As an application, an example to illustrate our results is given.

1   Introduction

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. This subject, as old as the problem of ordinary differential calculus, can go back to the times when Leibniz and Newton invented differential calculus. As is known to all, the problem for fractional derivative was originally raised by Leibniz in a letter, dated September 30, 1695. Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode analysis of feedback amplifiers, capacitor theory, electrical

circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, etc. [1-4]. For instance, Pereira et al. [5] considered the following fractional Van der Pol equation

$$D^\lambda x(t) + \alpha(x^2(t) - 1)x'(t) + x(t) = 0, \ \ 1 < \lambda < 2, \tag{1.1}$$

where D^λ is the fractional derivative of order λ and α is a control parameter that reflects the degree of nonlinearity of the system. Equation (1.1) is obtained by substituting the capacitance by a fractance in the nonlinear RLC circuit model.

Recently, fractional differential equations have been of great interest due to the intensive development of the theory of fractional calculus itself and its applications. For example, for fractional initial value problems, the existence and multiplicity of solutions (or positive solutions) were discussed in [6-9]. On the other hand, for fractional boundary value problems (FBVPs for short), Agarwal et al. [10] considered a two-point boundary value problem at nonresonance, and Bai [11] considered a m-point boundary value problem at resonance. For more papers on FBVPs, see [12-18] and the references therein.

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problems, Leibenson [19] introduced the p-Laplacian equation as follows

$$(\phi_p(x'(t)))' = f(t, x(t), x'(t)), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s, \ p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $1/p + 1/q = 1$.

In the past few decades, many important results relative to equation (1.2) with certain boundary value conditions have been obtained. We refer the reader to [20-26] and the references cited therein. However, to the best of our knowledge, there are relatively few results on boundary value problems for fractional p-Laplacian equations at resonance.

Motivated by the works mentioned previously, in this paper, we investigate the existence of solutions for fractional p-Laplacian equation of the form

$$D_{0+}^\beta \phi_p(D_{0+}^\alpha x(t)) = f(t, x(t), D_{0+}^\alpha x(t)), \ \ t \in [0, 1] \tag{1.3}$$

subject to either boundary value conditions

$$x(0) = 0, \ \ D_{0+}^\alpha x(0) = D_{0+}^\alpha x(1) \tag{1.4}$$

or

$$x(1) = 0, \ \ D_{0+}^\alpha x(0) = D_{0+}^\alpha x(1), \tag{1.5}$$

where $0 < \alpha, \beta \leq 1, \ 1 < \alpha + \beta \leq 2, D_{0+}^\alpha$ is a Caputo fractional derivative, and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Note that, the nonlinear operator $D_{0+}^\beta \phi_p(D_{0+}^\alpha)$ reduces to the linear operator $D_{0+}^\beta D_{0+}^\alpha$ when $p = 2$ and the additive index law

$$D_{0+}^\beta D_{0+}^\alpha u(t) = D_{0+}^{\alpha+\beta} u(t)$$

holds under some reasonable constraints on the function $u(t)$ [27]. Moreover, FBVP (1.3)(1.4) (or FBVP (1.3)(1.5)) happens to be at resonance in the sense that its associated linear homogeneous boundary value problem

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = 0, \ t \in [0,1], \\ x(0) = 0, \ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1) \ (\text{or } x(1) = 0, \ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1)) \end{cases}$$

has a nontrivial solution $x(t) = ct^{\alpha}$ (or $x(t) = c(1 - t^{\alpha})$), where $c \in \mathbb{R}$.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3 and Section 4, basing on the coincidence degree theory of Mawhin [28], we establish two theorems on existence of solutions for FBVP (1.3)(1.4) (Theorem 3.1) and FBVP (1.3)(1.5) (Theorem 4.1) under nonlinear growth restriction of f. Finally, in Section 5, an example is given to illustrate the main results. Our results are different from those of bibliographies listed in the previous texts.

2 Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory, which can be found, for instance, in [2,4].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha}u(t) = I_{0+}^{n-\alpha}\frac{d^n u(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.1 [1]. Let $\alpha > 0$. Assume that $u, D_{0+}^{\alpha}u \in L(0,1)$. Then the following equality holds

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$, here n is the smallest integer greater than or equal to α.

Now, we briefly recall some notations and an abstract existence result, which can be found in [28].

Let X, Y be real Banach spaces, $L : \mathrm{dom} L \subset X \to Y$ be a Fredholm operator with index zero, and $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\mathrm{Im} P = \mathrm{Ker} L, \ \ \mathrm{Ker} Q = \mathrm{Im} L, \ \ X = \mathrm{Ker} L \oplus \mathrm{Ker} P, \ \ Y = \mathrm{Im} L \oplus \mathrm{Im} Q.$$

It follows that

$$L|_{\mathrm{dom} L \cap \mathrm{Ker} P} : \mathrm{dom} L \cap \mathrm{Ker} P \to \mathrm{Im} L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X such that $\mathrm{dom} L \cap \overline{\Omega} \neq \varnothing$, then the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.2 [28]. Let $L : \mathrm{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied
   (1) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\mathrm{dom} L \setminus \mathrm{Ker} L) \cap \partial \Omega] \times (0, 1)$;
   (2) $Nx \notin \mathrm{Im} L$ for every $x \in \mathrm{Ker} L \cap \partial \Omega$;
   (3) $\deg(QN|_{\mathrm{Ker} L}, \Omega \cap \mathrm{Ker} L, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $\mathrm{Im} L = \mathrm{Ker} Q$.
Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom} L \cap \overline{\Omega}$.

In this paper, we take $Y = C[0, 1]$ with the norm $\|y\|_\infty = \max_{t \in [0,1]} |y(t)|$, and $X = \{x | x, D_{0+}^\alpha x \in Y\}$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|D_{0+}^\alpha x\|_\infty\}$. By means of the linear functional analysis theory, we can prove that X is a Banach space.

3 Existence of Solutions for FBVP (1.3)(1.4)

In this section, a theorem on existence of solutions for FBVP (1.3)(1.4) will be given.

Define the operator $L : \mathrm{dom} L \subset X \to Y$ by

$$Lx = D_{0+}^\beta \phi_p(D_{0+}^\alpha x), \tag{3.1}$$

where

$$\mathrm{dom} L = \{x \in X | D_{0+}^\beta \phi_p(D_{0+}^\alpha x) \in Y, \ x(0) = 0, D_{0+}^\alpha x(0) = D_{0+}^\alpha x(1)\}.$$

Let $N : X \to Y$ be the Nemytskii operator

$$Nx(t) = f(t, x(t), D_{0+}^\alpha x(t)), \ \ \forall t \in [0, 1]. \tag{3.2}$$

Then FBVP (1.3)(1.4) is equivalent to the operator equation

$$Lx = Nx, \ \ x \in \mathrm{dom} L.$$

Now, we begin with some lemmas that are useful in what follows.

Lemma 3.1. Let L be defined by (3.1), then

$$\text{Ker}L = \{x \in X | x(t) = ct^\alpha, \forall t \in [0,1], c \in \mathbb{R}\}, \tag{3.3}$$

$$\text{Im}L = \left\{ y \in Y | \int_0^1 (1-s)^{\beta-1} y(s) ds = 0 \right\}. \tag{3.4}$$

Proof. By Lemma 2.1, $D_{0+}^\beta \phi_p(D_{0+}^\alpha x(t)) = 0$ has solution

$$x(t) = c_0 + I_{0+}^\alpha \phi_q(c_1) = c_0 + \frac{\phi_q(c_1)}{\Gamma(\alpha+1)} t^\alpha, \quad c_0, c_1 \in \mathbb{R}.$$

Thus, from the boundary value condition $x(0) = 0$, one has that (3.3) holds.

If $y \in \text{Im}L$, then there exists a function $x \in \text{dom}L$ such that $y(t) = D_{0+}^\beta \phi_p(D_{0+}^\alpha x(t))$. Basing on Lemma 2.1, we have

$$\phi_p(D_{0+}^\alpha x(t)) = I_{0+}^\beta y(t) + c = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds + c, \quad c \in \mathbb{R}.$$

By condition $D_{0+}^\alpha x(0) = D_{0+}^\alpha x(1)$, one has

$$\int_0^1 (1-s)^{\beta-1} y(s) ds = 0. \tag{3.5}$$

Thus, we get (3.4).

On the other hand, suppose $y \in Y$ and satisfies (3.5). Let $x(t) = I_{0+}^\alpha \phi_q(I_{0+}^\beta y(t))$, then $x \in \text{dom}L$ and $Lx(t) = D_{0+}^\beta \phi_p(D_{0+}^\alpha I_{0+}^\alpha \phi_q(I_{0+}^\beta y(t))) = y(t)$. So that, $y \in \text{Im}L$. The proof is complete.

Lemma 3.2. Let L be defined by (3.1), then L is a Fredholm operator of index zero, and the linear continuous projector operators $P : X \to X$ and $Q : Y \to Y$ can be defined as

$$Px(t) = \frac{D_{0+}^\alpha x(0)}{\Gamma(\alpha+1)} t^\alpha, \quad \forall t \in [0,1],$$

$$Qy(t) = A \int_0^1 (1-s)^{\beta-1} y(s) ds \cdot t^\alpha, \quad \forall t \in [0,1],$$

where $A = \left(\int_0^1 (1-s)^{\beta-1} s^\alpha ds \right)^{-1} > 0$ is a constant. Furthermore, the operator $K_P : \text{Im}L \to \text{dom}L \cap \text{Ker}P$ can be written by

$$\begin{aligned} K_P y(t) &= I_{0+}^\alpha \phi_q(I_{0+}^\beta y(t)) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} y(\tau) d\tau \right) ds. \end{aligned}$$

Proof. For any $y \in Y$, we have

$$Q^2 y(t) = Qy(t) \cdot A \int_0^1 (1-s)^{\beta-1} s^\alpha ds = Qy(t). \tag{3.6}$$

Let $y_1 = y - Qy$, then we get from (3.6) that

$$\begin{aligned}
\int_0^1 (1-s)^{\beta-1} y_1(s) ds &= \int_0^1 (1-s)^{\beta-1} y(s) ds - \int_0^1 (1-s)^{\beta-1} Qy(s) ds \\
&= \frac{1}{At^\alpha} Qy(t) - \frac{1}{At^\alpha} Q^2 y(t) = 0,
\end{aligned}$$

which implies $y_1 \in \mathrm{Im}L$. Hence $Y = \mathrm{Im}L + \mathrm{Im}Q$. Since $\mathrm{Im}L \cap \mathrm{Im}Q = \{0\}$, we have $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$. Thus

$$\dim \mathrm{Ker}L = \dim \mathrm{Im}Q = \mathrm{codim}\, \mathrm{Im}L = 1.$$

This means that L is a Fredholm operator of index zero.

From the definitions of P, K_P, it is easy to see that the generalized inverse of L is K_P. In fact, for $y \in \mathrm{Im}L$, we have

$$LK_P y = D_{0+}^\beta \phi_p (D_{0+}^\alpha I_{0+}^\alpha \phi_q (I_{0+}^\beta y)) = y. \tag{3.7}$$

Moreover, for $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, we get $x(0) = D_{0+}^\alpha x(0) = D_{0+}^\alpha x(1) = 0$. By Lemma 2.1, we obtain that

$$I_{0+}^\beta Lx(t) = I_{0+}^\beta D_{0+}^\beta \phi_p (D_{0+}^\alpha x(t)) = \phi_p (D_{0+}^\alpha x(t)) + c_0, \quad c_0 \in \mathbb{R},$$

which together with $D_{0+}^\alpha x(0) = 0$ yields that

$$I_{0+}^\beta Lx(t) = \phi_p (D_{0+}^\alpha x(t)).$$

Thus, we have

$$I_{0+}^\alpha \phi_q (I_{0+}^\beta Lx(t)) = I_{0+}^\alpha D_{0+}^\alpha x(t) = x(t) + c_1, \quad c_1 \in \mathbb{R},$$

which together with $x(0) = 0$ yields that

$$K_P Lx = x. \tag{3.8}$$

Combining (3.7) with (3.8), we know that $K_P = (L|_{\mathrm{dom}L \cap \mathrm{Ker}P})^{-1}$. The proof is complete.

Lemma 3.3. Assume $\Omega \subset X$ is an open bounded subset such that $\mathrm{dom}L \cap \overline{\Omega} \neq \varnothing$, then N is L-compact on $\overline{\Omega}$.

Proof. By the continuity of f, we can get that $QN(\overline{\Omega})$ and $K_P(I - Q)N(\overline{\Omega})$ are bounded. Moreover, there exists a constant $T > 0$ such that $|I_{0+}^\beta (I - Q)Nx| \leq T, \forall x \in \overline{\Omega}, t \in [0,1]$. Thus, in view of the Arzelà-Ascoli theorem, we need only prove that $K_P(I - Q)N(\overline{\Omega}) \subset X$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq 1, x \in \overline{\Omega}$, we have

$$
\begin{aligned}
&|K_P(I - Q)Nx(t_2) - K_P(I - Q)Nx(t_1)| \\
= \ &\frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} \phi_q(I_{0+}^{\beta}(I - Q)Nx(s))ds \right. \\
&\left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} \phi_q(I_{0+}^{\beta}(I - Q)Nx(s))ds \right| \\
\leq \ &\frac{T^{q-1}}{\Gamma(\alpha)} \left\{ \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}]ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}ds \right\} \\
= \ &\frac{T^{q-1}}{\Gamma(\alpha+1)} [t_1^{\alpha} - t_2^{\alpha} + 2(t_2 - t_1)^{\alpha}] \\
\leq \ &\frac{T^{q-1}}{\Gamma(\alpha+1)} [t_2^{\alpha} - t_1^{\alpha} + 2(t_2 - t_1)^{\alpha}].
\end{aligned}
$$

Since t^{α} is uniformly continuous on $[0, 1]$, we can obtain that $K_P(I - Q)N(\overline{\Omega}) \subset Y$ is equicontinuous. Similar proof can show that $I_{0+}^{\beta}(I - Q)N(\overline{\Omega}) \subset Y$ is equicontinuous. This, together with the uniformly continuity of $\phi_q(s)$ on $[-T, T]$, yields that $D_{0+}^{\alpha}K_P(I - Q)N(\overline{\Omega})(= \phi_q(I_{0+}^{\beta}(I - Q)N)(\overline{\Omega})) \subset Y$ is also equicontinuous. Thus, we get that $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. The proof is complete.

Theorem 3.1. Let $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that
  (H_1) there exist nonnegative functions $a, b, c \in Y$ such that

$$
|f(t, u, v)| \leq a(t) + b(t)|u|^{p-1} + c(t)|v|^{p-1}, \quad \forall t \in [0, 1], (u, v) \in \mathbb{R}^2;
$$

  (H_2) there exists a constant $B > 0$ such that either

$$
vf(t, u, v) > 0, \quad \forall t \in [0, 1], u \in \mathbb{R}, |v| > B \tag{3.9}
$$

or

$$
vf(t, u, v) < 0, \quad \forall t \in [0, 1], u \in \mathbb{R}, |v| > B. \tag{3.10}
$$

Then FBVP (1.3)(1.4) has at least one solution, provided that

$$
\frac{2}{\Gamma(\beta+1)} \left(\frac{\|b\|_{\infty}}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_{\infty} \right) < 1. \tag{3.11}
$$

Proof. Set

$$
\Omega_1 = \{x \in \text{dom}L \backslash \text{Ker}L | Lx = \lambda Nx, \ \lambda \in (0, 1)\}.
$$

For $x \in \Omega_1$, we get $Lx = \lambda Nx$, $x(0) = 0$ and $Nx \in \text{Im}L$. From Lemma 2.1 and $x(0) = 0$, one has

$$
x(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} D_{0+}^{\alpha} x(s)ds.
$$

Thus, we have

$$
\begin{aligned}
|x(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |D_{0+}^\alpha x(s)| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \|D_{0+}^\alpha x\|_\infty \cdot \frac{1}{\alpha} t^\alpha \\
&\leq \frac{1}{\Gamma(\alpha+1)} \|D_{0+}^\alpha x\|_\infty, \quad \forall t \in [0,1].
\end{aligned}
$$

That is

$$
\|x\|_\infty \leq \frac{1}{\Gamma(\alpha+1)} \|D_{0+}^\alpha x\|_\infty. \tag{3.12}
$$

By $Nx \in \mathrm{Im}L$ and (3.4), we have

$$
\int_0^1 (1-s)^{\beta-1} f(s, x(s), D_{0+}^\alpha x(s)) ds = 0.
$$

Then, by the integral mean value theorem, there exists a constant $\xi \in (0,1)$ such that $f(\xi, x(\xi), D_{0+}^\alpha x(\xi)) = 0$. So, from (H_2), we get $|D_{0+}^\alpha x(\xi)| \leq B$, which implies that $|\phi_p(D_{0+}^\alpha x(\xi))| \leq B^{p-1}$. By (H_1) and (3.12), we have

$$
\begin{aligned}
&|I_{0+}^\beta Nx(t)|_{t=1}| \\
&= \left| \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s, x(s), D_{0+}^\alpha x(s)) ds \right| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (a(s) + b(s)|x(s)|^{p-1} + c(s)|D_{0+}^\alpha x(s)|^{p-1}) ds \\
&\leq \frac{1}{\Gamma(\beta)} (\|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1} + \|c\|_\infty \|D_{0+}^\alpha x\|_\infty^{p-1}) \cdot \frac{1}{\beta} \\
&\leq \frac{1}{\Gamma(\beta+1)} \left[\|a\|_\infty + \left(\frac{\|b\|_\infty}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_\infty \right) \|D_{0+}^\alpha x\|_\infty^{p-1} \right]. \tag{3.13}
\end{aligned}
$$

From $Lx = \lambda Nx$ and Lemma 2.1, one has

$$
\phi_p(D_{0+}^\alpha x(t)) = \phi_p(D_{0+}^\alpha x(\xi)) - \lambda I_{0+}^\beta Nx(t)|_{t=\xi} + \lambda I_{0+}^\beta Nx(t).
$$

Thus, from (3.13) and $|\phi_p(D_{0+}^\alpha x(\xi))| \leq B^{p-1}$, we have

$$
\begin{aligned}
&|\phi_p(D_{0+}^\alpha x(t))| \\
&\leq |\phi_p(D_{0+}^\alpha x(\xi))| + |I_{0+}^\beta Nx(t)|_{t=\xi}| + |I_{0+}^\beta Nx(t)| \\
&\leq B^{p-1} + \frac{2}{\Gamma(\beta+1)} \left[\|a\|_\infty + \left(\frac{\|b\|_\infty}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_\infty \right) \|D_{0+}^\alpha x\|_\infty^{p-1} \right],
\end{aligned}
$$

which, together with $|\phi_p(D_{0+}^\alpha x(t))| = |D_{0+}^\alpha x(t)|^{p-1}$, yields that

$$
\begin{aligned}
\|D_{0+}^\alpha x\|_\infty^{p-1} &\leq B^{p-1} + \frac{2}{\Gamma(\beta+1)} \\
&\cdot \left[\|a\|_\infty + \left(\frac{\|b\|_\infty}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_\infty \right) \|D_{0+}^\alpha x\|_\infty^{p-1} \right]. \tag{3.14}
\end{aligned}
$$

In view of (3.11), from (3.14), we can see that there exists a constant $M_1 > 0$ such that

$$\|D_{0+}^\alpha x\|_\infty \leq M_1. \tag{3.15}$$

Thus, from (3.12), we get

$$\|x\|_\infty \leq \frac{M_1}{\Gamma(\alpha + 1)} := M_2. \tag{3.16}$$

Combining (3.15) with (3.16), we have

$$\|x\|_X = \max\{\|x\|_\infty, \|D_{0+}^\alpha x\|_\infty\} \leq \max\{M_1, M_2\} := M.$$

Therefore, Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in \mathrm{Ker}L | Nx \in \mathrm{Im}L\}.$$

For $x \in \Omega_2$, we have $x(t) = ct^\alpha, c \in \mathbb{R}$ and $Nx \in \mathrm{Im}L$. Then we get

$$\int_0^1 (1 - s)^{\beta - 1} f(s, cs^\alpha, c\Gamma(\alpha + 1))ds = 0,$$

which together with (H_2) implies that $|c\Gamma(\alpha + 1)| \leq B$. Thus, we have

$$\|x\|_X \leq \max\{\frac{B}{\Gamma(\alpha + 1)}, B\} := R.$$

Hence, Ω_2 is bounded.

If (3.9) holds, set

$$\Omega_3 = \{x \in \mathrm{Ker}L | \lambda x + (1 - \lambda)QNx = 0, \ \lambda \in [0, 1]\}.$$

For $x \in \Omega_3$, we have $x(t) = ct^\alpha, c \in \mathbb{R}$ and

$$\lambda ct^\alpha + (1 - \lambda)A \int_0^1 (1 - s)^{\beta - 1} f(s, cs^\alpha, c\Gamma(\alpha + 1))ds \cdot t^\alpha = 0, \ \ \forall t \in [0, 1].$$

That is

$$\lambda c + (1 - \lambda)A \int_0^1 (1 - s)^{\beta - 1} f(s, cs^\alpha, c\Gamma(\alpha + 1))ds = 0. \tag{3.17}$$

If $\lambda = 0$, then $|c\Gamma(\alpha + 1)| \leq B$ because of (3.9). If $\lambda \in (0, 1]$, we can also obtain $|c\Gamma(\alpha + 1)| \leq B$. Otherwise, if $|c\Gamma(\alpha + 1)| > B$, in view of (3.9), one has

$$\lambda c^2\Gamma(\alpha + 1) + (1 - \lambda)A \int_0^1 (1 - s)^{\beta - 1} c\Gamma(\alpha + 1)f(s, cs^\alpha, c\Gamma(\alpha + 1))ds > 0,$$

which contradicts to (3.17). Therefore, Ω_3 is bounded.

If (3.10) holds, then define the set

$$\Omega_3' = \{x \in \mathrm{Ker}L| - \lambda x + (1 - \lambda)QNx = 0, \ \lambda \in [0,1]\}.$$

Similar to above argument, we can show that Ω_3' is also bounded.

In the following, we shall prove that all assumptions of Lemma 2.2 are satisfied.

Set

$$\Omega = \{x \in X| \|x\|_X < \max\{M, R\} + 1\}.$$

Obviously, $\Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \Omega$ (or $\Omega_1 \cup \Omega_2 \cup \Omega_3' \subset \Omega$). It follows from Lemma 3.2 and Lemma 3.3 that L (defined by (3.1)) is a Fredholm operator of index zero and N (defined by (3.2)) is L-compact on $\overline{\Omega}$. By above arguments, we get that the following two conditions are satisfied

(1) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\mathrm{dom}L \setminus \mathrm{Ker}L) \cap \partial\Omega] \times (0, 1)$;

(2) $Nx \notin \mathrm{Im}L$ for every $x \in \mathrm{Ker}L \cap \partial\Omega$.

The condition (3) of Lemma 2.2 remains to be verified. To do that, let

$$H(x, \lambda) = \pm\lambda x + (1 - \lambda)QNx.$$

According to the above argument, we know

$$H(x, \lambda) \neq 0, \ \forall x \in \partial\Omega \cap \mathrm{Ker}L.$$

Thus, by the homotopy property of degree, we have

$$
\begin{aligned}
\deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot, 0), \Omega \cap \mathrm{Ker}L, 0) \\
&= \deg(H(\cdot, 1), \Omega \cap \mathrm{Ker}L, 0) \\
&= \deg(\pm I, \Omega \cap \mathrm{Ker}L, 0) \neq 0.
\end{aligned}
$$

So that, the condition (3) of Lemma 2.2 is satisfied.

Consequently, by using Lemma 2.2, the operator equation $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$. Namely, FBVP (1.3)(1.4) has at least one solution in X. The proof is complete.

4 Existence of Solutions for FBVP (1.3)(1.5)

In this section, we will give a theorem on existence of solutions for FBVP (1.3)(1.5).

Define the operator $L_1 : \mathrm{dom}L_1 \subset X \to Y$ by

$$L_1 x = D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} x), \tag{4.1}$$

where

$$\mathrm{dom}L_1 = \{x \in X| D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} x) \in Y, \ x(1) = 0, D_{0^+}^{\alpha} x(0) = D_{0^+}^{\alpha} x(1)\}.$$

Then FBVP (1.3)(1.5) is equivalent to the operator equation

$$L_1 x = Nx, \ x \in \mathrm{dom}L_1,$$

where $N : X \to Y$ is the Nemytskii operator defined by (3.2).

Now, we begin with some lemmas that are useful in what follows.

Lemma 4.1. Let L_1 be defined by (4.1), then

$$\text{Ker} L_1 = \{x \in X | x(t) = c(1 - t^\alpha), \ \forall t \in [0,1], c \in \mathbb{R}\},$$

$$\text{Im} L_1 = \left\{y \in Y | \int_0^1 (1 - s)^{\beta-1} y(s) ds = 0\right\}.$$

Lemma 4.2. Let L_1 be defined by (4.1), then L_1 is a Fredholm operator of index zero, and the linear continuous projector operators $P_1 : X \to X$ and $Q_1 : Y \to Y$ can be defined as

$$P_1 x(t) = -\frac{D_{0+}^\alpha x(0)}{\Gamma(\alpha + 1)}(1 - t^\alpha), \quad \forall t \in [0,1],$$

$$Q_1 y(t) = A_1 \int_0^1 (1 - s)^{\beta-1} y(s) ds \cdot (1 - t^\alpha), \quad \forall t \in [0,1],$$

where $A_1 = \left(\int_0^1 (1 - s)^{\beta-1}(1 - s^\alpha) ds\right)^{-1} > 0$ is a constant. Furthermore, the operator $K_{P_1} : \text{Im} L_1 \to \text{dom} L_1 \cap \text{Ker} P_1$ can be written by

$$
\begin{aligned}
K_{P_1} y(t) &= I_{0+}^\alpha \phi_q(I_{0+}^\beta y(t)) - I_{0+}^\alpha \phi_q(I_{0+}^\beta y(t))|_{t=1} \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \tau)^{\beta-1} y(\tau) d\tau\right) ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \tau)^{\beta-1} y(\tau) d\tau\right) ds.
\end{aligned}
$$

Lemma 4.3. Assume $\Omega \subset X$ is an open bounded subset such that $\text{dom} L_1 \cap \overline{\Omega} \neq \varnothing$, then N is L_1-compact on $\overline{\Omega}$.

The proof of Lemma 4.1-4.3 are similar to the proof of Lemma 3.1-3.3, so we omit the details.

Theorem 4.1. Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Suppose (H_1) and (H_2) hold, then FBVP (1.3)(1.5) has at least one solution, provided that

$$\frac{2}{\Gamma(\beta + 1)}\left(\frac{2^{p-1}\|b\|_\infty}{(\Gamma(\alpha + 1))^{p-1}} + \|c\|_\infty\right) < 1. \tag{4.2}$$

Proof. Set

$$\Omega_{11} = \{x \in \text{dom} L_1 \backslash \text{Ker} L_1 | L_1 x = \lambda N x, \ \lambda \in (0,1)\}.$$

For $x \in \Omega_{11}$, we get $L_1 x = \lambda N x$ and $x(1) = 0$. From Lemma 2.1 and $x(1) = 0$, one has

$$
\begin{aligned}
x(t) &= I_{0+}^\alpha D_{0+}^\alpha x(t) - I_{0+}^\alpha D_{0+}^\alpha x(t)|_{t=1} \\
&= \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t - s)^{\alpha-1} D_{0+}^\alpha x(s) ds - \int_0^1 (1 - s)^{\alpha-1} D_{0+}^\alpha x(s) ds\right).
\end{aligned}
$$

Thus, we get

$$\|x\|_\infty \leq \frac{2}{\Gamma(\alpha+1)} \|D_{0+}^\alpha x\|_\infty. \tag{4.3}$$

By (H_1) and (4.3), we have

$$|I_{0+}^\beta Nx(t)|_{t=1}| \leq \frac{1}{\Gamma(\beta+1)} \left[\|a\|_\infty + \left(\frac{2^{p-1}\|b\|_\infty}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_\infty \right) \|D_{0+}^\alpha x\|_\infty^{p-1} \right].$$

Then, similar to the proof of (3.14), we obtain that

$$\|D_{0+}^\alpha x\|_\infty^{p-1} \leq B^{p-1} + \frac{2}{\Gamma(\beta+1)}$$
$$\cdot \left[\|a\|_\infty + \left(\frac{2^{p-1}\|b\|_\infty}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_\infty \right) \|D_{0+}^\alpha x\|_\infty^{p-1} \right]. \tag{4.4}$$

In view of (4.2), from (4.4), we can see that there exists a constant $M_{11} > 0$ such that

$$\|D_{0+}^\alpha x\|_\infty \leq M_{11}. \tag{4.5}$$

Thus, from (4.3) and (4.5), we get

$$\|x\|_X \leq \max\{\frac{2M_{11}}{\Gamma(\alpha+1)}, M_{11}\}.$$

Therefore, Ω_{11} is bounded.

The remainder of proof work are similar to the proof of Theorem 3.1, so we omit the details. The proof is complete.

5 An example

In this section, we will give an example to illustrate our main results.

Example 5.1. Consider the following fractional p-Laplacian equation

$$D_{0+}^{\frac{3}{4}} \phi_3 \left(D_{0+}^{\frac{1}{2}} x(t) \right) = -\frac{72}{5} + te^{-x^2(t)} + \frac{2}{5} \left(D_{0+}^{\frac{1}{2}} x(t) \right)^2, \quad t \in [0,1]. \tag{5.1}$$

Corresponding to equation (1.3), we get that $p = 3, \alpha = 1/2, \beta = 3/4$ and

$$f(t, u, v) = -\frac{72}{5} + te^{-u^2} + \frac{2}{5}v^2.$$

Choose $a(t) = 16, b(t) = 0, c(t) = 2/5, B = 6$. By a simple calculation, we can obtain that $\|b\|_\infty = 0, \|c\|_\infty = 2/5$ and

$$vf(t, u, v) = v \left[\frac{2}{5}(v^2 - 36) + te^{-u^2} \right] > 0 \text{ (or } < 0), \quad \forall t \in [0,1], u \in \mathbb{R}, |v| > 6,$$

$$\frac{2}{\Gamma\left(\frac{3}{4}+1\right)} \left(\frac{0}{\left(\Gamma\left(\frac{1}{2}+1\right)\right)^2} + \frac{2}{5} \right) < 1.$$

Obviously, equation (5.1) subject to boundary value conditions (1.4) (or (1.5)) satisfies all assumptions of Theorem 3.1 (or Theorem 4.1). Hence, FBVP (5.1)(1.4) (or FBVP (5.1)(1.5)) has at least one solution.

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