On the remarkable Lamperti representation of the inverse local time of a radial Ornstein-Uhlenbeck process

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Abstract

We give a description, in terms of "pseudo-stable increasing process", of the Lamperti process associated with the inverse local time of a radial Ornstein-Uhlenbeck process. Following Bertoin-Yor, we also express, in two particular cases, the law of the perpetuity associated with this inverse local time.

1 Introduction

This Introduction consists of three first Subsections 1.1, 1.2, 1.3, which are devoted respectively to the presentation of the necessary prerequisites about the Lamperti correspondence, the pseudo-stable increasing processes, and the radial Ornstein-Uhlenbeck processes. We then state an identity in law (12), between two subordinators, which is the main object of this paper.
In Subsection 1.4, we indicate the organization of the remainder of this paper, which aims at understanding (12) in depth.
Finally, in Subsection 1.5, we stress the necessity of a precise determination of the constant involved in the different definitions of the local times.

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1.1 Some preliminaries about the Lamperti correspondence

In 1972, Lamperti [10] established an extremely interesting correspondence between:

- on one hand, real-valued Lévy processes $(\xi_t, t \geq 0)$,

and,

- on the other hand, Feller processes $(X_u, u \geq 0)$ taking values in $(0, \infty)$, which furthermore satisfy the scaling property:

$$\forall c > 0, \quad (X_{x,cu}, u \geq 0) \stackrel{\text{(law)}}{=} (c\, X_{\frac{x}{c},u}, u \geq 0) \tag{1}$$

  where $(X_{y,u}, u \geq 0)$ denotes the Markov process X starting at y.

Likewise, $(\xi_t, t \geq 0)$ denotes the Lévy process starting at 0, and for $a \in \mathbb{R}$,

$$(\xi_{a,t}, t \geq 0) \stackrel{\text{(law)}}{=} (a + \xi_t, t \geq 0).$$

In the particular case where: $\int_0^\infty \exp(\xi_s)\, \mathrm{d}s = \infty$ a.s., Lamperti's correspondence may be presented very simply in the form of either of the following identities:

$$\exp(\xi_{a,t}) = X_{\exp a, A_t}, \quad \text{or:} \quad \log(X_{x,u}) = \xi_{\log x, H_u}, \tag{2}$$

$$\text{where:} \quad A_t = \int_0^t \exp(\xi_{a,s})\, \mathrm{d}s, \quad \text{and:} \ H_u = \int_0^u \frac{1}{X_{x,v}}\, \mathrm{d}v.$$

Put simply, Lamperti's correspondence expresses the fact that the independence and homogeneity properties of the increments of the Lévy process $(\xi_t, t \geq 0)$ translate, via (2), into the scaling property (1) of the process X.

In the particular case where $(\xi_t, t \geq 0)$ is a subordinator, the law of the perpetuity:

$$\mathcal{I} \equiv \mathcal{I}(\xi) = \int_0^\infty \exp(-\xi_t)\, \mathrm{d}t$$

has been of great interest for a number of "real-world" problems, and many properties of this law have been obtained; see, e.g., Bertoin-Yor [1, 2], Salminen-Yor [18], Khoshnevisan et al. [8]. We refer also to recent papers by Patie (e.g. [12]) and Kuznetsov et al. (e.g. [9]). In fact, for at least fifteen years, every month has seen the publication of at least one paper on this subject in the Probability journals. Among other results, the law of $\mathcal{I}$ is characterized by its integral moments:

$$\mathbb{E}[\mathcal{I}^n] = \frac{n!}{\Phi(1) \cdots \Phi(n)}, \quad n \geq 1, \tag{3}$$

where $(\Phi(s), s \geq 0)$ is the Laplace-Bernstein exponent of $(\xi_t, t \geq 0)$:

$$\mathbb{E}[\exp(-s\, \xi_t)] = \exp(-t\, \Phi(s)).$$

In fact, quoting Bertoin-Yor [1] again, the relation (3) may be complemented as follows: the standard exponential variable **e** may be factorized as:

$$\mathbf{e} \overset{\text{(law)}}{=} \mathcal{I} \cdot \mathcal{R}, \tag{4}$$

with $\mathcal{I}$ and $\mathcal{R}$ independent, and $\mathcal{R}$ (or rather its law) is characterized by:

$$\mathbb{E}_1\left(\frac{1}{X_t}\right) = \mathbb{E}[\exp(-t\,\mathcal{R})], \quad t \geq 0. \tag{5}$$

($\forall x > 0$, $\mathbb{P}_x$ indicates the law of the process $(X_{x,u})_{u \geq 0}$, while $\mathbb{P}$ is a generic probability.)

Combining (3) and (4), the integral moments of $\mathcal{R}$ are seen to be given by:

$$\mathbb{E}[\mathcal{R}^n] = \Phi(1) \cdots \Phi(n). \tag{6}$$

More generally, for every $p > 0$, Bertoin-Yor [1] show the existence of a random variable $\mathcal{R}_p$, taking values in $\mathbb{R}_+$, such that:

$$\mathbb{E}_1\left(\frac{1}{X_t^p}\right) = \mathbb{E}[\exp(-t\,\mathcal{R}_p)], \quad t \geq 0. \tag{7}$$

Actually, (7) is the starting point of our recent paper [7] in which we consider, more generally, functions f on $(0,\infty)$ such that, for every $x > 0$, the function:

$$t \longrightarrow \mathbb{E}_x[f(X_t)]$$

is a completely monotone function on $(0,\infty)$, and (X_t) is even replaced by a more general Markov process.

1.2 Pseudo-stable increasing processes

In [7], we considered a particular class of Lamperti processes which we called *pseudo-stable increasing processes*. A pseudo-stable increasing process of index $\alpha \in (0,1)$ and parameter $\lambda > 0$, is the $(0,\infty)$-valued process:

$$(X_{x,u}^{(\alpha,\lambda)};\ x > 0, u \geq 0) := ((x^{1/\alpha} + \lambda\,\tau_u^{(\alpha)})^\alpha;\ x > 0, u \geq 0)$$

where $(\tau_t^{(\alpha)}, t \geq 0)$ denotes the α-stable subordinator started at 0, defined from the Bernstein function $F_\alpha(s) = s^\alpha$:

$$\mathbb{E}[\exp(-s\,\tau_t^{(\alpha)})] = \mathrm{e}^{-t\,s^\alpha}, \;\; s > 0,\; t \geq 0.$$

Clearly, $(X_{x,u}^{(\alpha,\lambda)};\ x > 0, u \geq 0)$ is an increasing Lamperti process, whose associated subordinator is denoted $(\xi_t^{(\alpha,\lambda)}, t \geq 0)$. The next theorem describes the Lévy measure and the Laplace-Bernstein exponent of $\xi^{(\alpha,\lambda)}$.

Theorem 1.1 ([7] Theorem 7.1). *1. Let $\nu^{(\alpha,\lambda)}$ be the Lévy measure of $\xi^{(\alpha,\lambda)}$. Then:*

$$\nu^{(\alpha,\lambda)}(\mathrm{d}v) = \frac{\lambda^\alpha}{\Gamma(1-\alpha)}\, \mathrm{e}^{v/\alpha}\, (\mathrm{e}^{v/\alpha} - 1)^{-\alpha-1}\, \mathrm{d}v.$$

2. Let $F_{\alpha,\lambda}$ be the Laplace-Bernstein exponent of $\xi^{(\alpha,\lambda)}$. Then:

$$F_{\alpha,\lambda}(s) = \lambda^\alpha\, \frac{\Gamma(\alpha\,(s+1))}{\Gamma(\alpha\, s)}, \quad s > 0.$$

1.3 Radial Ornstein-Uhlenbeck processes

Let δ be a positive integer and $(B_t^{(\delta)}, t \geq 0)$ be a δ-dimensional standard Brownian motion starting from 0. For $\mu > 0$, the $\mathbb{R}^\delta$*-valued Ornstein-Uhlenbeck process* with parameter μ is the solution to:

$$U_t^{(\delta,\mu)} = B_t^{(\delta)} - \mu \int_0^t U_s^{(\delta,\mu)}\, \mathrm{d}s. \tag{8}$$

Now, consider $Z_t = |U_t^{(\delta,\mu)}|^2$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^\delta$. Then, $(Z_t, t \geq 0)$ is a nonnegative solution to:

$$Z_t = 2 \int_0^t \sqrt{Z_s}\, \mathrm{d}\beta_s - 2\mu \int_0^t Z_s\, \mathrm{d}s + \delta\, t \tag{9}$$

where $(\beta_s, s \geq 0)$ denotes a standard real-valued Brownian motion starting from 0. Of course, equation (9) makes sense for any real number $\delta > 0$. Hence, we adopt the following definition.

Definition 1.1. For every real numbers δ, $\mu > 0$, the *radial Ornstein-Uhlenbeck process of dimension δ and parameter μ* is the process:

$$R_t^{(\delta,\mu)} = \sqrt{Z_t}, \quad t \geq 0$$

where $(Z_t, t \geq 0)$ solves equation (9).

We refer to Pitman-Yor [15] and the references therein for further background and motivation for the study of these processes. The solutions to equation (9) are known, in mathematical finance, as Cox-Ingersoll-Ross processes, but the study of (9) has been a very natural topic to discuss families of diffusions with the additivity property (Shiga-Watanabe [17], Pitman-Yor [13][15]). By definition, for an integer δ and $\mu > 0$, we have:

$$R_t^{(\delta,\mu)} = |U_t^{(\delta,\mu)}|,$$

and if δ is a positive real number and $\mu = 0$, the solution to equation (9) is $\mathrm{BESQ}^{(\delta)}(0)$ (the squared Bessel process of dimension δ starting from 0). Henceforth, we shall denote by $(R_t^{(\delta)}, t \geq 0)$ the Bessel process of dimension δ starting from 0. Thus, $R^{(\delta)}$ may be viewed as $R^{(\delta,0)}$.

In the sequel, we restrict ourselves to the case $0 < \delta < 2$, *and we set* $\alpha = 1 - \frac{\delta}{2} \in (0,1)$.

Consequently, we change slightly our notation, replacing δ by α. Thus, $R_t^{(\alpha,\mu)}$ (resp. $R_t^{(\alpha)}$) will be set for $R_t^{(2(1-\alpha),\mu)}$ (resp. $R_t^{(2(1-\alpha))}$). We shall denote by $L^{(\alpha,\mu)} := (L_t^{(\alpha,\mu)}, t \geq 0)$ (resp. $L^{(\alpha)} := (L_t^{(\alpha)}, t \geq 0)$) "the" local time at 0 of $R^{(\alpha,\mu)}$ (resp. $R^{(\alpha)}$). Of course, these local times are defined up to a multiplicative constant. Finally, we denote by $I^{(\alpha,\mu)} := (I_l^{(\alpha,\mu)}, l \geq 0)$ (resp. $I^{(\alpha)} := (I_l^{(\alpha)}, l \geq 0)$) the inverse local time at 0 of $R^{(\alpha,\mu)}$ (resp. $R^{(\alpha)}$), i.e. the inverse function of $t \longrightarrow L_t^{(\alpha,\mu)}$ (resp. $t \longrightarrow L_t^{(\alpha)}$). (As a mnemonic for the fact that these processes are inverse local times, we use l for their time parameter, just as, e.g., $(e_l, l \geq 0)$ denotes Itô's excursion process ...) These inverse local times are subordinators without drift, and we denote respectively by $\Lambda^{(\alpha,\mu)}$ and $\Lambda^{(\alpha)}$ their Lévy measures. According to Pitman-Yor [15], the following representations hold.

Theorem 1.2 ([15]).

1. *The process* $I^{(\alpha)}$ *is an* α*-stable subordinator whose Lévy measure is:*

$$\Lambda^{(\alpha)}(\mathrm{d}v) = C_\alpha \, v^{-\alpha-1} \, \mathrm{d}v,$$

where C_α *denotes a positive constant, which depends on the choice of normalization of local time.*

2. *The Lévy measure of* $I^{(\alpha,\mu)}$ *is:*

$$\Lambda^{(\alpha,\mu)}(\mathrm{d}v) = C_\alpha \left(\frac{\mu}{\sinh(\mu\, v)} \right)^{1+\alpha} \mathrm{e}^{\mu\,(1-\alpha)\,v} \, \mathrm{d}v$$

1.4 A mere coincidence?

Inspection of the expressions of $\nu^{(\alpha,\lambda)}$ in Theorem 1.1 and $\Lambda^{(\alpha,\mu)}$ in Theorem 1.2, yields the identity:

$$\nu^{(\alpha,\lambda_\alpha)} = \Lambda^{(\alpha,\mu_\alpha)}, \tag{10}$$

with

$$\lambda_\alpha = \left(C_\alpha \, \Gamma(1-\alpha)\, \alpha^{-1-\alpha} \right)^{1/\alpha} \quad \text{and} \quad \mu_\alpha = \frac{1}{2\,\alpha}. \tag{11}$$

Expressed in terms of processes, (10) may be phrased as:

$$I^{(\alpha,\mu_\alpha)} \overset{(\text{law})}{=} \xi^{(\alpha,\lambda_\alpha)}. \tag{12}$$

The following question is now natural: what is the status of the equality (10) or its equivalent (12)? Is it a mere coincidence? One of the aims of this paper is to answer this question (as might be expected, it is not a "mere coincidence", whatever this may mean ...).

This led us to the following organization:

- In Section 2, we state an integral relationship between the local times $L^{(\alpha,\mu)}$ and $L^{(\alpha)}$, and we deduce therefrom a relationship between the corresponding inverse local times.
- In Section 3, we prove that the Lamperti process associated with the subordinator: $(2\mu\alpha\, I_l^{(\alpha,\mu)}, l \geq 0)$ is a pseudo-stable increasing process of index α. This allows to recover Theorem 1.2 from Theorem 1.1, and to explain the equality (10).
- In Section 4, we express the Lamperti process associated with $(I_l^{(\alpha,\mu)}, l \geq 0)$ as some power of a time changed pseudo-stable process.
- Finally, in Section 5 we present some results of Bertoin-Yor [1] concerning the laws of some perpetuities associated with the subordinators $I^{(\alpha,\mu)}$, which are closely related with those discussed in the previous sections.

1.5 On constants C_α, $\widehat{C}_\alpha$, c_α, ... $(0 < \alpha < 1)$

As the reader shall soon discover, it is necessary in order to state our results precisely, to determine exactly which local times we consider. This care was also needed in, e.g., Biane-Yor [3] and Pitman-Yor [14]. In particular, such precisions are essential when proceeding with various values of the parameter α.

2 An integral relationship between local times

2.1 A general result

We start with a general result (see also, for instance, Chaumont-Yor ([5, Exercise 6.11]). In the sequel, the local times which are being considered, are defined via Meyer-Tanaka formulae for continuous semi-martingales (see Meyer [11], and Revuz-Yor [16, Chapter 6, Theorem 1.2]).

Proposition 2.1. *Let $(U_t, t \geq 0)$ be a continuous semi-martingale with $U_0 = 0$. Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing continuous function with $\varphi(0) = 0$, and let $h : \mathbb{R}_+ \longrightarrow (0, \infty)$ be a strictly positive, continuous function, locally with bounded variation. We set:*

$$V(t) = h(t)\, U_{\varphi(t)}, \quad t \geq 0$$

and we denote by L^U (resp. L^V) the local time at 0 of the continuous semi-martingale U (resp. V). Then:

$$L_t^V = \int_0^t h(s)\; \mathrm{d}L^U_{\varphi(s)}.$$

Proof. By definition of the local time via Meyer-Tanaka formulae, one has:

$$|U_t| = \int_0^t \operatorname{sgn}(U_s)\; \mathrm{d}U_s + L_t^U$$

where the function sgn is defined by:

$$\operatorname{sgn}(x) = 1 \quad \text{if} \;\; x > 0 \quad \text{and} \quad \operatorname{sgn}(x) = -1 \quad \text{if} \;\; x \leq 0$$

(see, e.g., Meyer [11] for a first occurrence of this convention). Consequently:

$$\begin{aligned} |U_{\varphi(t)}| &= \int_0^{\varphi(t)} \operatorname{sgn}(U_s)\, \mathrm{d}U_s + L^U_{\varphi(t)} \\ &= \int_0^t \operatorname{sgn}(V_s)\, \mathrm{d}\left((h(s))^{-1}\, V_s\right) + L^U_{\varphi(t)} \\ &= \int_0^t \operatorname{sgn}(V_s)\, (h(s))^{-1}\, \mathrm{d}V_s - \int_0^t (h(s))^{-2}\, |V_s|\, \mathrm{d}h(s) + L^U_{\varphi(t)} \end{aligned}$$

(to avoid any confusion, let us insist that x^{-1} means $1/x$). Hence, we get:

$$\begin{aligned} |V_t| &= h(t)\, |U_{\varphi(t)}| \\ &= \int_0^t \operatorname{sgn}(V_s)\, \mathrm{d}V_s - \int_0^t (h(s))^{-1}\, |V_s|\, \mathrm{d}h(s) + \int_0^t h(s)\, \mathrm{d}L^U_{\varphi(s)} \\ &\quad + \int_0^t (h(s))^{-1}\, |V_s|\, \mathrm{d}h(s) \\ &= \int_0^t \operatorname{sgn}(V_s)\, \mathrm{d}V_s + \int_0^t h(s)\, \mathrm{d}L^U_{\varphi(s)}, \end{aligned}$$

which, by identification, yields the desired result. ■

2.2 A representation of $R^{(\alpha,\mu)}$

The following proposition is well-known (see e.g. Pitman-Yor [13, formula (6.b), p. 454]). Nevertheless, for convenience of the reader, a proof is now proposed.

Proposition 2.2. *We set, for $\mu > 0$,*

$$\mathrm{e}_\mu(t) = \frac{1}{2\mu}\,(\mathrm{e}^{2\mu t} - 1).$$

Then:

$$(R^{(\alpha,\mu)}_t, t \geq 0) \overset{(\text{law})}{=} (\mathrm{e}^{-\mu t}\, R^{(\alpha)}_{\mathrm{e}_\mu(t)}, t \geq 0).$$

Proof. We set:

$$Z_t = \mathrm{e}^{-2\mu t}\, \left[R^{(\alpha)}_{\mathrm{e}_\mu(t)}\right]^2$$

and we recall (see Subsection 1.3) that $\delta = 2\,(1-\alpha)$. Then we have:

$$\left(R^{(\alpha)}_t\right)^2 = 2\int_0^t R^{(\alpha)}_s\, \mathrm{d}\beta_s + \delta\, t.$$

Hence,

$$Z_t = 2\int_0^t \sqrt{Z_s}\, \mathrm{e}^{-\mu s}\, \mathrm{d}\beta_{\mathrm{e}_\mu(s)} - 2\mu \int_0^t Z_s\, \mathrm{d}s + \delta\, t.$$

We set:

$$\widetilde{\beta}_t = \int_0^t \mathrm{e}^{-\mu s}\, \mathrm{d}\beta_{\mathrm{e}_\mu(s)} = \int_0^{\mathrm{e}_\mu(t)} (1+2\mu\, u)^{-1/2}\, \mathrm{d}\beta_u.$$

Now, $(\widetilde{\beta}_t, t \geq 0)$ is a Brownian motion and:

$$Z_t = 2\int_0^t \sqrt{Z_s}\, \mathrm{d}\widetilde{\beta}_s - 2\mu \int_0^t Z_s\, \mathrm{d}s + \delta\, t.$$

It then suffices to use the weak uniqueness of solutions to (9). ∎

2.3 The local time at 0 of $R^{(\alpha,\mu)}$

Definition 2.1. The process $\left(\left[R_t^{(\alpha)}\right]^{2\alpha},\ t \geq 0\right)$ is a continuous semi-martingale, whose local time at 0 is called the *local time of the Bessel process* $R^{(\alpha)}$, and is denoted by $L^{(\alpha)}$. The process $\left(\left[R_t^{(\alpha,\mu)}\right]^{2\alpha},\ t \geq 0\right)$ is also a continuous semi-martingale, whose local time at 0 is called the *local time of the radial Ornstein-Uhlenbeck process* $R^{(\alpha,\mu)}$, and is denoted by $L^{(\alpha,\mu)}$.

Remark 2.1. In Donati-Martin et al. [6], it is shown that $\left(\left[R_t^{(\alpha)}\right]^{2\alpha},\ t \geq 0\right)$ is a submartingale, and the authors define the local time at 0 of $R^{(\alpha)}$ as the increasing process in the Doob-Meyer decomposition of $\left[R^{(\alpha)}\right]^{2\alpha}$. It is easy to see that the local time $L^{(\alpha)}$ as defined in Definition 2.1 is twice the local time as defined in [6]. In particular, by [6, Proposition 3.1],

$$L_t^{(\alpha)} = 4\,(1-\alpha)\,\alpha \lim_{\varepsilon \to 0} \varepsilon^{-2(1-\alpha)} \int_0^t 1_{[0,\varepsilon)}\left(R_s^{(\alpha)}\right)\, \mathrm{d}s.$$

As a consequence of Definition 2.1, Proposition 2.1 and Proposition 2.2, we obtain:

Proposition 2.3.

$$\left(L_t^{(\alpha,\mu)},\ t \geq 0\right) \overset{\text{(law)}}{=} \left(\int_0^t \mathrm{e}^{-2\mu\alpha s}\, \mathrm{d}L^{(\alpha)}_{\mathrm{e}_\mu(s)},\ t \geq 0\right).$$

Remark 2.2. The local times we consider here are also *diffusion local times* for the concerned diffusions, in the sense of Borodin-Salminen [4].

2.4 The inverse local time of $R^{(\alpha,\mu)}$

Now we introduce the process $I^{(\alpha,\mu)} := (I_l^{(\alpha,\mu)}, l \geq 0)$ (resp. $I^{(\alpha)} := (I_l^{(\alpha)}, l \geq 0)$), which is the right-continuous inverse of the local time: $t \longrightarrow L_t^{(\alpha,\mu)}$ (resp. $t \longrightarrow L_t^{(\alpha)}$). These are subordinators without drift.

Proposition 2.4. *The process $I^{(\alpha)}$ is an α-stable subordinator whose Lévy measure is:*

$$\Lambda^{(\alpha)}(\mathrm{d}v) = C_\alpha\, v^{-\alpha-1}\, \mathrm{d}v \tag{13}$$

with

$$C_\alpha = \frac{2^{-\alpha-1}}{\Gamma(\alpha)}. \tag{14}$$

We also have:

$$(I_l^{(\alpha)}, l \geq 0) \overset{\text{(law)}}{=} (\widetilde{C}_\alpha\, \tau_l^{(\alpha)}, l \geq 0) \tag{15}$$

with

$$\widetilde{C}_\alpha = \left[\frac{2^{-\alpha-1}\,\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\right]^{1/\alpha}. \tag{16}$$

In formula (15), $(\tau_l^{(\alpha)}, l \geq 0)$ denotes the α-stable subordinator defined in Subsection 1.2.

Proof. From Donati-Martin et al. [6, Proposition 3.2] and Remark 2.1, the Laplace-Bernstein exponent of $I^{(\alpha)}$ is:

$$\frac{2^{-\alpha-1}\,\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\, s^\alpha,$$

which yields (15), (16). Now, (13), (14) follow therefrom, since the Lévy measure of $\tau^{(\alpha)}$ is:

$$\sigma^{(\alpha)}(\mathrm{d}v) = \frac{\alpha}{\Gamma(1-\alpha)}\, v^{-\alpha-1}\, \mathrm{d}v.$$

■

Proposition 2.5. *Set:*

$$H_{\alpha,\mu}(t) = \int_0^t \left(1 + 2\mu\, I_l^{(\alpha)}\right)^{-\alpha} \mathrm{d}l$$

and let $K_{\alpha,\mu}$ be the (continuous) inverse of $H_{\alpha,\mu}$. Then:

$$(I_l^{(\alpha,\mu)}, l \geq 0) \overset{\text{(law)}}{=} \left(\frac{1}{2\mu} \log\left(1 + 2\mu\, I^{(\alpha)}_{K_{\alpha,\mu}(l)}\right),\ l \geq 0\right).$$

Proof. By Proposition 2.3, we may set:

$$L_t^{(\alpha,\mu)} = \int_0^t \mathrm{e}^{-2\mu\alpha s}\, \mathrm{d}L^{(\alpha)}_{\mathrm{e}_\mu(s)} = \int_0^{\mathrm{e}_\mu(t)} (1 + 2\mu\, u)^{-\alpha}\, \mathrm{d}L_u^{(\alpha)}.$$

Therefore, by change of variable,

$$L_t^{(\alpha,\mu)} = \int_0^{L^{(\alpha)}_{\mathrm{e}_\mu(t)}} (1 + 2\mu\, I_l^{(\alpha)})^{-\alpha}\, \mathrm{d}l = H_{\alpha,\mu}\left(L^{(\alpha)}_{\mathrm{e}_\mu(t)}\right).$$

Taking inverses, we get:

$$I_l^{(\alpha,\mu)} = \left(L^{(\alpha)} \circ \mathrm{e}_\mu\right)^{-1} (K_{\alpha,\mu}(l))$$

(where, here, $\theta^{-1}(u) \equiv \inf\{t;\ \theta(t) > u\}$). Now,

$$\left(L^{(\alpha)} \circ \mathrm{e}_\mu\right)^{-1} = (\mathrm{e}_\mu)^{-1} \circ I^{(\alpha)} = \frac{1}{2\mu} \log\left(1 + 2\mu\, I^{(\alpha)}\right),$$

and the desired result follows. ■

3 The Lamperti process associated with $2\mu\alpha\, I^{(\alpha,\mu)}$

Theorem 3.1. *Let $\widehat{I}^{(\alpha,\mu)} := 2\mu\alpha\, I^{(\alpha,\mu)}$ and denote by $\widehat{Y}^{(\alpha,\mu)}$ the Lamperti increasing process associated with $\widehat{I}^{(\alpha,\mu)}$. Then:*

$$(\widehat{Y}^{(\alpha,\mu)}_{x,u};\ x > 0, u \geq 0) = ((x^{1/\alpha} + 2\mu\alpha\lambda_\alpha\, \tau^{(\alpha)}_u)^\alpha;\ x > 0, u \geq 0)$$

where λ_α is defined in (11) by: $\lambda_\alpha = \left(C_\alpha\, \Gamma(1-\alpha)\, \alpha^{-1-\alpha}\right)^{1/\alpha}$, and C_α is defined in (14). Thus, with the notation in Subsection 1.2, $\widehat{Y}^{(\alpha,\mu)}$ is the pseudo-stable increasing process: $X^{(\alpha,2\mu\alpha\lambda_\alpha)}$ and

$$\lambda_\alpha = \frac{1}{2\,\alpha}\left(\frac{\Gamma(1-\alpha)}{2\,\Gamma(1+\alpha)}\right)^{1/\alpha} = \frac{1}{\alpha}\,\widetilde{C}_\alpha. \tag{17}$$

Proof. We keep the notation of Proposition 2.5. We may assume that:

$$I^{(\alpha,\mu)}_l = \frac{1}{2\mu}\log\left(1 + 2\mu\, I^{(\alpha)}_{K_{\alpha,\mu}(l)}\right).$$

Then, one has:

$$\begin{aligned} A_t \ &:= \int_0^t \exp\left(\widehat{I}^{(\alpha,\mu)}_l\right)\, \mathrm{d}l = \int_0^t \left(1 + 2\mu\, I^{(\alpha)}_{K_{\alpha,\mu}(l)}\right)^\alpha \mathrm{d}l \\ &= \int_0^{K_{\alpha,\mu}(t)} \left(1 + 2\mu\, I^{(\alpha)}_l\right)^\alpha \left(1 + 2\mu\, I^{(\alpha)}_l\right)^{-\alpha} \mathrm{d}l = K_{\alpha,\mu}(t). \end{aligned}$$

Therefore, the inverse function of $t \longrightarrow A_t$ is $H_{\alpha,\mu}$. This entails, by the definition, that:

$$\widehat{Y}^{(\alpha,\mu)}_{1,u} = \left(1 + 2\mu\, I^{(\alpha)}_u\right)^\alpha.$$

By Proposition 2.4 and (11),

$$(I^{(\alpha)}_l, l \geq 0) \overset{\text{(law)}}{=} (\widetilde{C}_\alpha\, \tau^{(\alpha)}_l, l \geq 0)$$

with

$$\widetilde{C}_\alpha = \left[C_\alpha\, \Gamma(1-\alpha)\, \alpha^{-1}\right]^{1/\alpha} = \alpha\, \lambda_\alpha.$$

Hence,

$$(\widehat{Y}^{(\alpha,\mu)}_{1,u},\ u \geq 0) \overset{\text{(law)}}{=} ((1 + 2\mu\alpha\lambda_\alpha\, \tau^{(\alpha)}_u)^\alpha,\ u \geq 0),$$

and the desired result follows from the Markov property. ∎

Remark 3.1. We set, as in (11), $\mu_\alpha = 1/2\alpha$. Then Theorem 3.1 entails that the Lamperti process associated with $I^{(\alpha,\mu_\alpha)}$ is the pseudo-stable process: $X^{(\alpha,\lambda_\alpha)}$. Consequently, we have:

$$(I^{(\alpha,\mu_\alpha)}_l, l \geq 0) \overset{\text{(law)}}{=} (\xi^{(\alpha,\lambda_\alpha)}_l, l \geq 0).$$

This is the equality (12), and the "coincidence" (10) is thus explained.

Now, we shall deduce Pitman-Yor's result (Theorem 1.2) from both Theorem 1.1 and Theorem 3.1.

Theorem 3.2. *1. Let $\Lambda^{(\alpha,\mu)}$ be the Lévy measure of $I^{(\alpha,\mu)}$. Then:*

$$\Lambda^{(\alpha,\mu)}(\mathrm{d}v) = C_\alpha \left(\frac{\mu}{\sinh(\mu\, v)}\right)^{1+\alpha} \mathrm{e}^{\mu\,(1-\alpha)\,v}\,\mathrm{d}v,$$

with C_α defined in (14).

2. The Laplace-Bernstein exponent of $I^{(\alpha,\mu)}$ is:

$$\Phi^{(\alpha,\mu)}(s) = \widehat{C}_\alpha\,\mu^\alpha \frac{\Gamma\left(\frac{s}{2\mu}+\alpha\right)}{\Gamma\left(\frac{s}{2\mu}\right)},$$

with

$$\widehat{C}_\alpha = \frac{\Gamma(1-\alpha)}{2\,\Gamma(1+\alpha)}. \tag{18}$$

Proof. 1) By Theorem 1.1 and Theorem 3.1, the Lévy measure $\widehat{\Lambda}^{(\alpha,\mu)}$ of $\widehat{I}^{(\alpha,\mu)}$ is given by:

$$\widehat{\Lambda}^{(\alpha,\mu)}(\mathrm{d}v) = \frac{(2\mu\alpha\lambda_\alpha)^\alpha}{\Gamma(1-\alpha)}\,\mathrm{e}^{v/\alpha}\,(\mathrm{e}^{v/\alpha}-1)^{-\alpha-1}\,\mathrm{d}v.$$

We deduce therefrom:

$$\begin{aligned}\Lambda^{(\alpha,\mu)}(\mathrm{d}v) &= 2\mu\alpha\,\frac{(2\mu\alpha\lambda_\alpha)^\alpha}{\Gamma(1-\alpha)}\,\mathrm{e}^{2\mu v}\,(\mathrm{e}^{2\mu v}-1)^{-\alpha-1}\,\mathrm{d}v \\ &= \frac{\alpha^{\alpha+1}(\lambda_\alpha)^\alpha}{\Gamma(1-\alpha)}\left(\frac{\mu}{\sinh(\mu\,v)}\right)^{1+\alpha}\mathrm{e}^{\mu\,(1-\alpha)\,v}\,\mathrm{d}v.\end{aligned}$$

Now, by (11),

$$\frac{\alpha^{\alpha+1}(\lambda_\alpha)^\alpha}{\Gamma(1-\alpha)} = C_\alpha.$$

2) By Theorem 1.1 and Theorem 3.1, the Laplace-Bernstein exponent $\widehat{\Phi}^{(\alpha,\mu)}$ of $\widehat{I}^{(\alpha,\mu)}$ is given by:

$$\widehat{\Phi}^{(\alpha,\mu)}(s) = (2\mu\alpha\lambda_\alpha)^\alpha\,\frac{\Gamma(\alpha\,(s+1))}{\Gamma(\alpha\,s)}.$$

Hence,

$$\Phi^{(\alpha,\mu)}(s) = \widehat{\Phi}^{(\alpha,\mu)}\left(\frac{s}{2\mu\alpha}\right) = (2\mu\alpha\lambda_\alpha)^\alpha\,\frac{\Gamma\left(\frac{s}{2\mu}+\alpha\right)}{\Gamma\left(\frac{s}{2\mu}\right)},$$

and, by (17),

$$(2\mu\alpha\lambda_\alpha)^\alpha = \frac{\Gamma(1-\alpha)}{2\,\Gamma(1+\alpha)}\,\mu^\alpha = \widehat{C}_\alpha\,\mu^\alpha.$$

■

4 The Lamperti process associated with $I^{(\alpha,\mu)}$

Theorem 4.1. *Set:*

$$M_{\alpha,\mu}(t) = \int_0^t \left(1 + 2\mu\, I_l^{(\alpha)}\right)^{\frac{1}{2\mu} - \alpha} \, \mathrm{d}l$$

and let $N_{\alpha,\mu}$ be the (continuous) inverse of $M_{\alpha,\mu}$. Then the Lamperti increasing process $Y^{(\alpha,\mu)}$ associated with $I^{(\alpha,\mu)}$ is given by:

$$\left(Y_{x,u}^{(\alpha,\mu)};\ x > 0, u \geq 0\right) = \left(x\,\left(1 + 2\mu\, I^{(\alpha)}_{N_{\alpha,\mu}(u/x)}\right)^{1/2\mu};\ x > 0, u \geq 0\right).$$

Thus, $\left[Y^{(\alpha,\mu)}\right]^{2\mu\alpha}$ is a time changed pseudo-stable process of index α.

Proof. By Proposition 2.5, one has:

$$\begin{aligned} \widetilde{A}_t &:= \int_0^t \exp\left(I_l^{(\alpha,\mu)}\right)\, \mathrm{d}l = \int_0^t \left(1 + 2\mu\, I^{(\alpha)}_{K_{\alpha,\mu}(l)}\right)^{1/2\mu} \mathrm{d}l \\ &= \int_0^{K_{\alpha,\mu}(t)} \left(1 + 2\mu\, I_l^{(\alpha)}\right)^{\frac{1}{2\mu} - \alpha} \mathrm{d}l = M_{\alpha,\mu} \circ K_{\alpha,\mu}(t). \end{aligned}$$

Therefore, the inverse function of $t \longrightarrow \widetilde{A}_t$ is $H_{\alpha,\mu} \circ N_{\alpha,\mu}$. This entails, by the definition, that:

$$Y_{1,u}^{(\alpha,\mu)} = \left(1 + 2\mu\, I^{(\alpha)}_{N_{\alpha,\mu}(u)}\right)^{1/2\mu}.$$

The desired result follows easily. ∎

5 The laws of some perpetuities associated with $I^{(\alpha,\mu)}$

In this section, we study some relationship existing between results in the previous sections, and the discussion of Bertoin-Yor [1, Section 3]. We keep the same notation as before. In particular, $\mu_\alpha = 1/2\alpha$.

5.1 The law of $\mathcal{I}\left(I^{(\alpha,\mu_\alpha)}\right)$

In this subsection, we shall determine the law of the perpetuity $\mathcal{I}_\alpha := \mathcal{I}\left(I^{(\alpha,\mu_\alpha)}\right)$, associated with the subordinator: $I^{(\alpha,\mu_\alpha)}$. We denote by $\mathcal{R}^{(\alpha)}$ the corresponding random variable defined via formula (5), namely:

$$\mathbb{E}\left[\frac{1}{Y_{1,t}^{(\alpha,\mu_\alpha)}}\right] = \mathbb{E}[\exp(-t\,\mathcal{R}^{(\alpha)})].$$

By Remark 3.1,

$$\Phi^{(\alpha,\mu_\alpha)}(s) = F_{\alpha,\lambda_\alpha}(s) = (\lambda_\alpha)^\alpha\, \frac{\Gamma(\alpha\,(s+1))}{\Gamma(\alpha\, s)}$$

where λ_α is given by (17). Therefore, by formula (6),

$$\forall n \geq 1, \quad \mathbb{E}[(\mathcal{R}^{(\alpha)})^n] = (c_\alpha)^{-n} \frac{\Gamma(\alpha\,(n+1))}{\Gamma(\alpha)}$$

with $c_\alpha = (\lambda_\alpha)^{-\alpha}$. Let γ_α denote a gamma variable with index α, i.e:

$$\mathbb{P}(\gamma_\alpha \in \mathrm{d}t) = \frac{1}{\Gamma(\alpha)}\,\mathrm{e}^{-t}\,t^{\alpha-1}\,\mathrm{d}t.$$

One has:

$$\forall n \geq 1, \quad \mathbb{E}[(\gamma_\alpha)^{\alpha n}] = \frac{\Gamma(\alpha\,(n+1))}{\Gamma(\alpha)}$$

and therefore

$$\mathcal{R}^{(\alpha)} \overset{\text{(law)}}{=} (c_\alpha)^{-1}\,(\gamma_\alpha)^\alpha. \tag{19}$$

We now introduce some further notation. If X is a positive r.v., we denote by $\widetilde{X}$ the r.v. whose law is defined by:

$$\mathbb{E}[f(\widetilde{X})] = \mathbb{E}\left[\frac{X}{\mathbb{E}[X]}\,f(X)\right].$$

(We call $\widetilde{X}$ the *length biased perturbation* of X.) We also set: $\tau_\alpha = \tau_1^{(\alpha)}$. Then, by Bertoin-Yor [1, Lemma 6], there is the factorization:

$$\mathbf{e} = (\gamma_\alpha)^\alpha \cdot \widetilde{(\tau_\alpha)^{-\alpha}}. \tag{20}$$

By identification, from (4), (19) and (20) we then obtain:

$$\mathcal{I}_\alpha \overset{\text{(law)}}{=} c_\alpha\,\widetilde{(\tau_\alpha)^{-\alpha}}.$$

5.2 The law of $\mathcal{I}\left(I^{(1-\alpha,\mu_\alpha)}\right)$

In this subsection, we shall determine the law of the perpetuity $\mathcal{I}'_\alpha := \mathcal{I}\left(I^{(1-\alpha,\mu_\alpha)}\right)$, associated with the subordinator: $I^{(1-\alpha,\mu_\alpha)}$. By Theorem 3.2,

$$\Phi^{(1-\alpha,\mu_\alpha)}(s) = \widehat{C}_{1-\alpha}\,(\mu_\alpha)^{1-\alpha}\,\frac{\Gamma(\alpha\,(s-1)+1)}{\Gamma(\alpha\,s)}.$$

We set:

$$c'_\alpha = \left[\alpha\,\widehat{C}_{1-\alpha}\,(\mu_\alpha)^{1-\alpha}\right]^{-1}.$$

Then,

$$\Phi^{(1-\alpha,\mu_\alpha)}(1) = (c'_\alpha)^{-1}\,\frac{1}{\alpha\,\Gamma(\alpha)}$$

and

$$\forall n \geq 2, \quad \Phi^{(1-\alpha,\mu_\alpha)}(n) = (c'_\alpha)^{-1}\,\frac{(n-1)\,\Gamma(\alpha\,(n-1))}{\Gamma(\alpha\,n)}.$$

Hence, by formula (3):

$$\forall n \geq 1, \quad \mathbb{E}\left[\left(\mathcal{I}'_\alpha\right)^n\right] = (c'_\alpha)^n\,\Gamma(\alpha\,n+1) = (c'_\alpha)^n\;\mathbb{E}[\mathbf{e}^{\alpha n}].$$

Consequently,

$$\mathcal{I}'_\alpha \overset{(\text{law})}{=} c'_a\;\mathbf{e}^\alpha.$$

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