Symplectic spectral geometry of semiclassical operators*

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Abstract

In the past decade there has been a flurry of activity at the intersection of spectral theory and symplectic geometry. In this paper we review recent results on semiclassical spectral theory for commuting Berezin-Toeplitz and $\hbar$-pseudodifferential operators. The paper emphasizes the interplay between spectral theory of operators (quantum theory) and symplectic geometry of Hamiltonians (classical theory), with an eye towards recent developments on the geometry of finite dimensional integrable systems.

1 Introduction

This paper gives a concise exposition of some recent results on spectral theory of $\hbar$-pseudodifferential and Berezin-Toeplitz operators. Most of what I will say is contained in my papers with L. Charles, L. Polterovich, and S. Vũ Ngọc [3, 17, 12]. I will also discuss some recent works on symplectic geometry of finite dimensional completely integrable Hamiltonian systems by the author and Vũ Ngọc [13, 14] because they are central to the spectral theory. The papers [13, 14] contain classification results for the so called completely integrable Hamiltonian systems of *semitoric type*, and are in the spirit of the seminal papers of Atiyah [1], Guillemin-Sternberg [10], and Delzant [8] on toric systems.

*Based on a scheduled plenary Talk by the author at the 2012 Joint Congress of the Belgian, Royal Spanish and Luxembourg Mathematical Societies. I was unable to attend the Congress and I thank the organizers for the invitation to write this paper. This paper is *not* a survey but rather a report on recent developments. Throughout we keep a more informal tone than in a regular research paper. We refer to the articles [15, 17] for details and further bibliographic references.

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In this paper we are going to emphasize the connection of symplectic geometry with spectral theory and microlocal analysis (see Guillemin-Sternberg [11] and Zworski [19]). In fact, the development of semiclassical microlocal analysis in the past four decades now allows a fruitful interplay between symplectic geometry (classical mechanics) and spectral theory (quantum mechanics). The literature on these subjects is vast and I refer to the aforementioned works for a more comprehensive list of references.

2 Symplectic geometry and integrable systems

The word "symplectic" was introduced by H. Weyl (Elmshorn 1885-Zürich 1955) in his book on classical groups [18]. It derives from a Greek word meaning *complex*. Symplectic geometry is concerned with the study of symplectic manifolds. A *symplectic manifold* is a pair (M,ω) consisting of a smooth C^∞-manifold M and a closed and non-degenerate differential 2-form ω on it, called a *symplectic form*. For instance, we can take M to be a surface, and ω to be an area form on it (in dimension 2, a symplectic form is the same as an area form). Another typical example is $\mathbb{R}^{2n}$ equipped with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ and symplectic form $\sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$. The cotangent bundle of any compact smooth manifold is also a symplectic manifold in a natural way.

Symplectic manifolds are even-dimensional (because the symplectic form is non-degenerate) and orientable (because $\omega^{\dim M/2}$ is a volume form). Let's write $2n$ for the dimension of M. If M is compact, then one can use Stokes' theorem to show that for every $0 \leqslant k \leqslant n$ we have that $0 \neq [\omega]^k \in \mathrm{H}^{2k}_{\mathrm{dR}}(M)$ so compact symplectic manifolds are topologically nontrivial. By a famous theorem of Darboux [4], near each point in (M, ω) there exist coordinates $(x_1, y_1, \ldots, x_n, y_n)$ in which ω has the form $\sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$, so symplectic manifolds have *no* local invariants, except the dimension.

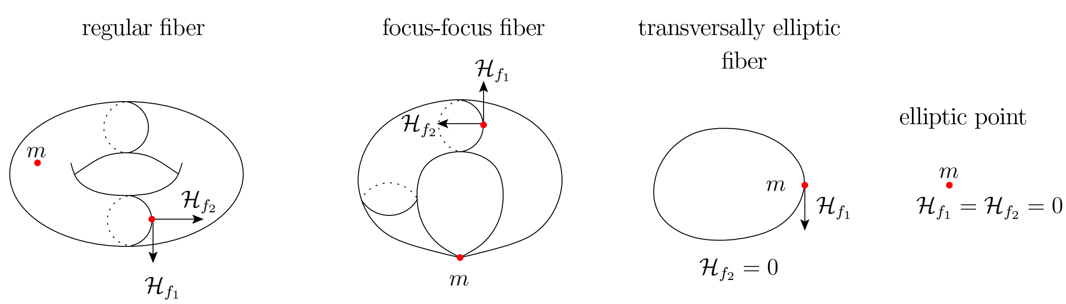

Figure 1: Some possible singularities of an integrable system.

One important class of dynamical systems which can be studied with the tools of symplectic geometry are those called *integrable*.

Definition 1. A *completely integrable system* (or simply an *integrable system*) on a $2n$-dimensional symplectic manifold (M, ω) is a smooth map $F := (f_1, \ldots, f_n)\colon M \longrightarrow \mathbb{R}^n$ such that each f_i is constant along the flow[1] of each *Hamiltonian vector field* $\mathcal{H}_{f_j}$, where $\mathcal{H}_{f_j}$ is defined by Hamilton's equation $\omega(\mathcal{H}_{f_j}, \cdot) = -\mathrm{d}f_j$ and, moreover, the vector fields $\mathcal{H}_{f_1}, \ldots, \mathcal{H}_{f_n}$ are linearly independent almost everywhere on M.

A *singularity* is a point $m \in M$ at which the vector fields $\mathcal{H}_{f_1}, \ldots, \mathcal{H}_{f_n}$ are linearly dependent. There are many mechanical systems which are integrable, for instance: the *coupled spin-oscillator* (also called Jaynes-Cummings model, see [2]), the *spherical pendulum*, the *two-body problem*, the *Lagrange top*, or the *Kowalevski top*. All of these systems have singularities.

While there are a few results on symplectic theory of integrable systems, the subject is largely not understood, we refer to [15, 17] for a more extensive discussion. In particular, [15] aims to give a more comprehensive description of the current state of the art of the symplectic theory of integrable systems. It is interesting to note that some features about the symplectic geometry of singularities can be detected using spectral theory, see for instance [16], where this is done for some of the singularities of the coupled spin-oscillator.

3 Notions of spectrum

3.1 Classical and quantum spectra

The self-adjoint operators $T_1, \ldots, T_d$ on a Hilbert space are mutually commuting if their spectral measures $\mu_1, \ldots, \mu_d$ pairwise commute. Then one can define the joint spectral measure on $\mathbb{R}^d$:

$$\mu := \mu_1 \otimes \cdots \otimes \mu_d.$$

Definition 2. The *joint spectrum* of $(T_1, \ldots, T_d)$ is the support of the joint spectral measure. It is denoted by $\mathrm{JointSpec}(T_1, \ldots, T_d)$.

For instance, if the T_j's are endomorphisms of a finite dimensional vector space, then the joint spectrum of $T_1, \ldots, T_d$ is the set

$$\Big\{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d \mid \exists v \neq 0 \text{ such that } P_j v = \lambda_j v \ \ \forall j = 1, \ldots, n\Big\}.$$

If $T_1, \ldots, T_d$ are pairwise commuting semiclassical operators, then of course the joint spectrum of $T_1, \ldots, T_d$ depends on the semiclassical parameter $\hbar$.

Following the physicists, we use the following definition.

[1] That is, any two f_i, f_j commute in the sense that the Poisson brackets vanish:

$$\{f_i, f_j\} := \omega(\mathcal{H}_{f_i}, \mathcal{H}_{f_j}) = 0, \quad \text{for all} \quad 1 \leqslant i, j \leqslant n.$$

Definition 3. We call *classical spectrum* of $(T_1, \ldots, T_d)$ the closure of the image

$$F(M) \subset \mathbb{R}^d,$$

where $F = (f_1, \ldots, f_d)$ is the map of principal symbols of $T_1, \ldots, T_d$.

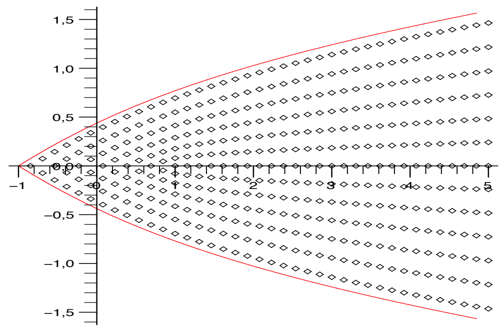

Figure 2: Joint spectrum of quantum *Jaynes-Cummings model*.

3.2 Jaynes-Cummings model

An interesting system given by two self-adjoint commuting operators is the quantum *Jaynes-Cummings model*, studied in detail in [17], and which is given as follows. For any $\hbar > 0$ such that $2 = \hbar(n+1)$, for some non-negative integer $n \in \mathbb{N}$, let $\mathcal{H} \subset \mathrm{L}^2(\mathbb{R})$ denote the standard $n+1$-dimensional Hilbert space quantizing the sphere S^2. Consider the operators:

$$\hat{x} := \frac{\hbar}{2}(a_1 a_2^* + a_2 a_1^*), \qquad \hat{y} := \frac{\hbar}{2\mathrm{i}}(a_1 a_2^* - a_2 a_1^*), \qquad \hat{z} := \frac{\hbar}{2}(a_1 a_1^* - a_2 a_2^*).$$

where

$$a_i := \frac{1}{\sqrt{2\hbar}}\Big(\hbar \frac{\partial}{\partial x_j} + x_j\Big), \quad i = 1, 2.$$

The operators on the Hilbert space $\mathcal{H} \otimes \mathrm{L}^2(\mathbb{R}) \subset \mathrm{L}^2(\mathbb{R}^2) \otimes \mathrm{L}^2(\mathbb{R})$

$$\hat{f}_1 := \mathrm{Id} \otimes \Big(-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}\,u^2} + \frac{u^2}{2}\Big) + (\hat{z} \otimes \mathrm{Id})$$

and

$$\hat{f}_2 := \frac{1}{2}(\hat{x} \otimes u + \hat{y} \otimes (\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial u})),$$

are unbounded, self-adjoint, and commute. The spectrum of $\hat{f}_1$ is discrete and consists of eigenvalues in

$$\hbar\Big(\frac{1-n}{2} + \mathbb{N}\Big).$$

The joint spectrum for a fixed value of $\hbar$ is depicted in Figure 2. This quantum model is in fact constructed by hands on quantization of the classical system given by the symplectic manifold $M = S^2 \times \mathbb{R}^2$, where S^2 is viewed as the unit sphere in $\mathbb{R}^3$ with coordinates (x, y, z), and the second factor $\mathbb{R}^2$ is equipped with coordinates (u, v), and the Hamiltonians $f_1 := (u^2 + v^2)/2 + z$ and $f_2 := \frac{1}{2}(ux + vy)$. So f_1 and f_2 are the principal symbols of $\hat{f}_1$ and $\hat{f}_2$.

4 Classical from semiclassical spectra

4.1 Compact case

Let (M, ω) be a compact symplectic manifold whose symplectic form represents an integral de Rham cohomology class of M. In what follows such symplectic manifolds will be called *prequantizable*. They admit a prequantum line bundle $\mathcal{L}$. Assume that M is endowed with a complex structure j compatible with ω, so that M is *Kähler* and $\mathcal{L}$ is *holomorphic*. Here the holomorphic structure of the prequantum bundle is the unique one compatible with the connection.

For a positive integer $k = 1/\hbar$, we write $\mathcal{H}_\hbar$ for the space $\mathrm{H}^0(M, \mathcal{L}^k)$ of holomorphic sections of $\mathcal{L}^k$. Since M is compact, $\mathcal{H}_\hbar$ is a closed finite dimensional subspace of the Hilbert space $\mathrm{L}^2(M, \mathcal{L}^k)$. Here the scalar product is defined by integrating the Hermitian pointwise scalar product of sections against the Liouville measure of M.

Denote by $\Pi_\hbar$ the orthogonal projector of $\mathrm{L}^2(M, \mathcal{L}^k)$ onto $\mathcal{H}_\hbar$. In this case, we have the following definition.

Definition 4. A *Berezin-Toeplitz operator* is a sequence

$$T := (T_\hbar \colon \mathcal{H}_\hbar \to \mathcal{H}_\hbar)_{\hbar = 1/k;\, k \in \mathbb{N}^*}$$

of operators of the form

$$(T_\hbar := \Pi_\hbar f(\cdot, k))_{k \in \mathbb{N}^*},$$

where $f(\cdot, k)$, viewed as a multiplication operator, is a sequence in $\mathrm{C}^\infty(M)$ with an asymptotic expansion

$$f_0 + k^{-1} f_1 + \dots$$

for the C^∞ topology. The coefficient f_0 is the *principal symbol* of $(T_\hbar)_{\hbar=1/k; k \in \mathbb{N}^*}$.

Before stating the result of this section, recall that the *Hausdorff distance* $\mathrm{d}_H(A, B)$ between two subsets A and B of $\mathbb{R}^n$ is the infimum of the $\epsilon > 0$ such that $A \subseteq B_\epsilon$ and $B \subseteq A_\epsilon$, where for any subset X of $\mathbb{R}^n$, the set X_ϵ is

$$X_\epsilon := \bigcup_{x \in X} \{m \in \mathbb{R}^n \mid \|x - m\| \leqslant \epsilon\}.$$

If $(A_k)_{k \in \mathbb{N}^*}$ and $(B_k)_{k \in \mathbb{N}^*}$ are sequences of subsets of $\mathbb{R}^n$, we say that $A_k = B_k + \mathcal{O}(k^{-\infty})$ if

$$\mathrm{d}_H(A_k, B_k) = \mathcal{O}(k^{-N}) \quad \forall N \in \mathbb{N}^*.$$

In the following theorem, the convergence is taken in the Hausdorff metric.

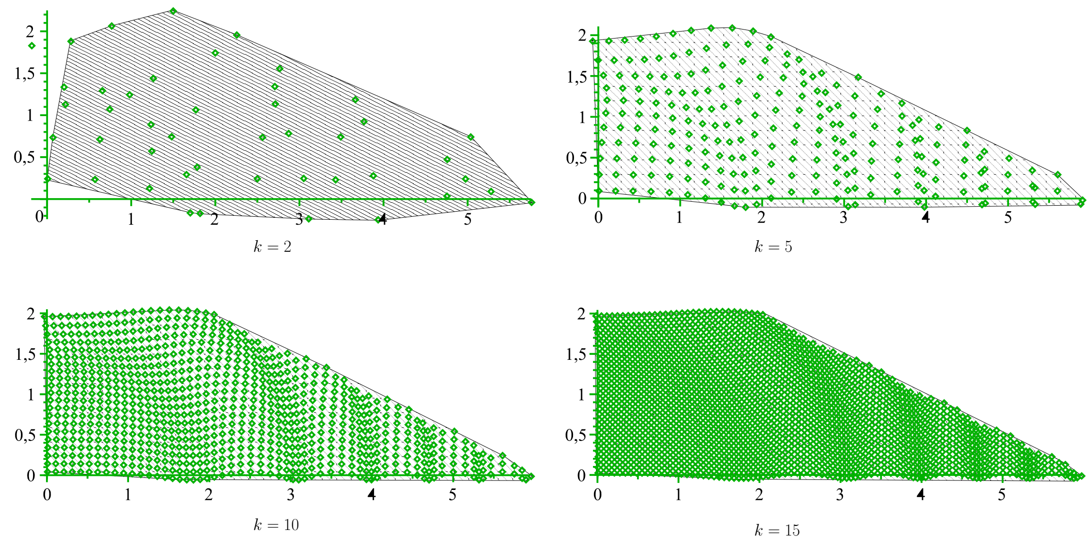

Figure 3: Convergence of convex hulls of spectra

Theorem 5 (Pelayo-Polterovich-Vũ Ngọc [12])**.** *Let $\mathcal{F}_d := (T_1, \dots, T_d)$ be a family of pairwise commuting self-adjoint Berezin-Toeplitz operators on M. Let $\mathcal{S}_d \subset \mathbb{R}^d$ be the classical spectrum of $\mathcal{F}_d$, and suppose that it is a convex set. Then* $\mathrm{JointSpec}(\mathcal{F}_d) \to \mathcal{S}_d$*, as $\hbar \to 0$.*

The proof of Theorem 5 uses microlocal techniques (the key lemma for the proof is [12, Lemma 5]). The result proven in [12] is stronger than Theorem 5: one does not need to assume that $\mathcal{S}_d$ is convex. In the general case, what we proved is that we have convergence (still in the Hausdorff metric) at the level of convex hulls

$$\text{Convex Hull}(\mathrm{JointSpec}(\mathcal{F}_d)) \to \overline{\text{Convex Hull}(\mathcal{S}_d)},$$

as $\hbar \to 0$ (see Figure 3). These results may be extended in a natural way to non-commuting operators, see [12, Section 9].

4.2 Noncompact case

Suppose that M is $\mathbb{R}^{2n}$, or the cotangent bundle T^*X of a compact smooth n--dimensional manifold X (with a smooth density μ). In these cases a semiclassical quantization of M is given by semiclassical $\hbar$-pseudodifferential operators, a well-known semiclassical version of the quantization given by homogeneous pseudodifferential operators (see for instance Dimassi-Sjöstrand [9]). Symbolic calculus of pseudodifferential operators holds when the symbols belong to a Hömander class, eg. take $\mathcal{A}_0$ consisting of functions $f \in \mathrm{C}^\infty(\mathbb{R}^{2n}_{(x,\xi)})$ such that there exists $m \in \mathbb{R}$ for which

$$\left|\partial^\alpha_{(x,\xi)} f\right| \leqslant C_\alpha \langle (x,\xi) \rangle^m$$

for all $\alpha \in \mathbb{N}^{2n}$. Here $\langle z \rangle := (1 + |z|^2)^{1/2}$.

If $f \in \mathcal{A}_0$, its *Weyl quantization* is defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$(\mathrm{Op}_\hbar(f)u)(x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}((x-y)\cdot\xi)} f(\tfrac{x+y}{2}, \xi) u(y) \mathrm{d}y \mathrm{d}\xi.$$

Let's cover X with charts $U_1, \ldots, U_N$, each of which is identified with a convex bounded domain of $\mathbb{R}^n$ equipped with the Lebesgue measure. Consider a partition of unity $\chi_1^2, \ldots, \chi_n^2$ subordinated to $U_1, \ldots, U_N$. Let $f \in \mathrm{C}^\infty(\mathrm{T}^*X)$ such that $\left|\partial_\xi^\alpha f(x)\right| \leqslant C_\alpha \langle\xi\rangle^m$ for all $(x, \xi) \in \mathrm{T}^*X$, $\alpha \in \mathbb{N}^n$, for some $m \in \mathbb{R}$. Let $\mathrm{Op}_\hbar^j(f)$ be the *Weyl quantization* calculated in U_j and define:

$$\mathrm{Op}_\hbar(f)u := \sum_{j=1}^{N} \chi_j \cdot \mathrm{Op}_\hbar^j(f)(\chi_j u)\ , \text{for } u \in \mathrm{C}^\infty(X)\,,$$

which is a pseudodifferential operator on X with principal symbol $f = \sum_{i=1}^N f\chi_j^2$.

In what follows we work with the standard Hörmander class of symbols depending on $\hbar$, $a(x, \xi, \hbar)$ on $\mathbb{R}^{2n}$ or T^*X with compact X. We say that *a mildly depends on* $\hbar$ if

$$a(x, \xi, \hbar) = a_0(x, \xi) + \hbar\, a_{1,\hbar}(x, \xi),$$

where all $a_{1,\hbar}(x, \xi)$ are uniformly bounded in $\hbar$ and supported in the same compact set.[2]

Definition 6. A *semiclassical $\hbar$-pseudodifferential operator on X* is any sequence of the form $T := (\mathrm{Op}_\hbar(f))_{\hbar\in(0,1]}$.

The analysis of $\hbar$-pseudodifferential operators is delicate due to the possible unboundedness of the operators.

Theorem 7 (Pelayo-Polterovich-Vũ Ngọc [12])**.** *Let X be either $\mathbb{R}^n$, or a closed manifold. Let $\mathcal{F}_d := (T_1, \ldots T_d)$ be a family of pairwise commuting self-adjoint semiclassical $\hbar$-pseudodifferential operators on X whose symbols mildly depend on $\hbar$. Let $\mathcal{S}_d \subset \mathbb{R}^d$ be the classical spectrum of $\mathcal{F}_d$, and suppose that it is a convex set. Then from the family* $\{\mathrm{JointSpec}(\mathcal{F}_d)\}_{\hbar\in J}$ *one can recover $\mathcal{S}_d$. If moreover each operator T_i is bounded, $1 \leqslant i \leqslant d$, then* $\mathrm{JointSpec}(\mathcal{F}_d) \to \mathcal{S}_d$, *as $\hbar \to 0$.*

As it was explained before, a more general result holds, where one does not need to assume convexity of the classical spectrum.

Example 8. The results in this section apply to a number of examples. For instance, Theorem 7 applies to the system given by a particle in a rotationally symmetric potential ([12, Section 9.2]). Theorem 5 applies to systems given by Hamiltonian torus actions ([12, Section 8.2]), and to the coupled-angular momenta system ([12, Section 8.3]). ⊘

[2] Note: the principal symbol a_0 can be unbounded.

5 Spectral theory for systems of toric type

5.1 Symplectic geometry

An n-tuple of smooth functions

$$(\mu_1, \ldots, \mu) \colon M \to \mathbb{R}^n$$

on a $2n$-dimensional symplectic manifold (M, ω) is called a *momentum map* for a Hamiltonian n-torus action if the Hamiltonian flows $t_j \mapsto \varphi_{\mu_j}^{t_j}$ are periodic of period 1, and pairwise commute:

$$\varphi_{\mu_j}^{t_j} \circ \varphi_{\mu_i}^{t_i} = \varphi_{\mu_i}^{t_i} \circ \varphi_{\mu_j}^{t_j}$$

so that they define an action of $\mathbb{R}^n/\mathbb{Z}^n$. We say that (M, ω, μ) is a *toric integrable system*, or simply a *toric system*, if in addition M is compact and connected, and the action of $\mathbb{R}^n/\mathbb{Z}^n$ is effective.

Two toric systems (M, ω, μ) and (M', ω', μ') are *isomorphic* if there exists a symplectomorphism $\varphi : M \to M'$ such that

$$\varphi^* \mu' = \mu.$$

The convexity theorem of Atiyah [1] and Guillemin-Sternberg [10] implies that $\mu(M)$ is a convex polytope in $\mathbb{R}^n$. By the Delzant classification theorem [8], $\mu(M)$ is a so called *Delzant polytope* (i.e. rational, simple, and smooth), and the toric integrable system (M, ω, μ) is classified, up to isomorphisms, by $\mu(M)$.

5.2 Semiclassical spectral theory

In the case of toric integrable systems, a complete description of the semiclassical spectral theory can be given. That is, we are going to make a much stronger assumption than in Section 4.1: "being toric"; but we are also going to obtain much more information (in fact, *all* the information). Let $(\mu_1, \ldots, \mu_n)$ be a toric integrable system on a compact prequantizable symplectic manifold M equipped with a prequantum bundle $\mathcal{L}$ and a compatible complex structure j.

Theorem 9 (Charles-Pelayo-Vũ Ngọc [3]). *Let $T_1, \ldots, T_n$ be commuting Berezein-Toeplitz operators with principal symbols $\mu_1, \ldots, \mu_n$. Then*

$$\mathrm{JointSpec}(T_1, \ldots, T_n)$$

is given by

$$g\Big(\mu(M) \cap \Big(v + \frac{2\pi}{k}\mathbb{Z}^n\Big); k\Big) + \mathcal{O}(k^{-\infty}),$$

where v is any vertex of $\mu(M)$ and $g(\cdot; k) : \mathbb{R}^n \to \mathbb{R}^n$ admits a C^∞-asymptotic expansion of the form $g(\cdot; k) = \mathrm{Id} + k^{-1} g_1 + k^{-2} g_2 + \cdots$ where each $g_j : \mathbb{R}^n \to \mathbb{R}^n$ is smooth.

Moreover, the multiplicity of the eigenvalues in Theorem 9 can be described: for all sufficiently large k, the multiplicity of the eigenvalues of $\mathrm{JointSpec}(T_1, \ldots, T_n)$ is 1, and there exists a small constant $\delta > 0$ such that each ball of radius $\frac{\delta}{k}$ centered at an eigenvalue contains precisely only that eigenvalue.

5.3 Semiclassical isospectrality

The following type of inverse problem is classical and belongs to the realm of questions in inverse spectral theory, going back to similar questions raised (and in many cases answered) by pioneer works of Colin de Verdière [5, 6]. Let $(M,\ \omega,\ \mu : M \to \mathbb{R}^n)$ be a toric integrable system with a prequantum bundle $\mathcal{L}$ and a compatible complex structure j.

Theorem 10 (Charles-Pelayo-Vũ Ngọc [3]). *Let $\mathcal{F}_n := (T_1, \ldots, T_n)$ be a family of commuting self-adjoint Berezin-Toeplitz operators with principal symbols $\mu_1, \ldots, \mu_n$. Then one can recover (M, ω, μ) from the limit of the joint spectrum of $T_1, \ldots, T_n$.*

Theorem 10 follows from combining the Delzant theorem with Theorem 9. In fact, combining Delzant's theorem with Theorem 5 gives an easier proof of Theorem 10 which does not use Theorem 9 which is a more difficult (but much more informative) result.

6 Spectral theory for systems of semitoric type

6.1 Symplectic geometry

A *semitoric system* consists of a connected symplectic four-dimensional manifold $(M,\ \omega)$ and two smooth functions $f_1 \colon M \to \mathbb{R}$ and $f_2 \colon M \to \mathbb{R}$ such that f_1 is constant along the flow of the Hamiltonian vector field $\mathcal{H}_{f_2}$ generated by f_2 or, equivalently, $\{f_1,\ f_2\} = 0$ and for almost all points $p \in M$, the vectors $\mathcal{H}_{f_1}(p)$ and $\mathcal{H}_{f_2}(p)$ are linearly independent. Moreover,[3] f_1 is the momentum map of an S^1-action on M, and it is a proper map. Finally, we require $F := (f_1, f_2) \colon M \to \mathbb{R}^2$ to have only non-degenerate singularities without hyperbolic components. Two semitoric systems

$$(M_1, \omega_1, F_1 := (f_1^1, f_1^1)) \text{ and } (M_2, \omega_2, F_2 := (f_1^2, f_2^2))$$

are *isomorphic* if there exists a symplectomorphism $\phi \colon M_1 \to M_2$, and a smooth map $\varphi \colon F_1(M_1) \to \mathbb{R}$ with $\partial_2\varphi \neq 0$, such that

$$\begin{cases} \phi^* f_1^1 = f_1^2 \\ \phi^* f_2 = \varphi(f_1^1, f_2^1). \end{cases}$$

Semitoric systems can be classified, up to isomorphisms, in terms of five symplectic invariants. This classification appeared in [13, 14]. Roughly speaking, these invariants are as follows: an integer m_f counting the number of isolated singularities, a collection of Taylor series classifying symplectically a saturated neighborhood of the singular fiber corresponding to these sigularities, a family of rational convex polygons

$$\left(\Delta,\ (\ell_j)_{j=1}^{m_f},\ (\epsilon_j)_{j=1}^{m_f}\right),$$

[3]this is the condition which gives rise to the name "semitoric"

which is constructed from the image $F(M) \subset \mathbb{R}^n$ of the system by performing a very precise "cutting" (the ℓ_j's are vertical lines cutting Δ with orientations $\epsilon_j = \pm 1$), an invariant measuring the volumes of certain submanifolds meeting at each of the isolated singularities, and, finally, a collection of integers measuring how twisted the Lagrangian fibration of the system is around the singularities.

Toric systems are a particular case of semitoric systems. If the system is toric, then four of the invariants do not appear (and the remaining one is simpler: a polygon, instead of a class of polygons).

6.2 Semiclassical spectral theory

The semiclassical spectral theory of semitoric systems is not yet understood. In [17, Section 9] it is conjectured that from the semiclassical joint spectrum of two self-adjoint commuting operators one can recover the integrable system given by the principal symbols, up to symplectic isomorphisms, provided these principal symbols form a semitoric system. A sketch of proof of this conjecture appeared in [17, Section 3.2].

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