The order of the commutator on $SU(3)$ and an application to gauge groups

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Abstract

We show that the commutator map on $SU(3)$ has order $2^3 \cdot 3 \cdot 5$. As an application, we give an upper bound on the number of homotopy types of gauge groups for principal $SU(3)$-bundles over an n-sphere.

1 Introduction

Let G be an H-group, defined as a homotopy associative H-space with a homotopy inverse. Let $\bar{c}\colon G \times G \longrightarrow G$ be the map defined pointwise by $\bar{c}(x, y) = xyx^{-1}y^{-1}$. Consider the cofibration sequence $G \vee G \longrightarrow G \times G \longrightarrow G \wedge G \xrightarrow{\delta} \Sigma G \vee \Sigma G$. Observe that $\bar{c}$ is null homotopic when restricted to $G \vee G$, implying that $\bar{c}$ factors through a map $c\colon G \wedge G \longrightarrow G$. Since $\Sigma(G \times G) \simeq (\Sigma G \vee \Sigma G) \vee (\Sigma G \wedge G)$, the cofibration connecting map δ is null homotopic. Thus the homotopy class of c is uniquely determined by the homotopy class of $\bar{c}$. Call the map c the *commutator* of G.

If G is finite then, rationally, it is homotopy equivalent to a product of Eilenberg-MacLane spaces as an H-space, implying that it is homotopy commutative. So, rationally, c is trivial, which implies that the order of c is finite. A fundamental problem is to determine the order of c. However, this is known only in extremely simple cases. For example, consider the case of $SU(n)$. If $n = 2$ then $SU(2) \simeq S^3$ and the order of $S^3 \wedge S^3 \xrightarrow{c} S^3$ is 12, a consequence of Toda's calculations [To].

Received by the editors March 2012.
Communicated by Y. Félix.
2010 *Mathematics Subject Classification* : Primary 55P15, 57T99, Secondary 54C35.
Key words and phrases : $SU(3)$, commutator, order, gauge group, homotopy type.

On the other hand, if $n > 2$ then the order of $SU(n) \wedge SU(n) \xrightarrow{c} SU(n)$ is unknown. It is not even clear what an upper bound should be. In this paper we consider the case of $SU(3)$ and show the following.

Theorem 1.1. *The commutator $SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ has order $120 = 2^3 \cdot 3 \cdot 5$.*

Theorem 1.1 can be used to help determine the homotopy types of certain gauge groups. In general, if G is a topological group, X is a space, and $P \longrightarrow X$ is a principal G-bundle, then the gauge group $\mathcal{G}(P)$ of the bundle is the group of G-equivariant automorphisms of P which fix X. Crabb and Sutherland [CS] showed that if G is a compact, connected Lie group and X is a connected, finite CW-complex, then while there may be infinitely many distinct principal G-bundles over X, their corresponding gauge groups have only finitely many distinct homotopy types. However, Crabb and Sutherland gave no upper bound on the number of distinct gauge groups, and precise numbers have been worked out only in a few cases of very low rank [K, HK, KKKT, Th]. We consider instead the p-local homotopy types for a prime p, and produce explicit upper bounds for the number of distinct p-local homotopy types of gauge groups in a more restricted setting. The possible existence of a kind of Zabrodsky mixing of homotopy types means that it is not really clear how our p-local result relates to Crabb and Sutherland's integral result. For a prime p and an integer m, let $\nu_p(m)$ be the largest integer r such that p^r divides m but p^{r+1} does not divide m.

Theorem 1.2. *Let G be a compact, connected Lie group and let Y be a space. Fix a homotopy class $[f] \in [\Sigma Y, BG]$. For an integer k, let $P_k \longrightarrow \Sigma Y$ be the principal G-bundle induced by kf, and let $\mathcal{G}_k$ be its gauge group. If the order of the commutator $G \wedge G \xrightarrow{c} G$ is m, then the number of distinct p-local homotopy types for the gauge groups $\{\mathcal{G}_k\}$ is at most $\nu_p(m) + 1$.*

When $G = SU(3)$ then for any homotopy class $[f] \in [\Sigma Y, BSU(3)]$, Theorems 1.1 and 1.2 imply that the number of distinct p-local homotopy types for the gauge groups $\{\mathcal{G}_k\}$ is at most 4 if $p = 2$, 2 if $p = 3$ or $p = 5$, and 1 if $p > 5$. These general upper bounds closely match known lower bounds. If $Y = S^3$ and $[f] \in [S^4, BSU(3)]$ represents an integral generator, then [HK] shows that there are exactly four 2-types, two 3-types and one p-type for $p \geq 5$. The advantage of Theorem 1.2 is that it works for any space Y, not just $Y = S^3$.

2 A lower bound on the order of c

Take homology with integer coefficients. Recall that $H^*(SU(3)) \cong \Lambda(x, y)$ where $|x| = 3$, $|y| = 5$. Let $\imath\colon \Sigma\mathbb{C}P^2 \longrightarrow SU(3)$ be the canonical map which induces the projection onto the generating set in cohomology. Since $\imath$ is the inclusion of the 7-skeleton of $SU(3)$, whose next cell is in dimension 10, it follows that the map $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath\wedge\imath} SU(3) \wedge SU(3)$ is the inclusion of the 10-skeleton.

In this section we will show that the composite

$$\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath\wedge\imath} SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$$

has order 120. This implies that the order of c must be at least 120. To do this we use a method due to Hamanaka and the first author [HK].

We begin with a preliminary lemma. For any integer $k \geq 0$, let $W_{3,k} = SU(3+k)/SU(3)$. Let η be the stable generator of $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Lemma 2.1. *There are isomorphisms $\pi_i(W_{3,\infty}) \cong \mathbb{Z}$ for $i \in \{7, 9, 11\}$.*

Proof. Since there are fibrations $W_{3,k} \longrightarrow W_{3,k+1} \longrightarrow S^{2k+7}$ for each $k \geq 0$, by stability we obtain $\pi_i(W_{3,\infty}) \cong \pi_i(W_{3,2})$ for $i \in \{7, 9\}$ and $\pi_{11}(W_{3,\infty}) \cong \pi_{11}(W_{3,3})$. Now consider $W_{3,2}$. By definition, $W_{3,2} = SU(5)/SU(3)$ so as a CW-complex we have $W_{3,2} = S^7 \cup_\eta e^9 \cup e^{16}$. The Hurewicz homomorphism then implies that $\pi_7(W_{3,2}) \cong \mathbb{Z}$. As well, by connectivity $\pi_m(W_{3,2}) \cong \pi_m(S^7 \cup_\eta e^9)$ for $9 \leq m \leq 11$. The cofibration $S^8 \xrightarrow{\eta} S^7 \longrightarrow S^7 \cup_\eta e^9$ induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_9(S^8) \xrightarrow{\eta^*} \pi_9(S^7) \longrightarrow \pi_9(S^7 \cup_\eta e^9) \longrightarrow \pi_9(S^9) \xrightarrow{\eta^*} \pi_9(S^8) \longrightarrow \cdots.$$

Observe that the first arrow is an isomorphism while the fourth arrow is an epimorphism. Thus $\pi_9(S^7 \cup_\eta e^9) \cong \mathbb{Z}$. Similar exact sequence arguments show that $\pi_{10}(S^7 \cup_\eta e^9) \cong \pi_{11}(S^7 \cup_\eta e^9) \cong 0$. Hence $\pi_9(W_{3,2}) \cong \mathbb{Z}$ and $\pi_{10}(W_{3,2}) \cong \pi_{11}(W_{3,2}) \cong 0$. Finally, the fibration $W_{3,2} \longrightarrow W_{3,3} \longrightarrow S^{11}$ induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_{11}(W_{3,2}) \longrightarrow \pi_{11}(W_{3,3}) \longrightarrow \pi_{11}(S^{11}) \longrightarrow \pi_{10}(W_{3,2}) \longrightarrow \cdots.$$

Since $\pi_{10}(W_{3,2}) \cong \pi_{11}(W_{3,2}) \cong 0$, we immediately obtain $\pi_{11}(W_{3,3}) \cong \pi_{11}(S^{11}) \cong \mathbb{Z}$. ∎

Consider the fibration sequence

$$\Omega SU(\infty) \xrightarrow{\Omega\pi} \Omega W_{3,\infty} \xrightarrow{\delta} SU(3) \longrightarrow SU(\infty) \xrightarrow{\pi} W_{3,\infty}.$$

Since $SU(\infty)$ is an infinite loop space it is homotopy commutative. Thus the commutator map $SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ lifts through δ to a map

$$\lambda \colon SU(3) \wedge SU(3) \longrightarrow \Omega W_{3,\infty}.$$

There may be many choices of such a lift. In [HK] a choice was made that satisfies the statement of Lemma 2.2. To describe this, recall that $H^*(SU(\infty)) \cong \Lambda(x_3, x_5, \ldots)$. The generating set may be chosen so that $x_{2k+1} = \sigma(c_{k+1})$, where $c_{k+1} \in H^{2k+2}(BSU(\infty))$ is the $(k+1)^{st}$ Chern class and σ is the cohomology suspension. Then $H^*(W_{3,\infty}) \cong \Lambda(\bar{x}_7, \bar{x}_9, \ldots)$ where $\pi^*(\bar{x}_{2k+1}) = x_{2k+1}$.

Lemma 2.2. *The lift γ may be chosen so that*

$$\lambda^*(a_{2k}) = \Sigma_{i+j=k+1} x_i \otimes x_j$$

where $a_{2k} = \sigma(\bar{x}_{2k+1}) \in H^{2k}(\Omega W_{3,\infty})$. ∎

Each $a_{2k} \in H^{2k}(\Omega W_{3,\infty})$ is represented by a map $\Omega W_{3,\infty} \longrightarrow K(\mathbb{Z}, 2k)$, which we also label as a_{2k}. Taking the product of such maps for $k \geq 3$, we obtain a map

$$a = \prod_{k\geq 3} a_{2k} \colon \Omega W_{3,\infty} \longrightarrow \prod_{k\geq 3} K(\mathbb{Z}, 2k).$$

Observe that a is both a loop map and a rational homotopy equivalence. Since, by Lemma 2.1, $[\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, \Omega W_{3,\infty}]$ is a free abelian group, we therefore obtain the following.

Lemma 2.3. *The induced map*

$$a_* \colon [\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, \Omega W_{3,\infty}] \longrightarrow \bigoplus_{k\geq 3} H^{2k}(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2)$$

is a monomorphism. ∎

Applying the functor $[\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, \ ]$ to the fibration sequence $\Omega SU(\infty) \xrightarrow{\Omega\pi} \Omega W_{3,\infty} \xrightarrow{\delta} SU(3) \longrightarrow SU(\infty)$ we obtain an exact sequence

$$\widetilde{K}^0(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2) \xrightarrow{(\Omega\pi)_*} [\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, \Omega W_{3,\infty}] \xrightarrow{\delta_*} [\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, SU(3)] \longrightarrow \widetilde{K}^{-1}(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2).$$

Since $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2$ is a CW-complex with cells only in even dimensions, we have $\widetilde{K}^{-1}(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2) = 0$. Thus $[\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, SU(3)]$ is the cokernel of $(\Omega\pi)_*$. Let C be the cokernel of the composite $a_* \circ (\Omega\pi)_*$, where a_* is the map in Lemma 2.3. Then we obtain a commutative diagram of exact sequences

$$\begin{tikzcd}
\widetilde{K}^0(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2) \arrow[r, "(\Omega\pi)_*"] \arrow[d, equal] & {[\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, \Omega W_{3,\infty}]} \arrow[r, "\delta_*"] \arrow[d, "a_*"] & {[\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, SU(3)]} \arrow[r] \arrow[d, "b"] & 0 \arrow[d, equal] \\
\widetilde{K}^0(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2) \arrow[r, "a_*\circ(\Omega\pi)_*"] & \bigoplus_{k\geq 3} H^{2k}(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2) \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

where b is the induced map of cokernels. By Lemma 2.3, a_* is a monomorphism. A diagram chase then implies that b is also a monomorphism. The composite $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath\wedge\imath} SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ represents an element of $[\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2, SU(3)]$. Since c lifts through δ to the map λ, the composite $\lambda \circ (\imath \wedge \imath)$ represents a lift of $c \circ (\imath \wedge \imath)$ through δ_*. The fact that b is a monomorphism then implies the following.

Lemma 2.4. *The order of $c \circ (\imath \wedge \imath)$ is the order of the image of $a_*(\lambda \circ (\imath \wedge \imath))$ in C.* ∎

Now we calculate the order of the image of $a_*(\lambda \circ (\imath \wedge \imath))$ in C. To understand the cokernel C of $a_* \circ (\Omega\pi)_*$, we first consider the image of $a_* \circ (\Omega\pi)_*$. Observe that $a_* \circ (\Omega\pi)_*$ is induced by the composite $\Omega SU(\infty) \xrightarrow{\Omega\pi} \Omega W_{3,\infty} \xrightarrow{a} \prod_{k\geq 3} K(\mathbb{Z}, 2k)$. The choice of the generating sets for $H^*(SU(\infty))$ and $H^*(\Omega W_{3,\infty})$ implies that $(\Omega\pi)^*(a_{2k}) = c_{2k}$ for $k \geq 3$. Thus $a \circ \Omega\pi$ corresponds to $\oplus_{k\geq 3} k!\, \mathrm{ch}_k$

where ch_k is the $2k$-dimensional part of the Chern character. Hence for any $\xi \in \widetilde{K}^0(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2)$ we have

$$a_* \circ (\Omega\pi)_*(\xi) = (3!\,\mathrm{ch}_3(\xi), 4!\,\mathrm{ch}_4(\xi), 5!\,\mathrm{ch}_5(\xi)) \in H^6 \oplus H^8 \oplus H^{10}. \tag{1}$$

Let x be a generator of $\widetilde{K}^0(\mathbb{C}P^2)$ such that $\mathrm{ch}(x) = t + \frac{t^2}{2}$ for a generator $t \in H^2(\mathbb{C}P^2)$. Then $\widetilde{K}^0(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2)$ is a free abelian group generated by $\Sigma^2(x^i \otimes x^j)$ for $i, j = 1, 2$. Let $\epsilon_{i,j} = a_* \circ (\Omega\pi)_*(\Sigma^2(x^i \otimes x^j))$. Then (1) implies that the image of $a_* \circ (\Omega\pi)_*$ is as follows. In the group $\mathbb{Z}\Sigma^2(t \otimes t) \oplus \mathbb{Z}\Sigma^2(t \otimes t^2) \oplus \mathbb{Z}\Sigma^2(t^2 \otimes t) \oplus \mathbb{Z}\Sigma^2(t^2 \otimes t^2)$ we have

$$\begin{aligned}
\epsilon_{1,1} &= (3!, 4!/2, 4!/2, 5!/4) \\
\epsilon_{2,1} &= (0, 4!, 0, 5!/2) \\
\epsilon_{1,2} &= (0, 0, 4!, 5!/2) \\
\epsilon_{2,2} &= (0, 0, 0, 5!).
\end{aligned}$$

Next, consider the image of $a_*(\lambda \circ (\imath \wedge \imath))$. By Lemma 2.2 and (1) we have

$$a_*(\lambda \circ (\imath \wedge \imath)) = (1, 1, 1, 1).$$

Finally, observe that $20\epsilon_{1,1} - 5(\epsilon_{1,2} + \epsilon_{2,1}) + \epsilon_{2,2} = 5!(1, 1, 1, 1)$ and no other combination of $\epsilon_{i,j}$'s gives a smaller multiple of $(1, 1, 1, 1)$. That is, if $y = a_*(\lambda \circ (\imath \wedge \imath))$, then $5!y$ is in the image of $a_* \circ (\Omega\pi)_*$, and no smaller multiple of y is in the image. Thus y passes to an element in the cokernel C of order $5!$. Lemma 2.4 therefore implies the following.

Proposition 2.5. *The composite $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath \wedge \imath} SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ has order $5!$.* ∎

3 The odd primary components of the order of c

The adjoint of the identity map on $SU(3)$ is the evaluation map $ev \colon \Sigma SU(3) \simeq \Sigma\Omega BSU(3) \longrightarrow BSU(3)$. Let j be the composite $j \colon \Sigma^2\mathbb{C}P^2 \xrightarrow{\Sigma\imath} \Sigma SU(3) \xrightarrow{ev} BSU(3)$.

For the remainder of the section we localize all spaces and maps at an odd prime p. By [MNT], $SU(3) \simeq S^3 \times S^5$, so restricting to 5-skeleta we also have $\Sigma\mathbb{C}P^2 \simeq S^3 \vee S^5$. The following lemma is a special case of an argument in [Mc].

Lemma 3.1. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Sigma SU(3) \arrow[r, "ev"] \arrow[d, "t"] & BSU(3) \arrow[d, equal] \\
\Sigma^2\mathbb{C}P^2 \arrow[r, "j"] & BSU(3)
\end{tikzcd}$$

where t is a left homotopy inverse for $\Sigma\imath$.

Proof. Since $SU(3) \simeq S^3 \times S^5$, we have $\Sigma SU(3) \simeq S^4 \vee S^6 \vee S^9$, so ev can be regarded as the wedge sum $\Sigma\imath + h$ for some map $h\colon S^9 \longrightarrow BSU(3))$. By [G], the homotopy fibre of ev is the Hopf construction $\Sigma SU(3) \wedge SU(3) \xrightarrow{\mu^*} \Sigma SU(3)$. Observe that the restriction of μ^* to $S^9 \simeq \Sigma S^3 \wedge S^5 \longrightarrow \Sigma SU(3) \wedge SU(3)$ is onto in mod-p homology. Thus the equivalence for $\Sigma SU(3)$ can be chosen so that the S^9 summand of $\Sigma SU(3)$ factors through μ^*. Doing so, we obtain $h \simeq *$. Hence ev factors through j and the lemma follows. ∎

Note that the adjoint of the commutator $SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ is $[ev, ev]$, and the adjoint of the composite $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath\wedge\imath} SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ is $[j, j]$.

Lemma 3.2. *The maps $SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ and $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath\wedge\imath} SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ have the same order.*

Proof. Since $c \circ (\imath \wedge \imath)$ factors through c, its order can be no larger than the order of c. To prove the converse, suppose that $c \circ (\imath \wedge \imath)$ has order m. Then adjointing, the composite $[ev, ev] \circ (\Sigma\imath \wedge \imath)$ has order m. That is, $[j, j]$ has order m. Observe that $[j, j]$ is homotopic to the composite $\Sigma^2\mathbb{C}P^2 \wedge \mathbb{C}P^2 \longrightarrow \Sigma\mathbb{C}P^2 \vee \Sigma^2\mathbb{C}P^2 \xrightarrow{j\vee j} BSU(3)$, so $[j, j]$ having order m implies that $(m \cdot j \vee j)$ extends to a map $\mu\colon \Sigma^2\mathbb{C}P^2 \times \Sigma^2\mathbb{C}P^2 \longrightarrow BSU(3)$. Lemma 3.1 therefore implies that there is a map $\overline{\mu}\colon \Sigma SU(3) \times \Sigma SU(3) \longrightarrow BSU(3)$ which restricts to $(m \cdot ev \vee ev)$ on $\Sigma SU(3) \vee \Sigma SU(3)$. Thus if $m' = (m \cdot j \vee j)$, then the composite $\Sigma SU(3) \wedge SU(3) \longrightarrow \Sigma SU(3) \vee SU(3) \xrightarrow{m'} BSU(3)$ is null homotopic. But this composite is homotopic to $m \cdot [ev, ev]$. Thus $[ev, ev]$ has order m, implying that its adjoint c also has order m. ∎

By Proposition 2.5 we know the order of $c \circ (\imath \wedge \imath)$. Thus Lemma 3.2 immediately implies the following.

Proposition 3.3. *The commutator $SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ satisfies:*

(a) *localized at* 3, *c has order* 3;

(b) *localized at* 5, *c has order* 5;

(c) *localized at p for $p \geq 7$, c is null homotopic.* ∎

4 The 2-component of the order of c

Throughout this section we localize all spaces and maps at 2. We will use a result of Mimura to reduce the calculation of the order of $SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ to calculating the order of the composite $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath\wedge\imath} SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$.

Recall from Section 2 that the map $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath\wedge\imath} SU(3) \wedge SU(3)$ is the inclusion of the 10-skeleton. The following theorem incorporates this skeletal identification with Mimura's [M] description of the cell structure of $SU(3) \wedge SU(3)$.

Recall that $\eta\colon S^{n+1} \longrightarrow S^n$ represents the stable generator of $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$. For a space X, let X_n be the n-skeleton of X and let $\nabla\colon X \vee X \longrightarrow X$ be the fold map.

Theorem 4.1. *There is a homotopy equivalence*

$$SU(3) \wedge SU(3) \simeq B \vee \Sigma^9\mathbb{C}P^2 \vee \Sigma^9\mathbb{C}P^2$$

where B satisfies the following:

(a) $B = (\Sigma\mathbb{C}P^2 \wedge \mathbb{C}P^2) \cup e^{16}$;

(b) $B/S^6 \simeq (S^8 \times S^8) \cup_{\overline{\eta}} e^{10}$, *where* $\overline{\eta} = \eta \times \eta$; ■

Now we begin the reduction procedure.

Lemma 4.2. *The group $[\Sigma^9\mathbb{C}P^2, SU(3)]$ has order ≤ 8.*

Proof. The cofibration $S^{11} \longrightarrow \Sigma^9\mathbb{C}P^2 \longrightarrow S^{13}$ induces an exact sequence $[S^{13}, SU(3)] \longrightarrow [\Sigma^9\mathbb{C}P^2, SU(3)] \longrightarrow [S^{11}, SU(3)]$. By [MT], $\pi_{11}(SU(3)) \cong \mathbb{Z}/4\mathbb{Z}$ and $\pi_{13}(SU(3)) \cong \mathbb{Z}/2\mathbb{Z}$. Thus, by exactness, $[\Sigma^9\mathbb{C}P^2, SU(3)]$ has order at most 8. ■

Corollary 4.3. *If the composite $B \longrightarrow SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ has order ≤ 8 then c has order ≤ 8.*

Proof. By Theorem 4.1, $SU(3) \wedge SU(3) \simeq B \vee \Sigma^9\mathbb{C}P^2 \vee \Sigma^9\mathbb{C}P^2$. By Lemma 4.2, the restriction of c to either copy of $\Sigma^9\mathbb{C}P^2$ has order ≤ 8. Therefore, if the restriction of c to B also has order ≤ 8, then c has order ≤ 8. ■

Lemma 4.4. *The map $(\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2)/S^6 \longrightarrow B/S^6$ has a left homotopy inverse after suspending, implying there is a homotopy equivalence $\Sigma B/S^6 \simeq (\Sigma^2\mathbb{C}P^2 \wedge \mathbb{C}P^2)/S^6 \vee S^{17}$.*

Proof. By Theorem 4.1 (b) , $B/S^6 = (S^8 \times S^8) \cup_{\overline{\eta}} e^{10}$. This implies that there is an inclusion $S^8 \times S^8 \longrightarrow B/S^6$ with the property that the pinch map $B/S^6 \longrightarrow S^{16}$ to the top cell extends the pinch map $S^8 \times S^8 \longrightarrow S^{16}$. After suspending, $\Sigma(S^8 \times S^8) \simeq S^9 \vee S^9 \vee S^{17}$. Thus the top cell splits off $\Sigma B/S^6$. The lemma now follows since by Theorem 4.1 (a), $\Sigma B/S^6 = (\Sigma^2\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2)/S^6 \cup e^{17}$. ■

In what follows we will have to distinguish between power maps and degree maps. In general, if X is an H-space let $k\colon X \longrightarrow X$ be the k^{th}-power map and if Y is a co-H-space let $\underline{k}\colon Y \longrightarrow Y$ be the map of degree k.

Lemma 4.5. *If the composite $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \longrightarrow B \longrightarrow SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ has order ≤ 8 then the composite $B \longrightarrow SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ has order ≤ 8.*

Proof. The proof of the lemma takes several steps.
Step 1. Let f be the composite $B \longrightarrow SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$. Let $A = \Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2$ and let $i\colon A \longrightarrow B$ be the inclusion. Let $i'\colon A/S^6 \longrightarrow B/S^6$ be the map induced by pinching out the bottom cell common to both A and B. By [HK], the

composite $S^6 \hookrightarrow A \xrightarrow{i} B \xrightarrow{f} SU(3)$ has order 2. Thus there is a homotopy commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[r, "f"] \arrow[d] & SU(3) \arrow[d, "2"] \\ A/S^6 \arrow[r, "i'"] & B/S^6 \arrow[r, "g"] & SU(3) \end{tikzcd} \tag{2}$$

for some map g.

Step 2. The lemma asserts that if $8 \circ f \circ i$ is null homotopic then so is $8 \circ f$. We claim that it suffices to show that if $4 \circ g \circ i'$ is null homotopic then so is $4 \circ g$. To see this, suppose that $8 \circ f \circ i$ is null homotopic. Consider the cofibration sequence $S^6 \longrightarrow A \xrightarrow{\pi} A/S^6 \longrightarrow S^7$. Since $8 \circ f \circ i \simeq *$, the homotopy commutativity of the outer rectangle in (2) implies that $4 \circ g \circ i' \circ \pi$ is null homotopic. Thus $4 \circ g \circ i'$ extends through the cofibre of π to a map $S^7 \longrightarrow SU(3)$. But by [MT], $\pi_7(SU(3)) \cong 0$. Thus $4 \circ g \circ i'$ is null homotopic. We assume that this condition implies that $4 \circ g$ is null homotopic. But then the homotopy commutativity of the right square in (2) implies that $8 \circ f$ is null homotopic.

Step 3. It remains to show that if $4 \circ g \circ i'$ is null homotopic then so is $4 \circ g$. In general, for a space X, let $E\colon X \longrightarrow \Omega\Sigma X$ be the suspension map. Applying the James construction [J] to the map g, we obtain an H-map $\overline{g}\colon \Omega\Sigma(B/S^6) \longrightarrow SU(3)$ such that $\overline{g} \circ E \simeq g$. Let $A' = A/S^6$. By Lemma 4.4 there is a homotopy equivalence $e\colon \Sigma A' \vee S^{17} \longrightarrow \Sigma B/S^6$ where the restriction of e to $\Sigma A'$ is $\Sigma i'$. Consider the diagram

$$\begin{tikzcd} A' \arrow[rr, "i'"] \arrow[d, "i_1"] & & B/S^6 \arrow[d, "E"] & & \\ A' \vee S^{16} \arrow[r, "E"] & \Omega(\Sigma A' \vee S^{17}) \arrow[r, "\Omega e"] & \Omega\Sigma(B/S^6) \arrow[r, "\overline{g}"] & SU(3) \arrow[r, "4"] & SU(3) \end{tikzcd}$$

where i_1 is the inclusion of the left wedge summand. The rectangle homotopy commutes since the restriction of e to $\Sigma A'$ is $\Sigma i'$ and E commutes with suspensions. Since $\overline{g} \circ E \simeq g$, the upper direction around the diagram is homotopic to $4 \circ g \circ i'$, which we are assuming is null homotopic. Thus the lower direction around the diagram is also null homotopic. In addition, by [MT], $\pi_{16}(SU(3)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, so in fact the entire bottom row of the diagram is null homotopic. On the other hand, since A' and S^{16} are suspensions, the bottom row is homotopic to the composite

$$A' \vee S^{16} \xrightarrow{\underline{4}} A' \vee S^{16} \xrightarrow{E} \Omega(\Sigma A' \vee S^{17}) \xrightarrow{\Omega e} \Omega\Sigma B/S^6 \xrightarrow{\overline{g}} SU(3).$$

Therefore we have a string of homotopies

$$* \simeq 4 \circ \overline{g} \circ \Omega e \circ E \simeq \overline{g} \circ \Omega e \circ E \circ \underline{4} \simeq \overline{g} \circ \Omega e \circ \Omega\Sigma\underline{4} \circ E$$

where the last homotopy is due to the naturality of E. A consequence of the James construction is that the homotopy class of an H-map $\Omega\Sigma X \xrightarrow{f} Y$ is determined

by the homotopy class of $f \circ E$. In our case, the null homotopy for $\overline{g} \circ \Omega e \circ \Omega\Sigma\underline{4} \circ E$ implies that $\overline{g} \circ \Omega e \circ \Omega\Sigma\underline{4}$ is null homotopic.
Step 4. By the Hilton-Milnor Theorem, there is a homotopy equivalence

$$\Omega(\Sigma A' \vee S^{17}) \simeq \Omega\Sigma A' \times \Omega S^{17} \times \Omega(\Sigma\Omega\Sigma A' \wedge \Omega S^{17}).$$

Observe that as A' is 7-connected, $\Omega(\Sigma\Omega\Sigma A' \wedge \Omega S^{17})$ is 23-connected. The distributivity formula (see [C, §4], for example), therefore implies that the 4^{th}-power map on $\Omega(\Sigma A' \vee S^{17})$ is homotopic to $\Omega\Sigma\underline{4}$ through dimension 23. Thus in dimensions ≤ 23, there is a string of homotopies

$$* \simeq \overline{g} \circ \Omega e \circ \Omega\Sigma\underline{4} \simeq \overline{g} \circ \Omega e \circ 4 \simeq 4 \circ \overline{g} \circ \Omega e$$

where the first homotopy is by Step 3 and last is due to the 4^{th}-power map commuting with H-maps.
Step 5. We now have $4 \circ \overline{g} \circ \Omega e \simeq *$ in dimensions ≤ 23. Since e is a homotopy equivalence, we can compose on the right with Ωe^{-1} to obtain $4 \circ \overline{g} \simeq *$ in dimensions ≤ 23. As B/S^6 is 16-dimensional, we therefore obtain $4 \circ \overline{g} \circ E \simeq *$ without any dimensional restriction. But $\overline{g}$ was defined so that $\overline{g} \circ E \simeq g$. Hence $4 \circ g$ is null homotopic, as required. ∎

Combining Lemma 4.5 and Corollary 4.3 immediately implies the following.

Proposition 4.6. *If the composite $\Sigma\mathbb{C}P^2 \wedge \Sigma\mathbb{C}P^2 \xrightarrow{\imath\wedge\imath} SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$ has order ≤ 8, then c has order ≤ 8.* ∎

Now we can combine the results of the previous three sections to prove Theorem 1.1.

Proof of Theorem 1.1. Let m be the order of the commutator $SU(3) \wedge SU(3) \xrightarrow{c} SU(3)$. By Proposition 4.6, the 2-component of the order of c equals the 2-component of the order of $c \circ (\imath \wedge \imath)$, which by Proposition 2.5 is 8. By Proposition 3.3, the 3-component of m is 3, the 5-component of m is 5, and the p component of m for $p \geq 7$ is 1. Thus $m = 2^3 \cdot 3 \cdot 5 = 120$. ∎

5 Counting gauge groups

In this section we revert to assuming that spaces and maps have not yet been localized. We begin by stating a general criterion proved in [Th] for determining when certain fibres are homotopy equivalent.

Lemma 5.1. *Let X be a space and Y be an H-space with a homotopy inverse. Let $X \xrightarrow{f} Y$ be a map of order m, where m is finite. Let F_k be the homotopy fibre of the composite $X \xrightarrow{f} Y \xrightarrow{k} Y$. If $(m,k) = (m,k')$ then F_k and $F_{k'}$ are homotopy equivalent when localized rationally or at any prime.* ∎

In order to use this to help count gauge groups, we need a context in which gauge groups arise as homotopy fibres. Let G be a topological group, let X be a space, and let $P \longrightarrow X$ be a principal G-bundle with gauge group $\mathcal{G}(P)$. Let BG and $B\mathcal{G}(P)$ be the classifying spaces of G and $\mathcal{G}(P)$ respectively. In [AB] it was shown that there is a homotopy equivalence $B\mathcal{G}(P) \simeq \mathrm{Map}_P(X, BG)$, where $\mathrm{Map}_P(X, BG)$ is the component of the space of continuous maps from X to BG which are freely homotopic to the map inducing P. Moreover, there is a fibration

$$\mathrm{Map}^*_P(X, BG) \longrightarrow \mathrm{Map}_P(X, BG) \xrightarrow{ev} BG$$

where $\mathrm{Map}^*_P(X, BG)$ is the component of the space of continuous maps from X to BG which are based homotopic to the map inducing P, and ev evaluates a map at the basepoint.

Now specialize to $X = \Sigma Y$. Observe that the components of $\mathrm{Map}(\Sigma Y, BG)$ and $\mathrm{Map}^*(\Sigma Y, BG)$ are in one-to-one correspondence with the homotopy classes of maps $[\Sigma Y, BG]$. Fix a homotopy class $[f] \in [\Sigma Y, BG]$. For an integer k, let $P_k \longrightarrow \Sigma Y$ be the principal G-bundle classified by the homotopy class of kf. Note that if $[f]$ has infinite order then the bundles $P_k \longrightarrow \Sigma Y$ are distinct, but if $[f]$ has order m then there are bundle equivalences between $P_{ms+k} \longrightarrow \Sigma Y$ and $P_k \longrightarrow \Sigma Y$ for every integer s. Let $\mathcal{G}_k$ be the gauge group of the principal G-bundle $P_k \longrightarrow \Sigma Y$. Then there is a homotopy equivalence $B\mathcal{G}_k = \mathrm{Map}_{kf}(\Sigma Y, BG)$. In the pointed case, the pointed exponential law implies that $\mathrm{Map}^*(\Sigma Y, BG)$ is homotopy equivalent to the loop space $\Omega\mathrm{Map}^*(Y, BG)$, and in general the components of a homotopy-associative H-space are homotopy equivalent. Explicitly in our case, the existence of a pointed wedge product $\Sigma Y \longrightarrow \Sigma Y \vee \Sigma Y$ lets us define a map $\overline{-kf}\colon \mathrm{Map}^*_{kf}(\Sigma Y, BG) \longrightarrow \mathrm{Map}^*_0(\Sigma Y, BG)$. by sending $g \in \mathrm{Map}^*_{kf}(\Sigma Y, BG)$ to the composite $\Sigma Y \longrightarrow \Sigma Y \vee \Sigma Y \xrightarrow{g \vee -kf} BG \vee BG \xrightarrow{\nabla} BG$, where ∇ is the fold map. Since the wedge product on ΣY is associative, it follows that $\overline{kf} \circ \overline{-kf}$ takes a map $g \in \mathrm{Map}^*_{kf}(\Sigma Y, BG)$ to itself, implying that $\mathrm{Map}^*_{kf}(\Sigma Y, BG)$ retracts off $\mathrm{Map}^*_0(\Sigma Y, BG)$. A similar argument shows that $\mathrm{Map}^*_0(\Sigma Y, BG)$ retracts off $\mathrm{Map}^*_{kf}(\Sigma Y, BG)$, so in fact the two are homotopy equivalent. Therefore the evaluation fibration determines a homotopy fibration sequence

$$G \xrightarrow{\partial_k} \mathrm{Map}^*_0(\Sigma Y, BG) \longrightarrow B\mathcal{G}_k \xrightarrow{ev} BG \tag{3}$$

which defines the map ∂_k. In [L] it was shown that the adjoint $\Sigma Y \wedge G \longrightarrow BG$ of ∂_k is homotopic to the Whitehead product $[kf, ev]$. As the Whitehead product is linear, we have $[kf, ev] \simeq k[f, ev]$, implying that $\partial_k \simeq k \circ \partial_1$. Hence the fibration sequence (3) implies that $\mathcal{G}_k$ is the homotopy fibre of the map $k \circ \partial_1$.

Observe that the classifying space BG is rationally homotopy equivalent to a product of Eilenberg-MacLane spaces. That is, BG is rationally homotopy equivalent to an H-space. Therefore the adjoint of ∂_1 - the Whitehead product $[f, ev]$ - is rationally trivial. This implies that ∂_1 has order m_f, where m_f is finite. Now we can apply Lemma 5.1 to obtain the following.

Proposition 5.2. *If $(m_f, k) = (m_f, k')$ then there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ after localizing rationally or at any prime.* ∎

Now we relate the order of $[f, ev]$ to that of the commutator c to prove Theorem 1.2.

Proof of Theorem 1.2. We are given that the order of the commutator $G \wedge G \xrightarrow{c} G$ is m. The adjoint of c is the Whitehead product $[ev, ev]$, so $[ev, ev]$ has order m. By definition, the order of $[f, ev]$ is m_f. Since $[f, ev]$ factors through $[ev, ev]$, we must have m_f dividing m. Thus (m_f, k) divides (m, k) for each k. So if $(m, k) = (m, k')$ then $(m_f, k) = (m_f, k')$ for each k. Proposition 5.2 therefore implies that there is a p-local homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$.

Localized at p, we only need be concerned with the p-component of the integers (m, k). These p-components range from 0 to $\nu_p(m)$. Thus the number of distinct p-local homotopy types of the gauge groups $\{\mathcal{G}_k\}$ is bounded above by $\nu_p(m) + 1$. ∎

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