A note on the lattices $DP(X)$ and $K(X)$

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Abstract

Using the order structure of the lattice $DP(X)$ of density preserving continuous maps on a Hausdorff space X without isolated points, we describe closed nowhere dense subsets of X and, for a subspace A of X, we also deduce topological properties of the space $X - A$ from the lattice theoretic properties of $DP(X, A)$. Finally, we use them to obtain Thrivikraman's results concerning $\beta X - X$ and $K(X)$ and, Magill's result concerning the automorphism group of the lattice $K(X)$.

1 Introduction

In [5], we have studied $DP(X)$, the poset of all equivalence classes of density preserving maps obtained by identifying equivalent density preserving maps on X. We observe that for a compact Hausdorff space X, $DP(X)$ is a complete lattice and we have characterized it by proving that for countably compact T_3 spaces X and Y without isolated points, lattice $DP(X)$ is isomorphic to lattice $DP(Y)$ if and only if X and Y are homeomorphic. In fact, if lattice $DP(X)$ is isomorphic to lattice $DP(Y)$ then we obtain a bijective map $F : X \to Y$ preserving closed nowhere dense sets, which turns out to be a homeomorphism if X and Y are countably compact T_3 spaces without isolated points. In this paper we describe closed nowhere dense subsets of a Hausdorff space X without isolated points using the order structure of the lattice $DP(X)$. Consequently, we obtain Thrivikraman's [6] and Magill's [3] results concerning Stone-Čech remainder. For survey article on such posets see [2].

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Throughout spaces considered are Hausdorff and maps are continuous. A map $f : X \longrightarrow Y$ is called a *density preserving map* if $IntClf(A) \neq \phi$, whenever $IntA \neq \phi$, $A \subseteq X$ [1]. Two density preserving maps f and g each having domain X and range Rf and Rg respectively are said to be *equivalent* ($f \approx g$) if there exists a homeomorphism $h : Rf \longrightarrow Rg$ satisfying $h \circ f = g$. We denote by $DP(X)$, the set of all equivalence classes of density preserving maps obtained by identifying equivalent density preserving maps on X [5]. The set $DP(X)$ is a partially ordered set with the partial order relation '$\leq$' defined by $g \leq f$ if there exists a continuous map $h : Rf \longrightarrow Rg$ such that $h \circ f = g$.

An f in $DP(X)$ is called *primary* if $\wp(f)$ contains at most one non-singleton member. A primary f in $DP(X)$ is called a *dual* if $\wp(f)$ contains exactly one non-singleton member which is a doubleton. Note that the quotient map f obtained by identifying two distinct points a, b in X is a density preserving dual map. Such a map is also denoted by $(f, \{a, b\})$. The set of all duals in $DP(X)$ is denoted by Σ. An f in $DP(X)$ obtained by collapsing a closed nowhere dense subset H of X to a point is denoted by (f, H).

Recall that for $A \subseteq X$, $DP(X, A)$=$\{f \in DP(X) : | f^{-1}(f(x)) |= 1$, for each $x \in A\}$. A perfect irreducible continuous surjection is called a *covering map*. A study of the poset $IP(X)$ of all equivalence classes of covering maps on X is done by Porter and Woods in [4]. The poset $DP(X)$ naturally contains the poset $IP(X)$ and in [5] we have proved that if X is compact and A is dense in X then $DP(X, A) = IP(X, A)$. In particular, if X is locally compact then $DP(\alpha X, X) = IP(\alpha X, X)$, where αX is a compactification of X. By Corollary 3.6 in [5] and Lemma 3.11 in [4], we obtain the following result.

Theorem 1.1. *Let X be a locally compact Hausdorff space. Then $DP(\beta X, X)$ is order isomorphic to $K(X)$, the lattice of all compactifications of X.*

For an f in $DP(X)$, denote the set $\{f^{-1}(y) \mid y \in Rf\}$ by $\wp(f)$. Note that for every $f \in DP(X)$, the set $\wp(f)$ forms a partition of X. The partial ordering on $DP(X)$ naturally induces a partial ordering on the family $\Im = \{\wp(f) | f \in DP(X)\}$ of partitions of X. In fact, the lattice $\Im$ is isomorphic to the lattice $DP(X)$. In [7], it is proved that $E(X)$, the collection of all Hausdorff partitions of X, is a complete lattice with the natural ordering for a normal space X. We recall that a partition π of X is said to be a *Hausdorff partition* if the quotient space X/π is Hausdorff. The lattice $DP(X)$ is naturally a sublattice of the lattice $E(X)$. It is proved that for a locally compact space X, $E(\beta X - X)$ is isomorphic to $K(X)$ [7]. Now by Theorem 1.1 one can deduce the following result.

Theorem 1.2. *Let X be a locally compact Hausdorff space. Then $DP(\beta X, X)$ is order isomorphic to $E(\beta X - X)$, the lattice of all Hausdorff partitions of X.*

We also note that using techniques similar to the proofs of Lemmas 3.2 to 3.7 in [7], lattice homomorphisms from $DP(X)$ to $DP(Y)$ will have the following property.

Theorem 1.3. *Let Φ be a lattice homomorphism from $DP(X)$ into $DP(Y)$. Then Φ is a bijection on the set Σ of all duals in $DP(X)$.*

In Section 2, we define the notion of 'hinged' and 'overlapping' for duals in $DP(X)$. The *hinged set* Λ consists of those members of the dual set Σ which are hinged with overlapping duals. We introduce the notion of Λ-closed sets for the subsets of hinged set Λ. The notion of hinged set Λ and that of Λ-closed set can be naturally extended to $DP(X, A)$ for any subset A of X. In particular, when X is a locally compact Hausdorff space then using Theorem 1.1, one can observe that Λ-closed sets for the hinged set $\Lambda \subseteq DP(\beta X, X)$ are precisely F-compact sets defined by Thrivikraman in [6]. We show here that for a Hausdorff space X without isolated points there is a bijection from Λ onto X which maps Λ-closed sets in Λ to closed nowhere dense sets in X. The well known results concerning the Stone-Čech remainder due to Thrivikraman [6] and Magill [3] follow as a consequence.

Our study about interplay of the order structure of $DP(X)$ and the topology of X is continued in Section 3. We prove that if $DP(X)$ is complemented then X is totally disconnected. Further, for a subset A of a Hausdorff space X, we deduce topological properties of $X - A$ using lattice theoretic properties of $DP(X, A)$. We also observe that the results obtained by Thrivikraman in [6] concerning topological properties of $\beta X - X$ and the lattice theoretic properties of $K(X)$ follow from our results. We note two anomalies in [6]. In fact, we prove that $DP(X, A)$ is modular if and only if $|X - A| < 4$. Consequently, we obtain $K(X)$ is modular if and only if $|\beta X - X| < 4$ establishing that the inequality in Result 3.2 of [6] should be strict. Further, while observing that $DP(X, A)$ is modular if and only if $|X - A| < 4$ we note that primary members of $K(X)$ need not satisfy modular law. Hence the Result 3.3 in [6] is incorrect.

In Section 4, we determine the automorphism group of the lattice $DP(X)$. As a consequence we obtain Magill's result concerning the automorphism group of the lattice $K(X)$.

2 Topology of X and order structure of $DP(X)$

Recall that for a Hausdorff space X, the *dual set* Σ consists of all duals in $DP(X)$. The *hinged set* Λ consist of those subsets of the dual set Σ which are hinged with overlapping duals.

Definition 2.1. Two members in the dual set Σ are said to be *overlapping* if there are precisely three dual members greater than their meet.

Definition 2.2. An h in the dual set Σ is said to be *hinged* with two overlapping duals f and g if there are precisely six dual members greater than $f \wedge g \wedge h$.

For two overlapping duals f and g, denote by $|fg|$ the set containing f and g along with duals hinged with f and g. Note that the set $|fg|$ determines a unique point of X. In fact, if f and g are overlapping duals, then there exists $a, b, c \in X$

such that $f \approx (f, \{a, b\})$ and $g \approx (g, \{a, c\})$. In this case the set $|fg|$ is said to determine the point a of X and we denote it by $|fg|_a$. The *hinged set* Λ denote the set of all subsets of the dual set Σ of the form $|fg|$, where f and g are overlapping duals.

Definition 2.3. An f in the dual set Σ is said to be *determined* by a subset A of the hinged set Λ if there exist distinct points $|hk|, |lm|$ in A satisfying $\{f\} = |hk| \cap |lm|$.

Definition 2.4. Let A be a subset of the hinged set Λ and $\lambda = \{d \in \Sigma \mid d \text{ is determined by } A\}$. Then A is said to be Λ-*closed* if $\wedge_{f\in\lambda} f$ exists and $\lambda = \lambda'$, where λ' is the collection of all duals $\geq \wedge_{f\in\lambda} f$.

Using the order structure of the poset $DP(X)$, the following Proposition describes closed nowhere dense subsets of X.

Proposition 2.5. *Let X be a Hausdorff space without isolated points and let Λ be the hinged set. Then there exists a bijective map from Λ to X which maps Λ-closed sets in Λ to closed nowhere dense sets in X.*

Proof. Define $\varphi : \Lambda \to X$ by $\varphi(|fg|)=a$, where a in X is the unique point determined by $|fg|$. Clearly the map φ is bijective. Let A be a Λ-closed subset of Λ. If $A = \{|fg|\}$, then $\varphi(A) = \{a\}$, where a is the unique point determined by $|fg|$. Let A be a non-singleton Λ-closed subset and λ be the set of all duals determined by A. Then observe that $\wedge_{f\in\lambda} f$ exists and it is a primary member of $DP(X)$ say (f, H), where H is a closed nowhere dense subset of X. Since A is Λ-closed, the collection of all duals $\geq \wedge_{f\in\lambda} f$ is precisely λ. Thus $\varphi(A) = H$, is a closed nowhere dense subset of X.

On the other hand if H is any closed nowhere dense subset of X then for each $a \in H$, consider unique set $|fg|_a$ such that $|fg|_a$ determines the point a. Let $A = \{|fg|_a \mid a \in H\}$ and $\lambda = \{d \in \Sigma \mid d \text{ is determined by } A\}$. Then observe that $\wedge_{f\in\lambda} f$ exists. In fact, $\wedge_{f\in\lambda} f \approx (k, H)$. Also $\lambda = \lambda'$, where $\lambda' = \{d \in \Sigma \mid d \geq \wedge_{f\in\lambda} f\}$. Thus A is F-closed and $\varphi(A) = H$.

Let the dual set Σ be the set of all duals in $DP(\beta X, X)$ and let the hinged set Λ be the set of all subsets of Σ of the form $|fg|$, where f and g are overlapping duals. Then in this case our notion of Λ-closed sets coincides with the notion of F-compact sets defined in [6] and hence we have $F = \Lambda$. The F-compact sets are used in [6] to recover topology of the space $\beta X - X$ using order structure of $K(X)$, for a locally compact space X. Proposition 2.5 and our observation about F-compact sets leads to following result due to Thrivikraman [6]. As a consequence, Magill's result follows [3].

Theorem 2.6 [6, Theorem 4.9]. *Let X be a completely regular Hausdorff space. Then there is bijection from F onto $\beta X - X$ which carries F-compact sets to compact subsets of $\beta X - X$ and vice-versa. Further, the complements of F-compact sets of F form a topology for F if and only if X is locally compact. In this case F is homeomorphic to $\beta X - X$.*

Corollary 2.7 [3, Theorem 12]. *Let X and Y be locally compact Hausdorff spaces. Then $K(X)$ and $K(Y)$ are order isomorphic if and only if $\beta X - X$ and $\beta Y - Y$ are homeomorphic.*

3 Lattice $DP(X, A)$ and space $X - A$

In this section we deduce topological properties of $X - A$ from the lattice theoretic properties of $DP(X, A)$, where A is a subset of X. As a consequence we obtain Thrivikraman's results concerning $K(X)$ and $\beta X - X$ [6]. The following Theorem establishes a relation between order structure of the poset $DP(X)$ and topology of a space X. A similar result is proved in [6] for $K(X)$, which follows as a consequence of our result.

Theorem 3.1. *Let X be a Hausdorff space. If $DP(X)$ is complemented then X is totally disconnected.*

Proof. Let $x, y \in X, x \neq y$. Then consider the dual member $(f, \{x, y\})$ in $DP(X)$. Since $DP(X)$ is complemented, there exists g in $DP(X)$ such that $f \wedge g = \omega$ and $f \vee g = I_X$, where ω is the minimum element in $DP(X)$. Since $f \wedge g = \omega$, $\wp(g)$ can contain at most two non-empty members. Further, $f \vee g = I_X$ implies that $\wp(g)$ contains exactly two non-empty members, say H and K such that $x \in H$ and $y \in K$. Since H and K are the only non-empty members of $\wp(g)$ we have $X = H \cup K$. Thus for every pair of distinct points in X we get a separation for X.

Corollary 3.2. *Let X be a Hausdorff space and A be a subset of X. If $DP(X, A)$ is complemented then $X - A$ is totally disconnected.*

Using Theorem 1.1 and Corollary 3.2, we can deduce the following result.

Corollary 3.3. [6, Result 3.7] *Let X be a locally compact Hausdorff space. If $K(X)$ is complemented then $\beta X - X$ is totally disconnected.*

Remark 3.4. Converse of the Corollary 3.2 is not true in general. Let $X = [0, 1]$ and let A be such that $X - A = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. Then $X - A$ is totally disconnected but $DP(X, A)$ is not complemented as it does not contain the universal lower bound.

Theorem 3.5. *Let X be a Hausdorff space and A be a subset of X. Then,*
- (i) *$DP(X, A)$ is distributive if and only if $\mid X - A \mid < 3$.*
- (ii) *$DP(X, A)$ has a minimum element but has no atom if and only if $X - A$ is connected.*
- (iii) *$DP(X, A)$ is modular if and only if $\mid X - A \mid \leq 3$.*

Proof.
- (i) One easily verifies that if $\mid X - A \mid < 3$, then $DP(X, A)$ is distributive. If $|X - A| \geq 3$, then choose distinct points $a, b, c \in X - A$. Then consider

the members $(f, \{a,b\})$, $(g, \{b,c\})$, $(h, \{a,c\})$ and $(k, \{a,b,c\})$ in $DP(X, A)$. One easily verifies $(f \vee g) \wedge h = h \neq k = (f \wedge h) \vee (g \wedge h)$.

(ii) If $DP(X, A)$ has an atom say f, then $\wp(f)$ contains precisely two non-singleton members H and K whose union is $X - A$. Thus $X - A$ is disconnected. Further, if $X - A$ is disconnected then $X - A = H \cup K$, where H and K are disjoint clopen sets. The natural quotient map obtained by identifying H and K to distinct points is an atom in $DP(X, A)$. Note that the minimum element in $DP(X, A)$ is the quotient map obtained by identifying $X - A$ to a point.

(iii) One easily verifies that if $\mid X - A \mid \leq 3$, then $DP(X, A)$ is modular. That $DP(X, A)$ is not modular if $\mid X - A \mid > 3$ follows by observing that for $a, b, c, d \in X - A$, the members I_X, $(f, \{a,b\})$, $(g, \{a,b,c\})$, $(h, \{c,d\})$, $(k, \{a,b,c,d\})$ of $DP(X, A)$ form a sublattice isomorphic to a pentagon.

Corollary 3.6 [6, Result 3.1]. *Let X be a locally compact Hausdorff space. Then, $K(X)$ is distributive if and only if $\mid \beta X - X \mid < 3$.*

Corollary 3.7 [6, Result 3.4]. *Let X be a locally compact Hausdorff space. Then, $K(X)$ has a minimum element but has no atom if and only if $\beta X - X$ is compact and connected.*

Corollary 3.8. *Let X be a locally compact Hausdorff space. Then, $K(X)$ is modular if and only if $\mid \beta X - X \mid \leq 3$.*

Remark 3.9.

(a) In view of Corollary 3.8 note that the inequality in Result 3.2 in [6] should be strict.

(b) Maps f, g and h defined in proof of Theorem 3.5(iii) are primary but they do not satisfy modular law. Thus in general primary members of $K(X)$ need not satisfy modular law. Consequently Result 3.3 in [6] is incorrect.

4 Automorphism groups of $DP(X)$

In this section we determine the automorphism group of the lattice $DP(X)$. As a consequence of this we derive Magill's result concerning the group of automorphisms of lattice $K(X)$. We abbreviate a bijective map preserving closed nowhere dense sets as cln-bijection.

Theorem 4.1. *Let X be a Hausdorff space and let $\mathcal{A}(DP(X))$ denote the automorphism group of the lattice $DP(X)$.*

(i) *If $|X| = 2$, then $\mathcal{A}(DP(X))$ is the group consisting of one element.*

(ii) *If X has no isolated points, then $\mathcal{A}(DP(X))$ is isomorphic to the group (under composition) of all cln-bijections from X to X.*

Proof.

(i) If X consists of two elements then $DP(X)$ consists of the identity map and the map which commutes the two elements. Thus $\mathcal{A}(DP(X))$ consists of one element.

(ii) Let X be a space without isolated points and let $\Psi \in \mathcal{A}(DP(X))$. Then by Lemma 2.4 in [5] there exists a cln-bijection $F : X \to Y$ such that if $\Psi(f) = g$, then $\wp(g) = \{F(A) | A \in \wp(f)\}$. One can easily prove that such an F is unique. Define a mapping $\Phi : \mathcal{A}(DP(X)) \to \mathcal{G}(X)$ by $\Phi(\Psi) = F$, where $\mathcal{G}(X)$ is the group of all cln-bijections from X to X. We first observe that Φ is a homomorphism. Suppose $\Phi(\Psi_1) = F_1$ and $\Phi(\Psi_2) = F_2$. Then for any $f \in DP(X)$, $\wp(\Psi_1(f)) = \{F_1(A) | A \in \wp(f)\}$ and $\wp(\Psi_2(f)) = \{F_2(A) | A \in \wp(f)\}$. Further $\wp((\Psi_1 \circ \Psi_2)(f)) = \{(F_1 \circ F_2)(A) | A \in \wp(f)\}$. Therefore we have $\Phi(\Psi_1 \circ \Psi_2) = F_1 \circ F_2 = \Phi(\Psi_1) \circ \Phi(\Psi_2)$. Clearly Φ maps $\mathcal{A}(DP(X))$ onto $\mathcal{G}(X)$ and the kernel of Φ is $\{I\}$, where I denotes the identity map on X. Hence Φ is an isomorphism of $\mathcal{A}(DP(X))$ onto $\mathcal{G}(X)$.

Corollary 4.2. *Let X be a compact Hausdorff space and let $\mathcal{A}(DP(X))$ denote the automorphism group of the lattice $DP(X)$.*

(i) *If $|X| = 2$, then $\mathcal{A}(DP(X))$ is a group consisting of one element.*

(ii) *If X has no isolated points, then $\mathcal{A}(DP(X))$ is isomorphic to the group (under composition) of all homeomorphisms from X to X.*

Proof. Follows from the Theorem 4.1 as a compact Hausdorff space X is a countably compact T_3 space and closed nowhere dense sets determine the topology for these spaces.

Corollary 4.3 [3, Corollary 15]. *Let X be a locally compact non-compact space and let $\mathcal{A}(K(X))$ denote the automorphism group of the lattice $K(X)$. If $|\beta X - X| = 2$, then $\mathcal{A}(K(X))$ is a group consisting of one element. If $|\beta X - X| \neq 2$, then $\mathcal{A}(K(X))$ is isomorphic to the group (under composition) of all homeomorphisms from $\beta X - X$ to $\beta X - X$.*

Proof. Follows since $DP(\beta X, X)$ is order isomorphic to $K(X)$.

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