Meromorphic functions sharing a nonzero polynomial with finite weight

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Abstract

In the paper, we study the uniqueness theorems of meromorphic function concerning nonlinear differential polynomials sharing a nonzero polynomial with finite weight and obtain two results which improve and generalize the results due to X.M Li and L. Gao [12].

1 Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7], [15] and [16]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

Let f and g be two nonconstant meromorphic functions, and let a be a finite complex number. We say that f and g share the value a CM (counting multiplicities), provided that $f - a$ and $g - a$ have the same set of zeros with the same multiplicities. Similarly, we say that f and g share the value a IM (ignoring multiplicities), provided that $f - a$ and $g - a$ have the same set of zeros ignoring multiplicities. Throughout the paper, we need the following definition.

$$\Theta(a,f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

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where a is a value in the extended complex plane.

In 1959, W.K. Hayman (see [6], Corollary of Theorem 9) proved the following theorem:

Theorem A. *Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

Corresponding to Theorem A, C.C. Yang and X.H. Hua [14] proved the following result.

Theorem B. *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2000, M.L. Fang [4] proved the following result:

Theorem C. *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z = 0$ has infinitely many solutions.*

Corresponding to Theorem C, the following result was proved by M.L. Fang and H.L. Qiu [5].

Theorem D. *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

In 2003, W. Bergweiler and X.C. Pang [3] proved the following theorem:

Theorem E. *Let f be a transcendental meromorphic function, and let $R \not\equiv 0$ be a rational function. If all zeros and poles of f are multiple, except possibly finitely many, then $f' - R = 0$ has infinitely many solutions.*

The question arises:

Question 1. Is there exists a uniqueness theorem corresponding to Theorem E, similar to Theorems B and D ?

In 2010, X.M. Li and L. Gao [12] answered the above questions and proved the following uniqueness theorems.

Theorem F. *Let f and g be two transcendental meromorphic functions, let $n \geq 11$ be a positive integer, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leq 11$. If $f^n f' - P$ and $g^n g' - P$ share 0 CM, then either $f = tg$ for a complex number t satisfying $t^{n+1} = 1$, or $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta) d\eta$.*

Theorem G. *Let f and g be two transcendental meromorphic functions, let $n \geq 15$ be a positive integer, and let $P \not\equiv 0$ be a polynomial. If $(f^n(f-1))' - P$ and $(g^n(g-1))' - P$ share 0 CM and $\Theta(\infty, f) > 2/n$, then $f = g$.*

However questions arise in one's mind as follows which are the motive of the authors.

Question 2. Is it really possible to relax in any way the nature of sharing the value 0 in Theorems F and G keeping the lower bound of n fixed?

Question 3. What happened if one consider kth derivative instead of first in Theorems F and G?

In the paper, we shall try to find out the possible solution of the above two questions. To state the main results we require the following notion of weighted sharing of values, introduced by I. Lahiri [8, 9] which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1. *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the aid of weighted value sharing, we will prove two theorems of which first one will improve and generalize Theorem F and second one will improve and generalize Theorem G. The following theorems are the main results of the paper.

Theorem 1. *Let f and g be two transcendental meromorphic functions, let n, k be two positive integers such that $n \geq 3k + 9$, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leq n - 1$. Let $(f^n)^{(k)} - P$ and $(g^n)^{(k)} - P$ share $(0, 2)$. Then*
(i) if $k = 1$, either $f = tg$ for a complex number t satisfying $t^n = 1$ or $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^n c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta) d\eta$;
(ii) if $k \geq 2$, either $(f^n)^{(k)}(g^n)^{(k)} = P^2$ or $f = tg$ for a complex number t satisfying $t^n = 1$.

Theorem 2. *Let f and g be two transcendental meromorphic functions, let n, m, k be three positive integers, and let $P \not\equiv 0$ be a polynomial. If $(f^n(f-1)^m)^{(k)} - P$ and $(g^n(g-1)^m)^{(k)} - P$ share $(0, 2)$, then each of the following holds:*
(i) when $m = 1$, $n \geq 3k + 12$ and $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, then either $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ or $f = g$;
(ii) when $m \geq 2$ and $n \geq 3k + m + 11$, then either $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ or $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^n(w_1 - 1)^m - w_2^n(w_2 - 1)^m.$$

The possibility $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ does not arise for $k = 1$.

Remark 1. *Theorem 1 improves and generalizes Theorem F.*

Remark 2. *Theorem 2 improves and generalizes Theorems G.*

We now explain some definitions and notations which are used in the paper.

Definition 2. *[10] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid= 1)$ the counting functions of simple a-points of f. For a positive integer p we denote by $N(r,a;f \mid\leq p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f \mid\leq p)$ we denote the corresponding reduced counting function.*

Analogously we can define $N(r,a;f \mid\geq p)$ and $\overline{N}(r,a;f \mid\geq p)$.

Definition 3. *[9] Let p be a positive integer or infinity. We denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then*

$$N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + ... + \overline{N}(r,a;f \mid\geq p).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 4. *Let a be any value in the extended complex plane, and let p be an arbitrary nonnegative integer. We define*

$$\delta_p(a,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N_p(r,a;f)}{T(r,f)}.$$

Remark 3. *From the definitions of $\delta_p(a,f)$ and $\Theta(a,f)$, it is clear that*

$$0 \leq \delta_p(a,f) \leq \delta_{p-1}(a,f) \leq \delta_1(a,f) \leq \Theta(a,f) \leq 1.$$

Definition 5. *[1, 2] Let f and g be two nonconstant meromorphic functions sharing the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those 1-points of f and g, where $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g, where $p = q = 1$, by $\overline{N}_E^{(2}(r,1;f)$ the reduced counting function of those 1-points of f and g, where $p = q \geq 2$. Similarly we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$ and $\overline{N}_E^{(2}(r,1;g)$.*

2 Lemmas

In this section we present some lemmas which will be needed later.

Lemma 1. *[17] Let f and g be two nonconstant meromorphic functions, and let p, k be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2. *[7] Let f be a nonconstant meromorphic function, k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\infty;f) + N(r,0;f) + N(r,c;f^{(k)}) - N(r,0;f^{(k+1)}) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) - N_0(r,0;f^{(k+1)}) + S(r,f), \end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 3. *[7, 15] Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r, a_i(z)) = S(r,f)$, i=1,2. Then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

Lemma 4. *Let f and g be two transcendental meromorphic functions such that $f^{(k)} - P$ and $g^{(k)} - P$ share $(0,2)$, where k is a positive integer, $P \not\equiv 0$ is a polynomial. If*

$$\begin{aligned} \Delta_1 &= (k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) \\ &\quad + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > k+7 \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \Delta_2 &= (k+2)\Theta(\infty,g) + 2\Theta(\infty,f) + \Theta(0,g) + \Theta(0,f) \\ &\quad + \delta_{k+1}(0,g) + \delta_{k+1}(0,f) > k+7, \end{aligned} \tag{2.2}$$

then either $f^{(k)}g^{(k)} = P^2$ or $f = g$.

Proof. Noting that f and g are two transcendental meromorphic functions, $f^{(k)}$ and $g^{(k)}$ are also two transcendental meromorphic functions. Let

$$F = \frac{f^{(k)}}{P}, \quad G = \frac{g^{(k)}}{P},$$

and let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.3}$$

Let $z_0 \notin \{z : P(z) = 0\}$ be a common simple zero of $f^{(k)} - P$ and $g^{(k)} - P$. Then z_0 is a common simple zero of $F - 1$ and $G - 1$. Substituting their Taylor series at z_0 into (2.3), we see that z_0 is a zero of H. Thus we have

$$\begin{aligned} N_E^{1)}(r,1;F) &\leq N(r,0;H) \leq T(r,H) + O(1) \\ &\leq N(r,\infty;H) + S(r,F) + S(r,G). \end{aligned} \tag{2.4}$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a pole of H. Then from (2.3) we can see that H have poles only at the zeros of F' and G', 1-points of F whose multiplicities are not equal to the multiplicities of the corresponding 1-points of G, and poles of f and g. Hence we have

$$\begin{aligned} N(r,\infty;H) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}_L(r,1;F) \\ &\quad + \overline{N}_L(r,1;G) + N_0(r,0;F') + N_0(r,0;G') + O(\log r), \end{aligned} \tag{2.5}$$

where $N_0(r,0;F')$ denotes the counting function of those zeros of F' which are not the zeros of $f(F-1)$, $N_0(r,0;G')$ is similarly defined. Since f is a transcendental meromorphic functions we have

$$T(r,P) = o\{T(r,f)\}. \tag{2.6}$$

By Lemma 2, we have

$$\begin{aligned} T(r,f) &\le \overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + \overline{N}(r,1;F) \\ &\quad -N_0(r,0;F') + S(r,f). \end{aligned} \tag{2.7}$$

Similarly

$$\begin{aligned} T(r,g) &\le \overline{N}(r,\infty;g) + N_{k+1}(r,0;g) + \overline{N}(r,1;G) \\ &\quad -N_0(r,0;G') + S(r,g). \end{aligned} \tag{2.8}$$

Since $f^{(k)} - P$ and $g^{(k)} - P$ share 0 IM, therefore using (2.4) and (2.5) we obtain

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1;G) &= 2N_E^{1)}(r,1;F) + 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) \\ &\quad +2\overline{N}_E^{(2}(r,1;F) \\ &\le N_E^{1)}(r,1;F) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) \\ &\quad +\overline{N}(r,0;f) + \overline{N}(r,0;g) + 3\overline{N}_L(r,1;F) \\ &\quad +3\overline{N}_L(r,1;G) + N_0(r,0;F') + N_0(r,0;G') \\ &\quad +2\overline{N}_E^{(2}(r,1;F) + S(r,f) + S(r,g). \end{aligned} \tag{2.9}$$

Noting that $f^{(k)} - P$ and $g^{(k)} - P$ share $(0,2)$ we have

$$\begin{aligned} &N_E^{1)}(r,1;F) + 2\overline{N}_E^{(2}(r,1;F) + 3\overline{N}_L(r,1;F) + 3\overline{N}_L(r,1;G) \\ &\le N(r,1;G) + S(r,f) + S(r,g) \\ &\le T(r,G) + S(r,g) \\ &\le T(r,g) + k\overline{N}(r,\infty;g) + S(r,g). \end{aligned} \tag{2.10}$$

From (2.7) - (2.10), we obtain

$$\begin{aligned} T(r,f) &\le 2\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) + \overline{N}(r,0;f) + \overline{N}(r,0;g) \\ &\quad +N_{k+1}(r,0;f) + N_{k+1}(r,0;g) + S(r,f) + S(r,g). \end{aligned} \tag{2.11}$$

Similarly

$$\begin{aligned} T(r,g) &\le 2\overline{N}(r,\infty;g) + (k+2)\overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,0;f) \\ &\quad +N_{k+1}(r,0;g) + N_{k+1}(r,0;f) + S(r,f) + S(r,g). \end{aligned} \tag{2.12}$$

Suppose that there exists a subset $I \subseteq R^+$ satisfying $mes I = \infty$ such that $T(r,g) \le T(r,f)$, $r \in I$. Hence from (2.11) we have

$$\begin{aligned} \Delta_2 &= (k+2)\Theta(\infty,g) + 2\Theta(\infty,f) + \Theta(0,g) + \Theta(0,f) \\ &\quad +\delta_{k+1}(0,g) + \delta_{k+1}(0,f) \le k+7, \end{aligned}$$

contradicting (2.2). Similarly if there exists a subset $I \subseteq R^+$ satisfying $mes I = \infty$ such that $T(r,f) \leq T(r,g)$, $r \in I$, from (2.12) we can obtain

$$\begin{aligned}\Delta_1 &= (k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) \\ &\quad +\delta_{k+1}(0,f) + \delta_{k+1}(0,g) \leq k+7,\end{aligned}$$

contradicting (2.1). We now assume that $H = 0$. That is

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{2.13}$$

where $A(\neq 0)$ and B are finite complex constants. We now discuss the following three cases.

Case 1. *Let $B \neq 0$ and $A = B$. If $B = -1$, we obtain from (2.13) $FG = 1$, i.e., $f^{(k)}g^{(k)} = P^2$. If $B \neq -1$, from (2.13) we get*

$$\frac{1}{F} = \frac{BG}{(1+B)G-1} \text{ and } G = \frac{-1}{b(F - \frac{1+B}{B})}.$$

So by Lemma 1 we obtain

$$\begin{aligned}\overline{N}\left(r, \frac{1}{1+B}; G\right) &\leq \overline{N}(r,0;F) \leq N_{k+1}(r,0;f) + k\overline{N}(r,\infty;f) \\ &\quad +O(\log r) + S(r,f)\end{aligned} \tag{2.14}$$

and

$$\overline{N}\left(r, \frac{1+B}{B}; F\right) \leq \overline{N}(r,\infty;g) + O(\log r). \tag{2.15}$$

Using Lemma 2, (2.14) and (2.15) we obtain

$$\begin{aligned}T(r,g) &\leq N_{k+1}(r,0;g) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;g) \\ &\quad -N_0(r,0;G') + S(r,g) \\ &\leq N_{k+1}(r,0;g) + N_{k+1}(r,0;f) + k\overline{N}(r,\infty;f) \\ &\quad +\overline{N}(r,\infty;g) + S(r,f) + S(r,g)\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}T(r,f) &\leq N_{k+1}(r,0;f) + \overline{N}\left(r, \frac{1+B}{B}; F\right) + \overline{N}(r,\infty;f) \\ &\quad -N_0(r,0;F') + S(r,f) \\ &\leq N_{k+1}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f).\end{aligned} \tag{2.17}$$

Suppose that there exists a subset $I \subseteq R^+$ satisfying $mes I = \infty$ such that $T(r, f) \leq T(r, g)$, $r \in I$. So from (2.16) we obtain

$$k\Theta(\infty, f) + \Theta(\infty, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) \leq k + 2,$$

which by (2.1) gives

$$2\Theta(\infty, f) + \Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) > 5,$$

a contradiction by Remark 3. If there exists a subset $I \subseteq R^+$ satisfying $mes I = \infty$ such that $T(r, g) \leq T(r, f)$, $r \in I$, by the same argument we obtain a contradiction from (2.1) and (2.17).

Case 2. *Let $B \neq 0$ and $A \neq B$. If $B = -1$, from (2.13) we obtain $F = \frac{A}{-(G-(a+1))}$. If $B \neq -1$, from (2.13) we obtain $F - \frac{1+B}{B} = \frac{-A}{B^2(G+\frac{A-B}{B})}$. Using the same argument as in case 1 we obtain a contradiction in both cases.*

Case 3. *Let $B = 0$. Then from (2.13) we get*

$$g = Af + (1 - A)P_1, \tag{2.18}$$

where P_1 is a polynomial of degree $\gamma_{P_1} \geq k$. If $A \neq 1$, by Lemma 3 and (2.18) we get

$$\begin{aligned} T(r, g) &\leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + \overline{N}(r, (1 - A)P_1; g) + S(r, g) \\ &\leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; f) + S(r, g). \end{aligned} \tag{2.19}$$

Since f and g are transcendental meromorphic function from (2.18) we have

$$T(r, f) = T(r, g) + O(\log r).$$

So from (2.19), we obtain

$$\Theta(0, f) + \Theta(0, g) + \Theta(\infty, g) \leq 2,$$

which gives by (2.1)

$$(k + 2)\Theta(\infty, f) + \Theta(\infty, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k + 5,$$

a contradiction by Remark 3. Thus $A = 1$ and so $f = g$. This completes the proof of the lemma. ■

Lemma 5. *[12] Let f and g be two transcendental meromorphic functions, let $n \geq 2$ be a positive integer, and let P be a nonconstant polynomial with its degree $\gamma_P \leq n$. If $f^n f' g^n g' = P^2$, then f and g are expressed as $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$ respectively, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta) d\eta$.*

Lemma 6. *Let f and g be two transcendental meromorphic functions, let n, m be two positive integers and let P be a nonconstant polynomial. If $m = 1$, $n \geq 6$ or if $m \geq 2$, $n \geq m + 3$, then*

$$(f^n(f - 1)^m)'(g^n(g - 1)^m)' \not\equiv P^2.$$

Proof. If possible, let

$$(f^n(f-1)^m)'(g^n(g-1)^m)' \equiv P^2. \tag{2.20}$$

We discuss the following two cases.
Case 1. Let $m \geq 2$. Then from (2.20) we obtain

$$f^{n-1}(f-1)^{m-1}(cf-d)f'g^{n-1}(g-1)^{m-1}(cg-d)g' \equiv P^2, \tag{2.21}$$

where $c = n+m$ and $d = n$.
Let $z_0 \notin \{z : P(z) = 0\}$ be a 1-point of f with multiplicity $p_0(\geq 1)$. Then from (2.21) it follows that z_0 is a pole of g. Suppose that z_0 is a pole of g of order $q_0(\geq 1)$. Then we have $mp_0 - 1 = (n+m)q_0 + 1$, i.e., $mp_0 = (n+m)q_0 + 2 \geq n+m+2$, and so

$$p_0 \geq \frac{n+m+2}{m}.$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a zero of $cf - d$ with multiplicity $p_1(\geq 1)$. Then from (2.21) it follows that z_1 is a pole of g. Suppose that z_1 is a pole of g of order $q_1(\geq 1)$. Then we have $2p_1 - 1 = (n+m)q_1 + 1$, and so

$$p_1 \geq \frac{n+m+2}{2}.$$

Let $z_2 \notin \{z : P(z) = 0\}$ be a zero of f with multiplicity $p_2(\geq 1)$. Then it follows from (2.21) that z_2 is a pole of g. Suppose that z_2 is a pole of g of order $q_2(\geq 1)$. Then we have

$$np_2 - 1 = (n+m)q_2 + 1. \tag{2.22}$$

From (2.22) we get $mq_2 + 2 = n(p_2 - q_2) \geq n$, i.e., $q_2 \geq \frac{n-2}{m}$. Thus from (2.22) we obtain $np_2 = (n+m)q_2 + 2 \geq \frac{(n+m)(n-2)}{m} + 2$, and so

$$p_2 \geq \frac{n+m-2}{m}.$$

Let $z_3 \notin \{z : P(z) = 0\}$ be a pole of f. Then it follows from (2.21) that z_3 is a zero of $g(g-1)(cg-d)$ or a zero of g'. So we have

$$\begin{aligned}
\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,1;g) + \overline{N}\left(r,\frac{d}{c};g\right) + \overline{N}_0(r,0;g') \\
&\quad + S(r,f) + S(r,g) \\
&\leq \left(\frac{m+2}{n+m+2} + \frac{m}{n+m-2}\right) T(r,g) + \overline{N}_0(r,0;g') \\
&\quad + S(r,f) + S(r,g),
\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)(cg-d)$.

By the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}2T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}\left(r,\frac{d}{c};f\right)+\overline{N}(r,\infty;f)\\ &\quad -\overline{N}_0(r,0;f')+S(r,f)\\ &\leq \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}\\ &\quad -\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \qquad (2.23)\end{aligned}$$

Similarly

$$\begin{aligned}2T(r,g) &\leq \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}\\ &\quad +\overline{N}_0(r,0;f')-\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \qquad (2.24)\end{aligned}$$

Adding (2.23) and (2.24) we obtain

$$\left(1-\frac{m+2}{n+m+2}-\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

contradicting the fact that $n \geq m+3$.

Case 2. Let $m=1$. Then from (2.20) we obtain

$$f^{n-1}(af-b)f'g^{n-1}(ag-b)g' \equiv P^2, \qquad (2.25)$$

where $a=n+1$ and $b=n$.

Let $z_4 \notin \{z : P(z)=0\}$ be a pole of f. Then it follows from (2.25) that z_4 is a zero of $g(ag-b)$ or a zero of g'. Then proceeding in a like manner as Case 1 we obtain

$$\left(1-\frac{2}{n-1}-\frac{4}{n+3}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which contradicts with the fact that $n \geq 6$. This proves the lemma. ■

Lemma 7. *[13] Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form*

$$P_n(f)=a_nf^n(z)+a_{n-1}f^{n-1}(z)+...+a_1f(z)+a_0,$$

where $a_n(\neq 0)$, a_{n-1}, ... , a_1, a_0 are complex numbers. Then

$$T(r,P_n(f))=nT(r,f)+O(1).$$

Lemma 8. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b)\equiv g^n(ag+b)$$

implies $f\equiv g$, where a, b are two nonzero constants.

Proof. We omit the proof since it can be carried out in the line of Lemma 6 [11]. ■

3 Proof of the Theorem

Proof of Theorem 1. We consider $F_1 = f^n$ and $G_1 = g^n$. Then we see that $F_1^{(k)} - P$ and $G_1^{(k)} - P$ share the value 0 with weight two. Using Lemma 7, we have

$$\begin{aligned}
\Theta(0, F_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; F_1)}{T(r, F_1)} \\
&= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; f)}{nT(r, f)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{nT(r, f)} \\
&\geq \frac{n-1}{n}.
\end{aligned} \tag{3.1}$$

Similarly

$$\Theta(0, G_1) \geq \frac{n-1}{n}. \tag{3.2}$$

$$\begin{aligned}
\Theta(\infty, F_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; F_1)}{T(r, F_1)} \\
&= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; f)}{nT(r, f)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{nT(r, f)} \\
&\geq \frac{n-1}{n}.
\end{aligned} \tag{3.3}$$

Similarly

$$\Theta(\infty, G_1) \geq \frac{n-1}{n}. \tag{3.4}$$

$$\begin{aligned}
\delta_{k+1}(0, F_1) &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; F_1)}{T(r, F_1)} \\
&= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; f^n)}{nT(r, f)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{(k+1)T(r, f)}{nT(r, f)} \\
&\geq \frac{n-k-1}{n}.
\end{aligned} \tag{3.5}$$

Similarly

$$\delta_{k+1}(0, G_1) \geq \frac{n-k-1}{n}. \tag{3.6}$$

In view of (2.1)-(2.2) and (3.1)-(3.6) we obtain

$$\Delta_1 \geq (k+8) - \frac{3k+8}{n} \textit{ and } \Delta_2 \geq (k+8) - \frac{3k+8}{n}.$$

This gives $\Delta_1 > k+7$ and $\Delta_2 > k+7$ as $n \geq 3k+9$. So by Lemma 4 we obtain either $F_1^{(k)} G_1^{(k)} = P^2$ or $F_1 = G_1$. Suppose that $F_1^{(k)} G_1^{(k)} = P^2$, i.e.,

$$(f^n)^{(k)} (g^n)^{(k)} = P^2. \tag{3.7}$$

If $k = 1$, then from (3.7) we have $f^{n-1} f' g^{n-1} g' = P^2/n^2$. Applying Lemma 5 we obtain $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^n c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta) d\eta$.

If $F_1 = G_1$, then $f = tg$ for a complex number t such that $t^n = 1$. This completes the proof of Theorem 1. ∎

Proof of Theorem 2. Let $F_2 = f^n (f-1)^m$ and $G_2 = g^n (g-1)^m$. Then $F_2^{(k)} - P$ and $G_2^{(k)} - P$ share $(0, 2)$. Using Lemma 7, we obtain

$$\begin{aligned}
\Theta(0, F_2) &= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, 0; F_2)}{T(r, F_2)} \\
&= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, 0; f^n (f-1)^m)}{(n+m) T(r, f)} \\
&\geq 1 - \limsup_{r \to \infty} \frac{2T(r, f)}{(n+m) T(r, f)} \\
&\geq \frac{n+m-2}{n+m}.
\end{aligned} \tag{3.8}$$

Similarly

$$\Theta(0, G_2) \geq \frac{n+m-2}{n+m}. \tag{3.9}$$

$$\begin{aligned}
\Theta(\infty, F_2) &= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \infty; F_2)}{T(r, F_2)} \\
&= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \infty; f)}{(n+m) T(r, f)} \\
&\geq 1 - \limsup_{r \to \infty} \frac{T(r, f)}{(n+m) T(r, f)} \\
&\geq \frac{n+m-1}{n+m}.
\end{aligned} \tag{3.10}$$

Similarly

$$\Theta(\infty, G_2) \geq \frac{n+m-1}{n+m}. \tag{3.11}$$

$$\begin{aligned}
\delta_{k+1}(0,F_2) &= 1-\limsup_{r\to\infty}\frac{N_{k+1}(r,0;F_2)}{T(r,F_2)}\\
&= 1-\limsup_{r\to\infty}\frac{N_{k+1}(r,0;f^n(f-1)^m)}{(n+m)T(r,f)}\\
&\geq 1-\limsup_{r\to\infty}\frac{(k+m+1)T(r,f)}{(n+m)T(r,f)}\\
&\geq \frac{n-k-1}{n+m}. \qquad (3.12)
\end{aligned}$$

Similarly

$$\delta_{k+1}(0,G_2)\geq\frac{n-k-1}{n+m}. \qquad (3.13)$$

Using (2.1), (2.2) and (3.8)-(3.13) we obtain

$$\Delta_1\geq(k+6)+\frac{2n-3k-10}{n+m}\ \textit{and}\ \Delta_2\geq(k+6)+\frac{2n-3k-10}{n+m}.$$

Since $n\geq 3k+m+11$, we get $\Delta_1>k+7$ and $\Delta_2>k+7$. So by Lemma 4, either $F_2^{(k)}G_2^{(k)}=P^2$ or $F_2=G_2$ holds. Suppose that $F_2^{(k)}G_2^{(k)}=P^2$. Then

$$(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)}=P^2. \qquad (3.14)$$

Also by Lemma 6, (3.14) does not arise when $k=1$.

Next we suppose that $F_2=G_2$, i.e.,

$$f^n(f-1)^m=g^n(g-1)^m. \qquad (3.15)$$

Let $m=1$. Then in view of Lemma 8 and (3.15) we obtain $f=g$.
Let $m\geq 2$. Then from (3.15) we obtain

$$\begin{aligned}
f^n[f^m+...+(-1)^iC_m^if^{m-i}+...+(-1)^m]&=g^n[g^m\\
&+...+(-1)^iC_m^ig^{m-i}+...+(-1)^m]. \qquad (3.16)
\end{aligned}$$

Let $h=\frac{f}{g}$. If h is a constant, then substituting $f=gh$ in (3.16) we obtain

$$\begin{aligned}
g^{n+m}(h^{n+m}-1)+...+(-1)^iC_m^ig^{n+m-i}(h^{n+m-i}-1)&\\
+...+(-1)^mg^n(h^n-1)=0&,
\end{aligned}$$

which implies $h=1$. Hence $f=g$.
If h is not a constant, then from (3.15) we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(w_1,w_2)=w_1^n(w_1-1)^m-w_2^n(w_2-1)^m.$$

This completes the proof of theorem 2. ■

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