Bilinear factorization of algebras

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Abstract

We study the (so-called bilinear) factorization problem answered by a weak wreath product (of monads and, more specifically, of algebras over a commutative ring) in the works by Street and by Caenepeel and De Groot. A bilinear factorization of a monad R turns out to be given by monad morphisms $A \to R \leftarrow B$ inducing a split epimorphism of B-A bimodules $B \otimes A \to R$. We prove a biequivalence between the bicategory of weak distributive laws and an appropriately defined bicategory of bilinear factorization structures. As an illustration of the theory, we collect some examples of algebras over commutative rings which admit a bilinear factorization; i.e. which arise as weak wreath products.

Introduction

A *distributive law* (in a bicategory) consists of two monads A and B together with a 2-cell $A \otimes B \to B \otimes A$ which is compatible with the monad structures, see [2].

A distributive law $A \otimes B \to B \otimes A$ is known to be equivalent to a monad structure on the composite $B \otimes A$ such that the multiplication commutes with the actions by B on the left and by A on the right. The monad $B \otimes A$ is known as a *wreath product*, a *twisted product*, or a *smash product* of A and B.

Given a monad R, one may ask under what conditions it is isomorphic to a wreath product of A and B. This question is known as a *(strict) factorization* problem and the answer is this. A monad R is isomorphic to a wreath product of A and B if and only if there are monad morphisms $A \to R \leftarrow B$ such that composing $B \otimes A \to R \otimes R$ with the multiplication $R \otimes R \to R$ yields an isomorphism

$B \otimes A \cong R$. In the particular bicategory of spans this and related questions were studied in [14]. In the monoidal category (i.e. one object bicategory) of modules over a commutative ring; and also in its opposite, such questions were investigated in [7], see also [10] and [19].

In the papers [8] and [17], the notion of distributive law was generalized by weakening the compatibility conditions with the units of the monads. A so defined *weak distributive law* $A \otimes B \to B \otimes A$ also induces an associative multiplication on $B \otimes A$ but it fails to be unital. However, there is a canonical idempotent on $B \otimes A$. Whenever it splits, the corresponding retract is a monad, known as the *weak wreath product* or *weak smash product* of A and B, see [17] and [8].

The aim of this paper is to study the factorization problem answered by a weak wreath product. In the particular bicategory of spans, this problem has already been studied in [4]. In the paper [9] addressing questions of similar motivation, a more general notion of *weak crossed product monad* was considered. Such weak crossed products are not induced by weak distributive laws but by more general 1-cells in an extended bicategory of monads introduced in [3]. The factorization problem corresponding to weak crossed products is fully described in [9].

We use the term *strict factorization* in the same sense as in [14]. Motivated by it, when we want to stress the difference from the weak generalizations, we refer to Beck's distributive laws as *strict* distributive laws and to their induced wreath products as *strict* wreath products.

We start Section 1 by recalling from [17] the notion of weak distributive law and the corresponding construction of weak wreath product. We show that a monad R is isomorphic to a weak wreath product of some monads A and B if and only if there are monad morphisms (with trivial 1-cell parts) $A \to R \leftarrow B$ such that composing $B \otimes A \to R \otimes R$ with the multiplication $R \otimes R \to R$ yields a split epimorphism of B-A bimodules $B \otimes A \to R$. What is more, in Theorem 1.12, for any bicategory in which idempotent 2-cells split, we prove a biequivalence of the bicategory of weak distributive laws and an appropriately defined bicategory of bilinear factorization structures. This extends [4, Theorem 3.12].

Section 2 is devoted to collecting examples of algebras over commutative rings which admit a bilinear factorization.

The algebra homomorphisms $A \to R \leftarrow B$ in a bilinear factorization structure are not injective in general. In Paragraph 2.1 we show, however, that if R admits any bilinear factorization then it admits also a bilinear factorization with injective algebra homomorphisms $\tilde{A} \to R \leftarrow \tilde{B}$. In general the latter factorization is still non-strict and we characterize those cases when it happens to be strict.

In Paragraph 2.2 we consider an algebra A and an element e of it such that $ea = eae$ for all $a \in A$ (so that eA is an algebra with unit e). Assuming that there is a strict distributive law $eA \otimes B \to B \otimes eA$, we extend it to a weak distributive law $A \otimes B \to B \otimes A$. The corresponding weak wreath product is isomorphic to the strict wreath product of eA and B; hence it admits a strict factorization in terms of them.

The Ore extension of an algebra B over a commutative ring k is the wreath product of B with the algebra $k[X]$ of polynomials of a formal variable X, see [7, Example 2.11 (1)]. In Paragraph 2.3, generalizing Ore extensions, we construct

a weak wreath product of B with $k[X]$, that we regard as a weak Ore extension of B (although it turns out to be isomorphic in a nontrivial way to a strict Ore extension of an appropriate subalgebra $\tilde{B}$).

For any commutative ring k, there is a bicategory Bim of k-algebras, their bimodules and bimodule maps. In Paragraph 2.4 we consider strict distributive laws in Bim. Taking a 0-cell (i.e. k-algebra) R which admits a separable Frobenius structure, we show that any distributive law over R induces a weak distributive law over k. The corresponding weak wreath product is isomorphic to the R-module tensor product. We also present a morphism between these (weak) distributive laws over the respective objects R and k. The examples in Paragraph 2.5 and Paragraph 2.6 belong to this class of examples.

In Paragraph 2.5 we start with a finite collection of strict distributive laws $A_i \otimes B_i \to B_i \otimes A_i$ and construct a weak distributive law $(\oplus_i A_i) \otimes (\oplus_i B_i) \to (\oplus_i B_i) \otimes (\oplus_i A_i)$. The corresponding weak wreath product is isomorphic to the direct sum of the wreath product algebras $B_i \otimes A_i$.

In Paragraph 2.6 we take a weak bialgebra H and an H-module algebra A. We show that their smash product is a weak wreath product.

In Paragraph 2.7 we present explicitly a bilinear factorization of the three dimensional noncommutative algebra of 2×2 upper triangle matrices with entries in a field k whose characteristic is different from 2, in terms of two copies of the commutative algebra $k \oplus k$. This example does not belong to any of the previously discussed classes.

1   Weak wreath products and bilinear factorizations

The aim of this section is to prove an equivalence between weak wreath product monads on one hand, and monads admitting a bilinear factorization on the other. As a first step to that, under the assumption that an appropriate idempotent 2-cell splits, in Theorems 1.6 and 1.8 we show that a monad (in an arbitrary bicategory) admits a bilinear factorization if and only if it is isomorphic to a weak wreath product monad. Certainly, if we stopped at this point then it would not be necessary to work in a bicategory: a monoidal hom category singled out by one object would suffice. But we aim at more. As the main result of the section, for a bicategory $\mathcal{K}$ in which idempotent 2-cells split, in Theorem 1.12 we prove the biequivalence of the bicategory of bilinear factorizations of monads, and the bicategory of weak distributive laws in $\mathcal{K}$. Remarkably, in this way the same bicategory of weak distributive laws occurs that was introduced (in a dual form) in [5] on different grounds.

Concerning examples, as long as we are interested only in objects (i.e. weak wreath product monads), the bicategorical formulation plays no role. It becomes

important if we are interested also in morphisms between weak wreath product monads, possibly over different base objects. An example of this kind is presented in Example 2.4.

Throughout this section we work in a bicategory $\mathcal{K}$, whose coherence isomorphisms will be omitted in our notation. The horizontal composition is denoted by $\otimes$ and the vertical composition is denoted by juxtaposition. Our motivating example is the one-object bicategory (i.e. monoidal category) of modules over a commutative ring (where $\otimes$ is the module tensor product).

1.1. Weak distributive laws. Let (A, μ_A, η_A) and (B, μ_B, η_B) be (associative and unital) monads in $\mathcal{K}$ on the same object, with multiplications μ_A, μ_B and units η_A, η_B. Following [8, Theorem 3.2] and [17, Definition 2.1], a 2-cell $\Psi : A \otimes B \to B \otimes A$ is said to be a *weak distributive law of A over B* if the following diagrams commute. (Throughout, in the labels of diagrams we omit indices A and B of η and μ since they can be uniquely determined from the domains and codomains of the respective arrows.)

$$
\begin{array}{ccc}
\begin{array}{ccc}
A \otimes A \otimes B & \xrightarrow{\mu \otimes B} & A \otimes B \\
{\scriptstyle A \otimes \Psi}\downarrow & & \downarrow \\
A \otimes B \otimes A & & \Psi \\
{\scriptstyle \Psi \otimes A}\downarrow & & \downarrow \\
B \otimes A \otimes A & \xrightarrow[B \otimes \mu]{} & B \otimes A
\end{array}
&
\begin{array}{ccc}
A \otimes B \otimes B & \xrightarrow{A \otimes \mu} & A \otimes B \\
{\scriptstyle \Psi \otimes B}\downarrow & & \downarrow \\
B \otimes A \otimes B & & \Psi \\
{\scriptstyle B \otimes \Psi}\downarrow & & \downarrow \\
B \otimes B \otimes A & \xrightarrow[\mu \otimes A]{} & B \otimes A
\end{array}
&
\begin{array}{ccc}
B \otimes A & \xrightarrow{B \otimes A \otimes \eta} & B \otimes A \otimes B \\
{\scriptstyle \eta \otimes B \otimes A}\downarrow & & \downarrow {\scriptstyle B \otimes \Psi} \\
A \otimes B \otimes A & & B \otimes B \otimes A \\
{\scriptstyle \Psi \otimes A}\downarrow & & \downarrow {\scriptstyle \mu \otimes A} \\
B \otimes A \otimes A & \xrightarrow[B \otimes \mu]{} & B \otimes A
\end{array}
\end{array}
$$

$$\tag{1}$$

Lemma 1.2. *[17, Proposition 2.2] The third diagram in* (1) *is equivalent to the following two diagrams.*

$$
\begin{array}{ccc}
\begin{array}{ccc}
B & \xrightarrow{\eta \otimes B} & A \otimes B \\
{\scriptstyle B \otimes \eta \otimes \eta}\downarrow & & \downarrow \\
B \otimes A \otimes B & & \Psi \\
{\scriptstyle B \otimes \Psi}\downarrow & & \downarrow \\
B \otimes B \otimes A & \xrightarrow[\mu \otimes A]{} & B \otimes A
\end{array}
&
\begin{array}{ccc}
A & \xrightarrow{A \otimes \eta} & A \otimes B \\
{\scriptstyle \eta \otimes \eta \otimes A}\downarrow & & \downarrow \\
A \otimes B \otimes A & & \Psi \\
{\scriptstyle \Psi \otimes A}\downarrow & & \downarrow \\
B \otimes A \otimes A & \xrightarrow[B \otimes \mu]{} & B \otimes A
\end{array}
\end{array}
$$

$$\tag{2}$$

Proof. The following diagram shows that commutativity of the third diagram in (1) implies commutativity of the first diagram in (2),

$$
\begin{array}{ccccccc}
B & \xrightarrow{B \otimes \eta} & B \otimes A & \xrightarrow{B \otimes A \otimes \eta} & B \otimes A \otimes B & \xrightarrow{B \otimes \Psi} & B \otimes B \otimes A \\
{\scriptstyle \eta \otimes B}\downarrow & & \downarrow {\scriptstyle \eta \otimes B \otimes A} & & & & \\
A \otimes B & \xrightarrow{A \otimes B \otimes \eta} & A \otimes B \otimes A & & & & \downarrow {\scriptstyle \mu \otimes A} \\
{\scriptstyle \Psi}\downarrow & & \downarrow {\scriptstyle \Psi \otimes A} & & & & \\
B \otimes A & \xrightarrow[B \otimes A \otimes \eta]{} & B \otimes A \otimes A & \xrightarrow[\quad B \otimes \mu \quad]{} & & & B \otimes A
\end{array}
$$

where the region on the right commutes by the third diagram in (1). Commutativity of the second diagram in (2) is verified symmetrically. Conversely, if both

diagrams in (2) commute then so does

$$
\begin{array}{ccc}
B \otimes A & \xrightarrow{\;\;\eta \otimes B \otimes A\;\;} & A \otimes B \otimes A \quad (3)
\end{array}
$$

with diagonal arrows $B \otimes \eta \otimes \eta \otimes A$ to $B \otimes A \otimes B \otimes A$, then $B \otimes \Psi \otimes A$ to $B \otimes B \otimes A \otimes A \xrightarrow{\mu \otimes A \otimes A} B \otimes A \otimes A$; left vertical $B \otimes A \otimes \eta$; right vertical $\Psi \otimes A$ then $B \otimes \mu$; $B \otimes B \otimes \mu$ down to $B \otimes B \otimes A$; $\mu \otimes \mu$ diagonal to $B \otimes A$; bottom $B \otimes A \otimes B \xrightarrow{B \otimes \Psi} B \otimes B \otimes A \xrightarrow{\mu \otimes A} B \otimes A$. ∎

1.3. Weak wreath product. Let $\psi : A \otimes B \to B \otimes A$ be a weak distributive law. Define $\mu : B \otimes A \otimes B \otimes A \to B \otimes A$ as

$$
B \otimes A \otimes B \otimes A \xrightarrow{\;B \otimes \Psi \otimes A\;} B \otimes B \otimes A \otimes A \xrightarrow{\;\mu \otimes \mu\;} B \otimes A.
$$

It follows from the first two diagrams in (1) that μ is an associative multiplication. From now on, we consider $B \otimes A$ as an associative monad with the multiplication μ – possibly without a unit. (In fact, $B \otimes A$ can be seen to possess a preunit $\eta_B \otimes \eta_A$ in the sense discussed in [8].)

Proposition 1.4. *(See [17, Proposition 2.3].)* *For any weak distributive law* $\Psi : A \otimes B \to B \otimes A$, *define* $\overline{\Psi} : B \otimes A \to B \otimes A$ *by*

$$
B \otimes A \xrightarrow{\;B \otimes \eta \otimes \eta \otimes A\;} B \otimes A \otimes B \otimes A \xrightarrow{\;\mu\;} B \otimes A. \tag{4}
$$

Then $\overline{\Psi}$ *is an idempotent endomorphism of monads (without unit), and of B-A bimodules. Moreover,* $\overline{\Psi}\Psi = \Psi$.

Proof. Note that $\overline{\Psi}$ stands in the diagonal of the diagram (3). Hence it has the equivalent forms

$$
\overline{\Psi} = (B \otimes \mu_A)(\Psi \otimes A)(\eta_A \otimes B \otimes A) \tag{5}
$$
$$
= (\mu_B \otimes A)(B \otimes \Psi)(B \otimes A \otimes \eta_B). \tag{6}
$$

Since the expression (5) is evidently a right A-module map and (6) is a left B-module map, this proves the bilinearity of $\overline{\Psi}$, i.e.

$$
(\mu_B \otimes A)(B \otimes \overline{\Psi}) = \overline{\Psi}(\mu_B \otimes A) \quad \text{and} \quad (B \otimes \mu_A)(\overline{\Psi} \otimes A) = \overline{\Psi}(B \otimes \mu_A). \tag{7}
$$

By commutativity of

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{\;\Psi\;} & B \otimes A \\
\downarrow{\scriptstyle \eta \otimes A \otimes B} & & \downarrow{\scriptstyle \eta \otimes B \otimes A} \\
A \otimes A \otimes B & \xrightarrow{A \otimes \Psi} & A \otimes B \otimes A \\
\downarrow{\scriptstyle \mu \otimes B} & & \downarrow{\scriptstyle \Psi \otimes A} \\
& (1) \quad B \otimes A \otimes A & \\
& & \downarrow{\scriptstyle B \otimes \mu} \\
A \otimes B & \xrightarrow{\;\Psi\;} & B \otimes A
\end{array}
$$

and (5), we obtain $\overline{\Psi}\Psi = \Psi$. This implies

$$\overline{\Psi}\mu = \overline{\Psi}(\mu_B \otimes \mu_A)(B \otimes \Psi \otimes A) \overset{(7)}{=} (\mu_B \otimes \mu_A)(B \otimes \overline{\Psi} \otimes A)(B \otimes \Psi \otimes A) = \mu, \quad (8)$$

hence also $\overline{\Psi}^2 = \overline{\Psi}$. Moreover, by commutativity of

and (5), we obtain $(B \otimes \mu_A)(\Psi \otimes A)(A \otimes \overline{\Psi}) = (B \otimes \mu_A)(\Psi \otimes A)$. This implies that

$$\mu(B \otimes A \otimes \overline{\Psi}) = (\mu_B \otimes A)(B \otimes B \otimes \mu_A)(B \otimes \Psi \otimes A)(B \otimes A \otimes \overline{\Psi}) = \mu.$$

Combining it with the symmetrical counterpart, we conclude that

$$\mu(\overline{\Psi} \otimes \overline{\Psi}) = \mu. \tag{9}$$

From (9) and (8) we get that $\overline{\Psi}$ is multiplicative with respect to μ. ∎

1.5. Splitting idempotents. Assume that the idempotent 2-cell $\overline{\Psi}$ associated in Proposition 1.4 to a weak distributive law Ψ splits. That is, there is a (unique up-to isomorphism) 1-cell $B \otimes_\Psi A$ and 2-cells $\pi : B \otimes A \to B \otimes_\Psi A$ and $\iota : B \otimes_\Psi A \to B \otimes A$ such that $\pi\iota = B \otimes_\Psi A$ and $\iota\pi = \overline{\Psi}$. Since $\overline{\Psi}$ is a morphism of B-A bimodules, there is a unique B-A bimodule structure on $B \otimes_\Psi A$ such that both π and ι are morphisms of B-A bimodules (i.e. $B \otimes_\Psi A$ is a B-A bimodule retract of $B \otimes A$).

Theorem 1.6. *(See [17, Theorem 2.4].) Let* $\Psi : A \otimes B \to B \otimes A$ *be a weak distributive law in a bicategory* $\mathcal{K}$*, such that the associated idempotent 2-cell* $\overline{\Psi}$ *splits. Then there is a retract monad* $(B \otimes_\Psi A, \mu_\Psi)$ *of* $(B \otimes A, \mu)$ *which is unital. Moreover, the 2-cells*

$$\beta := \pi(B \otimes \eta_A) : B \to B \otimes_\Psi A, \qquad \alpha := \pi(\eta_B \otimes A) : A \to B \otimes_\Psi A$$

are homomorphisms of unital monads such that $\mu_\Psi(\beta \otimes \alpha) : B \otimes A \to B \otimes_\Psi A$ *is equal to* π*; and the left* B*- and right* A*-actions on* $B \otimes_\Psi A$ *can be written as* $\mu_\Psi(\beta \otimes (B \otimes_\Psi A))$ *and* $\mu_\Psi((B \otimes_\Psi A) \otimes \alpha)$*, respectively.*

Proof. Equip $B \otimes_\Psi A$ with the multiplication

$$\mu_\Psi := \left((B \otimes_\Psi A) \otimes (B \otimes_\Psi A) \overset{\iota \otimes \iota}{\longrightarrow} B \otimes A \otimes B \otimes A \overset{\mu}{\longrightarrow} B \otimes A \overset{\pi}{\longrightarrow} B \otimes_\Psi A \right).$$

By (9), $\pi\mu = \mu_\Psi(\pi \otimes \pi)$ and by (8), $\mu(\iota \otimes \iota) = \iota\mu_\Psi$. Since ι is a (split) monomorphism and π is a (split) epimorphism, any of these equalities implies associativity of μ_Ψ. It is also unital with $\eta_\Psi := \pi(\eta_B \otimes \eta_A)$ since

$$\mu_\Psi((B \otimes_\Psi A) \otimes \pi)((B \otimes_\Psi A) \otimes \eta_B \otimes \eta_A)\pi \overset{(9)}{=} \pi\mu(B \otimes A \otimes \eta_B \otimes \eta_A) \overset{(6)}{=} \pi\overline{\Psi} = \pi,$$

and symmetrically on the other side. Unitality of β is evident. We have $\iota\beta\mu_B = \overline{\Psi}(B \otimes \eta_A)\mu_B$ and by (8) and (9), $\iota\mu_\Psi(\beta \otimes \beta) = \mu(B \otimes \eta_A \otimes B \otimes \eta_A)$. Hence multiplicativity of β follows by commutativity of

That α is an algebra homomorphism follows by symmetry. Finally,

$$\iota\mu_\Psi(\beta \otimes (B \otimes_\Psi A))(B \otimes \pi) \overset{(8)(9)}{=} \mu(B \otimes \eta_A \otimes B \otimes A) \overset{(5)}{=}$$
$$(\mu_B \otimes A)(B \otimes \overline{\Psi}) = (\mu_B \otimes A)(B \otimes \iota\pi)$$

so that $\mu_\Psi(\beta \otimes (B \otimes_\Psi A)) = \pi(\mu_B \otimes A)(B \otimes \iota)$ as stated, and symmetrically for the right A-action. Therefore,

$$\mu_\Psi(\beta \otimes \alpha) = \pi(\mu_B \otimes A)(B \otimes \overline{\Psi})(B \otimes \eta_B \otimes A) \overset{(7)}{=} \pi\overline{\Psi} = \pi. \qquad \blacksquare$$

The situation in the above theorem motivates the following notion.

1.7. Bilinear factorization structures. In an arbitrary bicategory $\mathcal{K}$, consider unital monads (A, μ_A, η_A), (B, μ_B, η_B) and (R, μ_R, η_R) on the same object k. Let $\alpha : A \to R \leftarrow B : \beta$ be 2-cells which are compatible with the monad structures in the sense of the diagrams

i.e. α and β be morphisms of (unital) monads. (They are monad morphisms with trivial 1-cell parts in the sense of [15].) Regarding R as a left B-module via $\mu_R(\beta \otimes R) : B \otimes R \to R$ and a right A-module via $\mu_R(R \otimes \alpha) : R \otimes A \to R$,

$$\pi := \left(B \otimes A \xrightarrow{\beta \otimes \alpha} R \otimes R \xrightarrow{\mu} R \right) \tag{10}$$

is a homomorphism of B-A bimodules. If π has a B-A bimodule section ι, then we call the datum $(\alpha : A \to R \leftarrow B : \beta, \iota : R \to B \otimes A)$ a *bilinear factorization structure on R* or, shortly, a *bilinear factorization of R*.

By Theorem 1.6, any weak distributive law $\Psi : A \otimes B \to B \otimes A$ for which the idempotent 2-cell $\overline{\Psi}$ splits, determines a bilinear factorization structure $(\alpha : A \to B \otimes_\psi A \leftarrow B : \beta, \iota : B \otimes_\Psi A \to B \otimes A)$. We turn to proving the converse.

Theorem 1.8. *For a bilinear factorization structure* $(\alpha : A \to R \leftarrow B : \beta, \iota : R \to B \otimes A)$ *in an arbitrary bicategory* $\mathcal{K}$,
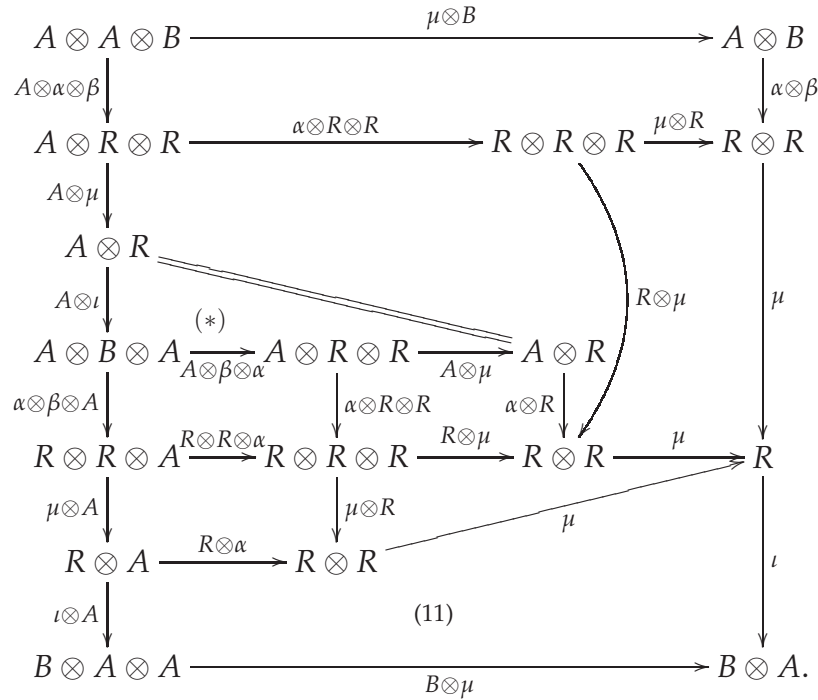
$$\Psi := \left(A \otimes B \xrightarrow{\alpha \otimes \beta} R \otimes R \xrightarrow{\mu} R \xrightarrow{\iota} B \otimes A \right)$$

is a weak distributive law of A over B such that the corresponding idempotent 2-cell $\overline{\Psi}$ *splits. Moreover, R is isomorphic to the corresponding unital monad* $B \otimes_{\Psi} A$.

Proof. The assumption that ι is a morphism of B-A bimodules means the equalities

$$\iota \mu_R (R \otimes \alpha) = (B \otimes \mu_A)(\iota \otimes A) \qquad \text{and} \qquad \iota \mu_R (\beta \otimes R) = (\mu_B \otimes A)(B \otimes \iota). \quad (11)$$

Compatibility of Ψ with the multiplication of A (i.e. the first diagram in (1)) follows by commutativity of

The top region commutes by the multiplicativity of α and the region labelled by $(*)$ commutes since ι is a section of π (occurring at the bottom of this region). It follows by symmetrical considerations that Ψ renders commutative also the second diagram in (1). As for the third one concerns, in the diagram
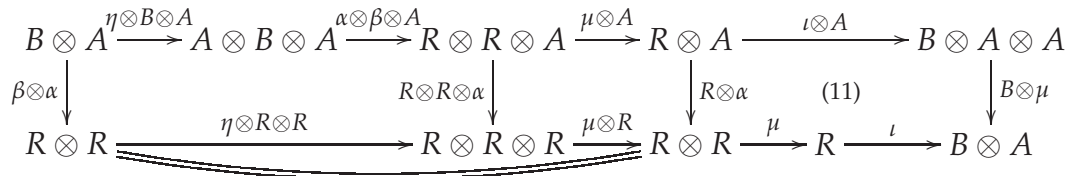
the region on the left commutes by the unitality of α. Commutativity of this diagram yields the equality

$$(B \otimes \mu_A)(\Psi \otimes A)(\eta_A \otimes B \otimes A) = \iota \pi. \quad (12)$$

Symmetrically,
$$(\mu_B \otimes A)(B \otimes \Psi)(B \otimes A \otimes \eta_B) = \iota\pi$$
which proves that Ψ renders commutative the third diagram in (1), so that Ψ is a weak distributive law.

By (5), the expression on the left hand side of (12) is $\overline{\Psi}$ which clearly splits. The corresponding 1-cell $B \otimes_\Psi A$ is defined (uniquely up-to isomorphism) via some splitting of it as $\pi_\Psi : B \otimes A \to B \otimes_\Psi A$ and $\iota_\Psi : B \otimes_\Psi A \to B \otimes A$. By uniqueness up-to isomorphism of the splitting of an idempotent 2-cell, (12) implies that $B \otimes_\Psi A$ and R are isomorphic 1-cells in $\mathcal{K}$ via the mutually inverse isomorphisms $\pi_\Psi\iota : R \to B \otimes_\Psi A$ and $\pi\iota_\Psi : B \otimes_\Psi A \to R$.

Composing both equal paths in

$$
\begin{array}{ccccc}
B \otimes R \otimes R \otimes A & \xrightarrow{\beta \otimes R \otimes R \otimes \alpha} & R \otimes R \otimes R \otimes R & \xrightarrow{\mu \otimes \mu} & R \otimes R \\
{\scriptstyle B \otimes \mu \otimes A}\downarrow & & {\scriptstyle R \otimes \mu \otimes R}\downarrow & & \downarrow{\scriptstyle \mu} \\
B \otimes R \otimes A & \xrightarrow{\beta \otimes R \otimes \alpha} & R \otimes R \otimes R & \xrightarrow{\mu^2} & R \\
{\scriptstyle B \otimes \iota \otimes A}\downarrow & & {\scriptstyle (11)} & & \downarrow{\scriptstyle \iota} \\
B \otimes B \otimes A \otimes A & \xrightarrow{\qquad \mu \otimes \mu \qquad} & & & B \otimes A,
\end{array}
$$

by $B \otimes \alpha \otimes \beta \otimes A$ on the right, we obtain

$$\iota\mu_R(\pi \otimes \pi) = (\mu_B \otimes \mu_A)(B \otimes \Psi \otimes A), \tag{13}$$

hence multiplicativity of π (and ι). Since ι_Ψ is multiplicative by (8), so is $\pi\iota_\Psi$. Finally,

$$\pi\iota_\Psi\eta_\Psi = \pi\iota_\Psi\pi_\Psi(\eta_B \otimes \eta_A) \overset{(12)}{=} \pi(\eta_B \otimes \eta_A) \overset{(10)}{=} \mu_R(\eta_R \otimes \eta_R) = \eta_R. \qquad \blacksquare$$

We close this section by proving that the constructions in Theorem 1.6 and Theorem 1.8 can be regarded as the object maps of a biequivalence between appropriately defined bicategories.

The bicategory of *mixed* weak distributive laws was studied in [5]. Taking the dual notion, we obtain the following.

1.9. The bicategory of weak distributive laws. The 0-cells of the bicategory $\mathsf{Wdl}(\mathcal{K})$ are weak distributive laws $\Psi : A \otimes B \to B \otimes A$ in the bicategory $\mathcal{K}$. The 1-cells between them consist of monad morphisms (in the sense of [15]) $\xi : A' \otimes V \to V \otimes A$ and $\zeta : B' \otimes V \to V \otimes B$ with a common 1-cell V such that the following diagram commutes.

$$
\begin{array}{ccccccc}
A' \otimes B' \otimes V & \xrightarrow{A' \otimes \zeta} & A' \otimes V \otimes B & \xrightarrow{\xi \otimes B} & V \otimes A \otimes B & = & V \otimes A \otimes B \\
{\scriptstyle \Psi' \otimes V}\downarrow & & & & & & \downarrow{\scriptstyle V \otimes \Psi} \\
B' \otimes A' \otimes V & \xrightarrow[B' \otimes \xi]{} & B' \otimes V \otimes A & \xrightarrow[\zeta \otimes A]{} & V \otimes B \otimes A & \xrightarrow[V \otimes \overline{\Psi}]{} & V \otimes B \otimes A
\end{array} \tag{14}
$$

The 2-cells are those 2-cells $\omega : V \to V'$ in $\mathcal{K}$ which are monad transformations (in the sense of [15]) $(V, \xi) \to (V', \xi')$ and $(V, \zeta) \to (V', \zeta')$. Horizontal and vertical compositions are induced by those in $\mathcal{K}$.

1.10. The bicategory of bilinear factorization structures. The 0-cells of the bicategory $\mathsf{Bf}(\mathcal{K})$ are the bilinear factorization structures $(\alpha : A \to R \leftarrow B : \beta, \iota : R \to B \otimes A)$ in the bicategory $\mathcal{K}$. The 1-cells between them are triples of monad morphisms (in the sense of [15]) $\xi : A' \otimes V \to V \otimes A$, $\zeta : B' \otimes V \to V \otimes B$ and $\varrho : R' \otimes V \to V \otimes R$ with a common 1-cell V such that the following diagrams commute.

$$
\begin{array}{ccc}
A' \otimes V & \xrightarrow{\alpha' \otimes V} & R' \otimes V \\
\xi \downarrow & & \downarrow \varrho \\
V \otimes A & \xrightarrow[V \otimes \alpha]{} & V \otimes R
\end{array}
\qquad
\begin{array}{ccc}
B' \otimes V & \xrightarrow{\beta' \otimes V} & R' \otimes V \\
\zeta \downarrow & & \downarrow \varrho \\
V \otimes B & \xrightarrow[V \otimes \beta]{} & V \otimes R
\end{array}
\qquad (15)
$$

The 2-cells are those 2-cells $\omega : V \to V'$ in $\mathcal{K}$ which are monad transformations (in the sense of [15]) $(V, \xi) \to (V', \xi')$, $(V, \zeta) \to (V', \zeta')$ and $(V, \varrho) \to (V', \varrho')$. Horizontal and vertical compositions are induced by those in $\mathcal{K}$.

1.11. A pseudofunctor $F : \mathsf{Bf}(\mathcal{K}) \to \mathsf{Wdl}(\mathcal{K})$. The pseudofunctor F takes a bilinear factorization structure $(\alpha : A \to R \leftarrow B : \beta, \iota : R \to B \otimes A)$ to the corresponding weak distributive law $\Psi := \iota \mu_R (\alpha \otimes \beta) : A \otimes B \to B \otimes A$ in Theorem 1.8. It takes a 1-cell (ξ, ζ, ϱ) to (ξ, ζ). On the 2-cells F acts as the identity map.

The only non-trivial point to see is that (ξ, ζ) is indeed a 1-cell in $\mathsf{Wdl}(\mathcal{K})$ by commutativity of the following diagram.

The middle region commutes since ϱ is a monad morphism. The bottom region commutes by commutativity of

which, in light of (10), means $(V \otimes \pi)(\zeta \otimes A)(B' \otimes \xi) = \varrho(\pi' \otimes V)$.

Theorem 1.12. *If idempotent 2-cells in a bicategory $\mathcal{K}$ split, then the pseudofunctor $F : \mathsf{Bf}(\mathcal{K}) \to \mathsf{Wdl}(\mathcal{K})$ in Paragraph 1.11 is a biequivalence.*

Proof. First of all, F is surjective on the objects. In order to see that, take a weak distributive law $\Psi : A \otimes B \to B \otimes A$ and evaluate F on the associated bilinear factorization structure $(\alpha : A \to B \otimes_\Psi A \leftarrow B : \beta, \iota : B \otimes_\Psi A \to B \otimes A)$ in Theorem 1.6. The resulting weak distributive law occurs in the top-right path of

$$
\begin{array}{ccccc}
A \otimes B & \xrightarrow{\eta \otimes A \otimes B \otimes \eta} & B \otimes A \otimes B \otimes A & \xrightarrow{\pi \otimes \pi} & (B \otimes_\Psi A) \otimes (B \otimes_\Psi A) \\
\Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle B\otimes\Psi\otimes A} \quad (8)(9) & & \Big\downarrow{\scriptstyle\mu_\Psi} \\
& & & & B \otimes_\Psi A \\
& & & & \Big\downarrow{\scriptstyle\iota} \\
B \otimes A & \xrightarrow{\eta \otimes B \otimes A \otimes \eta} & B \otimes B \otimes A \otimes A & \xrightarrow{\mu \otimes \mu} & B \otimes A.
\end{array}
$$

Thus by commutativity of this diagram, it is equal to Ψ.

Next we show that F induces an equivalence of the hom categories. The induced functor of the hom categories is also surjective on the objects. In order to see that, take a 1-cell $(\xi : A' \otimes V \to V \otimes A, \zeta : B' \otimes V \to V \otimes B)$ in $\mathsf{Wdl}(\mathcal{K})$ from the image under F of a bilinear factorization structure $(\alpha : A \to R \leftarrow B : \beta, \iota : R \to B \otimes A)$ to the image of $(\alpha' : A' \to R' \leftarrow B' : \beta', \iota' : R' \to B' \otimes A')$; that is, from the weak distributive law $\Psi := \iota\mu_R(\alpha \otimes \beta)$ to $\Psi' := \iota'\mu_{R'}(\alpha' \otimes \beta')$. We show that together with

$$
\varrho := \left(R' \otimes V \xrightarrow{\iota' \otimes V} B' \otimes A' \otimes V \xrightarrow{B' \otimes \xi} B' \otimes V \otimes A \xrightarrow{\zeta \otimes A} V \otimes B \otimes A \xrightarrow{V \otimes \pi} V \otimes R \right)
$$

they constitute a 1-cell in $\mathsf{Bf}(\mathcal{K})$. Unitality of ϱ follows by commutativity of

$$
\begin{array}{c}
\text{(diagram (10), (6), (14), (12))}
\end{array}
$$

The triangular region commutes by the unitality of the monad morphisms ξ and ζ and the bottom left square commutes by $\Psi'(\eta_{A'} \otimes \eta_{B'}) = \iota'\mu_{R'}(\alpha' \otimes \beta')(\eta_{A'} \otimes \eta_{B'}) = \iota'\mu_{R'}(\eta_{R'} \otimes \eta_{R'}) = \iota'\eta_{R'}$. Multiplicativity of ϱ is checked on page 233. The regions marked by (m) on page 233 commute since ξ and ζ are monad morphisms. This proves that ϱ is a monad morphism.

The first diagram in (15) commutes by commutativity of

The triangular region at the top left commutes by the unitality of ζ. The regions marked by $(*)$ commute by

$$\Psi(A \otimes \eta_B) \overset{(6)}{=} \overline{\Psi}(\eta_B \otimes A) \overset{(12)}{=} \iota\pi(\eta_B \otimes A) \overset{(10)}{=} \iota\alpha.$$

The second diagram in (15) commutes by symmetrical considerations.

The functor induced by F between the hom categories acts on the morphisms as the identity map, hence it is evidently faithful. It is also full since any 2-cell $\omega : (\xi, \zeta) \to (\xi', \zeta')$ in $\mathsf{Wdl}(\mathcal{K})$ is a 2-cell $(\xi, \zeta, \varrho) \to (\xi', \zeta', \varrho')$ in $\mathsf{Bf}(\mathcal{K})$ by commutativity of

The regions in the middle commute since ω is a 2-cell in $\mathsf{Wdl}(\mathcal{K})$. ∎

Remark 1.13. For an arbitrary bicategory $\mathcal{K}$ – not necessarily with split idempotents –, the pseudofunctor F in Paragraph 1.11 induces a biequivalence between $\mathsf{Bf}(\mathcal{K})$ and the full subbicategory of $\mathsf{Wdl}(\mathcal{K})$ whose 0-cells are those weak distributive laws Ψ for which the idempotent 2-cell $\overline{\Psi}$ splits. It induces in particular a biequivalence between the bicategory of distributive laws in $\mathcal{K}$ (as a full subbicategory of $\mathsf{Wdl}(\mathcal{K})$) and the bicategory of strict factorization structures (as a full subbicategory of $\mathsf{Bf}(\mathcal{K})$), cf. [14].

$$R' \otimes R' \otimes V \xrightarrow{R' \otimes \iota' \otimes V} R' \otimes B' \otimes A' \otimes V \xrightarrow{R' \otimes B' \otimes \xi} R' \otimes B' \otimes V \otimes A \xrightarrow{R' \otimes \zeta \otimes A} R' \otimes V \otimes B \otimes A \xrightarrow{R' \otimes V \otimes \pi} R' \otimes V \otimes R$$

$$\downarrow{\iota' \otimes B' \otimes A' \otimes V} \qquad \downarrow{\iota' \otimes V \otimes R}$$

$$B' \otimes A' \otimes B' \otimes A' \otimes V \xrightarrow{B' \otimes A' \otimes B' \otimes \xi} B' \otimes A' \otimes B' \otimes V \otimes A \xrightarrow{B' \otimes A' \otimes \zeta \otimes A} B' \otimes A' \otimes V \otimes B \otimes A \xrightarrow{B' \otimes A' \otimes V \otimes \pi} B' \otimes A' \otimes V \otimes R$$

$$\downarrow{B' \otimes \Psi' \otimes A' \otimes V} \qquad \downarrow{B' \otimes \Psi' \otimes V \otimes A} \qquad \downarrow{B' \otimes \xi \otimes B \otimes A} \qquad \downarrow{B' \otimes \xi \otimes R}$$

$$B' \otimes B' \otimes A' \otimes A' \otimes V \xrightarrow{B' \otimes B' \otimes A' \otimes \xi} B' \otimes B' \otimes A' \otimes V \otimes A \qquad (14) \qquad B' \otimes V \otimes A \otimes B \otimes A = B' \otimes V \otimes A \otimes B \otimes A \xrightarrow{B' \otimes V \otimes A \otimes \pi} B' \otimes V \otimes A \otimes R$$

$$\downarrow{B' \otimes B' \otimes \zeta \otimes A} \qquad \downarrow{B' \otimes V \otimes \Psi \otimes A} \qquad \downarrow{\zeta \otimes A \otimes B \otimes A} \qquad \downarrow{\zeta \otimes A \otimes R}$$

$$B' \otimes B' \otimes \mu' \otimes V \;\text{(m)}\; B' \otimes B' \otimes V \otimes A \otimes A \xrightarrow{B' \otimes \zeta \otimes A \otimes A} B' \otimes V \otimes B \otimes A \otimes A \xrightarrow{B' \otimes V \otimes \overline{\Psi} \otimes A} B' \otimes V \otimes B \otimes A \otimes A \qquad V \otimes B \otimes A \otimes B \otimes A \xrightarrow{V \otimes B \otimes A \otimes \pi} V \otimes B \otimes A \otimes R$$

$$\mu' \otimes V \quad (13) \qquad \downarrow{B' \otimes B' \otimes V \otimes \mu} \qquad \downarrow{B' \otimes V \otimes B \otimes \mu} \;(7)\; \qquad \downarrow{B' \otimes V \otimes B \otimes \mu} \qquad \downarrow{V \otimes B \otimes \Psi \otimes A} \qquad \downarrow{V \otimes \pi \otimes R}$$

$$B' \otimes B' \otimes A' \otimes V \xrightarrow{B' \otimes B' \otimes \xi} B' \otimes B' \otimes V \otimes A \xrightarrow{B' \otimes \zeta \otimes A} B' \otimes V \otimes B \otimes A \xrightarrow{B' \otimes V \otimes \overline{\Psi}} B' \otimes V \otimes B \otimes A \qquad V \otimes B \otimes B \otimes A \otimes A \qquad V \otimes R \otimes R$$

$$\downarrow{\zeta \otimes B \otimes A} \qquad \downarrow{\zeta \otimes B \otimes A} \qquad (13)$$

$$\downarrow{\mu' \otimes A' \otimes V} \qquad \downarrow{\mu' \otimes V \otimes A} \;\text{(m)}\; V \otimes B \otimes B \otimes A \xrightarrow{V \otimes B \otimes \overline{\Psi}} V \otimes B \otimes B \otimes A \qquad \downarrow{V \otimes \mu \otimes \mu} \qquad \downarrow{V \otimes \mu}$$

$$\downarrow{V \otimes \mu \otimes A} \quad (7) \qquad \downarrow{V \otimes \mu \otimes A}$$

$$R' \otimes V \xrightarrow{\iota' \otimes V} B' \otimes A' \otimes V \xrightarrow{B' \otimes \xi} B' \otimes V \otimes A \xrightarrow{\zeta \otimes A} V \otimes B \otimes A \xrightarrow{V \otimes \overline{\Psi}} V \otimes B \otimes A = V \otimes B \otimes A \xrightarrow{V \otimes \pi} V \otimes R$$

$$\underset{V \otimes \pi}{\underbrace{\qquad\qquad\qquad\qquad}} \;(12)$$

1.14. Morphisms with trivial underlying 1-cells. For the algebraists, particularly interesting are those 1-cells in $\mathsf{Bf}(\mathcal{K})$ and $\mathsf{Wdl}(\mathcal{K})$ whose 1-cell part is trivial – these are algebra homomorphisms in the usual sense. Such 1-cells form a subcategory of the respective horizontal category.

In $\mathsf{Bf}(\mathcal{K})$, this means monad morphisms $\varrho : R' \to R$ which restrict to monad morphisms $\xi : A' \to A$ and $\zeta : B' \to B$, i.e. for which $\varrho\alpha' = \alpha\xi$ and $\varrho\beta' = \beta\zeta$.

The corresponding 1-cells in $\mathsf{Wdl}(\mathcal{K})$ are pairs of monad morphisms $\xi : A' \to A$ and $\zeta : B' \to B$ such that $\overline{\Psi}(\zeta \otimes \xi)\Psi' = \Psi(\xi \otimes \zeta)$.

2   Examples: Bilinear factorizations of algebras

The aim of this section is to apply the results in the previous section to the particular monoidal category – i.e. one-object bicategory – of modules over a commutative ring. (Clearly, in this bicategory idempotent 2-cells split.) More precisely, we collect here some examples of associative and unital algebras over a commutative ring k which admit a bilinear factorization. Some of these algebras admit a strict factorization as well but the most interesting ones are those which do not.

2.1. Bilinear factorization via subalgebras. The algebra homomorphisms $\alpha : A \to R \leftarrow B : \beta$, occurring in a bilinear factorization of an algebra R, are not injective in general. In this paragraph we show however that, for any bilinear factorization structure $(\alpha : A \to R \leftarrow B : \beta, \iota : R \to B \otimes A)$, there is another bilinear factorization of R with injective homomorphisms $\tilde{\alpha} : \tilde{A} \to R \leftarrow \tilde{B} : \tilde{\beta}$. We give sufficient and necessary conditions for the latter factorization to be strict.

Consider a weak distributive law $\Psi : A \otimes B \to B \otimes A$, with corresponding algebra homomorphisms $\alpha : A \to B \otimes_\Psi A \leftarrow B : \beta$ obtained by the corestrictions of $\Psi(A \otimes \eta) : A \to B \otimes A \leftarrow B : \Psi(\eta \otimes B)$, cf. Theorem 1.6. Put $\tilde{A} := \operatorname{Im}(\alpha) \subseteq B \otimes_\Psi A \supseteq \operatorname{Im}(\beta) =: \tilde{B}$. Then α factorizes through an epimorphism $A \twoheadrightarrow \tilde{A}$ and a monomorphism $\tilde{\alpha} : \tilde{A} \rightarrowtail B \otimes_\Psi A$ of algebras. Similarly, β factorizes through an epimorphism $B \twoheadrightarrow \tilde{B}$ and a monomorphism $\tilde{\beta} : \tilde{B} \rightarrowtail B \otimes_\Psi A$ of algebras. By Theorem 1.6, $\iota : B \otimes_\Psi A \to B \otimes A$ is a B-A-bimodule section of $\mu_\Psi(\beta \otimes \alpha)$, so that

$$\tilde{\iota} := \left(B \otimes_\Psi A \xrightarrow{\ \iota\ } B \otimes A \longrightarrow \tilde{B} \otimes \tilde{A} \right)$$

is a $\tilde{B}$-$\tilde{A}$-bimodule section of $\mu_\Psi(\tilde{\beta} \otimes \tilde{\alpha})$. Therefore $(\tilde{\alpha} : \tilde{A} \to B \otimes_\Psi A \leftarrow \tilde{B} : \tilde{\beta}, \tilde{\iota} : B \otimes_\Psi A \to \tilde{B} \otimes \tilde{A})$ is a bilinear factorization of $B \otimes_\Psi A$ via subalgebras.

By Theorem 1.8 there is a weak distributive law $\tilde{\Psi} := \tilde{\iota}\mu_\Psi(\tilde{\alpha} \otimes \tilde{\beta})$ such that $\tilde{B} \otimes_{\tilde{\Psi}} \tilde{A}$ is isomorphic to $B \otimes_\Psi A$. The weak distributive law $\tilde{\Psi}$ is a strict distributive law if and only if both unitality conditions $\tilde{\Psi}(\tilde{A} \otimes \tilde{\eta}) = \tilde{\eta} \otimes \tilde{A}$ and $\tilde{\Psi}(\tilde{\eta} \otimes \tilde{B}) = \tilde{B} \otimes \tilde{\eta}$ hold. They amount to commutativity of the following diagrams.

$$
\begin{array}{ccccc}
A & \xrightarrow{\ A\otimes\eta\ } & A\otimes B & \xrightarrow{\ \Psi\ } & B\otimes A \\
{\scriptstyle \eta\otimes A}\downarrow & & & & \downarrow{\scriptstyle \eta\otimes B\otimes A\otimes\eta} \\
& & & & (A\otimes B)^{\otimes 2} \\
& & & & \downarrow{\scriptstyle \Psi^{\otimes 2}} \\
B\otimes A & \xrightarrow[\ \eta\otimes B\otimes A\otimes\eta\ ]{} & (A\otimes B)^{\otimes 2} & \xrightarrow[\ \Psi^{\otimes 2}\ ]{} & (B\otimes A)^{\otimes 2}
\end{array}
$$

$$
\begin{array}{ccccc}
B & \xrightarrow{\ \eta\otimes B\ } & A\otimes B & \xrightarrow{\ \Psi\ } & B\otimes A \\
{\scriptstyle B\otimes\eta}\downarrow & & & & \downarrow{\scriptstyle \eta\otimes B\otimes A\otimes\eta} \\
& & & & (A\otimes B)^{\otimes 2} \\
& & & & \downarrow{\scriptstyle \Psi^{\otimes 2}} \\
B\otimes A & \xrightarrow[\ \eta\otimes B\otimes A\otimes\eta\ ]{} & (A\otimes B)^{\otimes 2} & \xrightarrow[\ \Psi^{\otimes 2}\ ]{} & (B\otimes A)^{\otimes 2}
\end{array}
\tag{16}
$$

2.2. Extension of a distributive law. Let A be an (associative and unital) algebra over a commutative ring k; and let $e \in A$ such that $ea = eae$ for all $a \in A$ (so that in particular $e^2 = e$). Then eA is a subalgebra of A though with a different unit element e.

Assume that $\Phi : eA \otimes B \to B \otimes eA$ is a distributive law. It induces an algebra structure on $B \otimes eA$ with unit $1 \otimes e$ and multiplication $(b' \otimes ea')(b \otimes ea) = b'\Phi(ea' \otimes b)ea = b'\Phi(ea' \otimes b)a$. The maps

$$
\alpha : A \to B \otimes eA, \qquad a \mapsto 1 \otimes ea \qquad \text{and} \qquad \beta : B \to B \otimes eA, \qquad b \mapsto b \otimes e
$$

are clearly algebra homomorphisms inducing the B-A bimodule map

$$
\pi : B \otimes A \to B \otimes eA, \qquad b \otimes a \mapsto b \otimes ea.
$$

Since π possesses a B-A bimodule section $\iota : b \otimes ea \mapsto b \otimes ea$, the datum $(\alpha : A \to B \otimes eA \leftarrow B : \beta, \iota : B \otimes eA \to B \otimes A)$ is a bilinear factorization structure. Hence by Theorem 1.8 there is a weak distributive law

$$
\Psi : A \otimes B \to B \otimes A, \qquad a \otimes b \mapsto \Phi(ea \otimes b)
$$

such that the weak wreath product algebra $B \otimes_\Psi A$ is isomorphic the the strict wreath product $B \otimes_\Phi eA$.

By the above considerations, for any element e of A satisfying $ea = eae$ for all $a \in A$, and for any algebra B, there is a weak distributive law

$$
A \otimes B \to B \otimes A, \qquad a \otimes b \mapsto b \otimes ea
$$

such that the corresponding weak wreath product is the tensor product algebra $B \otimes eA$ with the factorwise multiplication. If B is the trivial k-algebra k, this gives a weak distributive law

$$
A \cong A \otimes k \to k \otimes A \cong A, \qquad a \mapsto ea
$$

and the corresponding weak wreath product algebra eA.

2.3. Weak Ore extension. Recall (e.g. from [11]) that a *quasi-derivation* on an (associative and unital) algebra B over a commutative ring k, consists of a (unital) algebra homomorphism $\sigma : B \to B$ and a k-module map $\delta : B \to B$ such that

$$\delta(bb') = \sigma(b)\delta(b') + \delta(b)b', \qquad \text{for } b, b' \in B.$$

Associated to any quasi-derivation, there is an *Ore extension* $B[X, \sigma, \delta]$ of B. As a k-module it is the tensor product of B with the algebra $k[X]$ of polynomials in a formal variable X, equipped with the B-$k[X]$ bilinear associative and unital multiplication determined by

$$(1 \otimes X)(b \otimes 1) = \sigma(b) \otimes X + \delta(b) \otimes 1, \qquad \text{for } b \in B.$$

Clearly, the Ore extension is a wreath product of B and $k[X]$ with respect to a distributive law defined iteratively, see [7, Example 2.11 (1)]. The following characterization can be found e.g. in [11, Section 1.2]. An algebra T is an Ore extension of B if and only if the following hold.

- T has a subalgebra isomorphic to B;

- there is an element X of T such that the powers of X are linearly independent over B and they span T as a left B-module;

- $XB \subseteq BX + B$.

In what follows, we generalize the notion of a quasi-derivation on B and the corresponding construction of Ore extension of B. The resulting algebra $B[X, \sigma, \delta]$ will be a weak wreath product of B with $k[X]$. However, we also show that it is a proper Ore extension of the image of B in $B[X, \sigma, \delta]$.

Let B be an (associative and unital) algebra over a commutative ring k, and let p and q be elements of B such that

$$p^2 = p, \quad q^2 = 0, \quad pq = q, \quad qp = 0, \quad \text{and} \quad pbp = bp, \quad \text{for all } b \in B.$$

Then by a (p, q)-*quasi-derivation* we mean a couple of k-linear maps $\sigma, \delta : B \to B$ such that the following identities hold for all $b, b' \in B$:

$$\sigma(bb') = \sigma(b)\sigma(b'), \qquad \sigma(1) = \sigma(p) = p, \qquad \sigma(q) = 0,$$
$$\delta(bb') = \sigma(b)\delta(b') + \delta(b)b'p, \qquad \delta(1) = \delta(p) = q, \qquad \delta(q) = 0.$$

So that a $(1, 0)$-quasi-derivation coincides with the classical notion of quasi-derivation recalled above. For example, if B is the algebra of 2×2 upper triangle matrices of entries in k, we may take

$$p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad q := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

$$\delta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

In terms of a (p,q)-quasi-derivation (σ,δ) on an algebra B, define a k-module map $\Psi : k[X] \otimes B \to B \otimes k[X]$ iteratively as

$$\Psi(1 \otimes b) := bq \otimes X + bp \otimes 1,$$
$$\Psi(X \otimes b) := \sigma(b)q \otimes X^2 + (\sigma(b) + \delta(b)q) \otimes X + \delta(b)p \otimes 1,$$
$$\Psi(X^{n+1} \otimes b) := \Psi(X^n \otimes \sigma(b))X + \Psi(X^n \otimes \delta(b)),$$

for $n > 0$ and $b \in B$. By induction in n and m, one easily checks the following properties for all $b, b' \in B$ and $n, m \geq 0$.

- $\Psi(X^n \otimes bp) = \Psi(X^n \otimes b)$ and $\Psi(X^n \otimes bq) = 0$;

- $b\Psi(X^n \otimes 1) = \Psi(1 \otimes b)X^n$,

- $(B \otimes \mu)(\Psi \otimes k[X])(k[X] \otimes \Psi)(X^n \otimes X^m \otimes b) = \Psi(X^{n+m} \otimes b)$,

- $(\mu \otimes k[X])(B \otimes \Psi)(\Psi \otimes B)(X^n \otimes b \otimes b') = \Psi(X^n \otimes bb')$.

That is, Ψ is a weak distributive law and we may regard the corresponding weak wreath product $B \otimes_\Psi k[X]$ as a *weak Ore extension* of B.

Note however, that Ψ renders commutative both diagrams in (16). Hence $B \otimes_\Psi k[X]$ is a strict wreath product of the subalgebras $\tilde{B} = \{b(q \otimes X + p \otimes 1) \mid b \in B\}$ and $\widetilde{k[X]}$, the latter having the set of powers $\{\Psi(X \otimes 1)^n = \Psi(X^n \otimes 1) \mid n \geq 0\}$ as a k-basis. In fact, by the characterization of Ore extensions recalled above, the weak Ore extension $B \otimes_\Psi k[X]$ is isomorphic to an Ore extension of $\tilde{B}$.

2.4. Distributive laws over separable Frobenius algebras. An (associative and unital) algebra R over a commutative ring k is said to possess a *Frobenius* structure if it is a finitely generated and projective k-module and there is an isomorphism of (say) left R-modules from R to $\hat{R} := \mathrm{Hom}(R, k)$. A more categorical characterization is this. Any k-algebra R can be regarded as an R-k bimodule; that is, a 1-cell $k \to R$ in the bicategory Bim of k-algebras, bimodules and bimodule maps. It possesses a right adjoint, the k-R bimodule (i.e. 1-cell $R \to k$) R. Whenever R is a finitely generated and projective k-module, the 1-cell $R : k \to R$ possesses also a left adjoint $\hat{R} : R \to k$ (with right R-action $\varphi \leftharpoonup r = \varphi(r-)$). A Frobenius structure is then an isomorphism between the right adjoint $R : R \to k$ and the left adjoint $\hat{R} : R \to k$. In technical terms, a Frobenius structure is given by an element $\psi \in \hat{R}$ (called a Frobenius functional) and an element $\sum_i e_i \otimes f_i \in R \otimes R$ (called a Frobenius basis) such that $\sum_i \psi(re_i)f_i = r = \sum_i e_i\psi(f_ir)$, for all $r \in R$. Note that a Frobenius algebra R possesses a canonical (Frobenius) coalgebra structure with R-bilinear comultiplication $r \mapsto \sum_i re_i \otimes f_i = \sum_i e_i \otimes f_ir$ and counit ψ. For more on Frobenius algebras we refer e.g. to [1] and [16].

A *separable* structure on a k-algebra R is an R-bilinear section of the multiplication map $R \otimes R \to R$. Categorically, this means a section of the counit of the adjunction $R \dashv R : R \to k$.

Finally, a *separable Frobenius* structure on R is a Frobenius structure $(\psi, \sum_i e_i \otimes f_i)$ such that the multiplication $R \otimes R \to R$ is split by the R-bilinear comultiplication $r \mapsto \sum_i re_i \otimes f_i = \sum_i e_i \otimes f_ir$. In other words, a Frobenius structure

$(\psi, \sum_i e_i \otimes f_i)$ such that $\sum_i e_i f_i = 1_R$. Categorically, the counit of the adjunction $R \dashv R : R \to k$ is split by the unit of the adjunction $\hat{R} \cong R \dashv R : k \to R$.

For a separable Frobenius algebra R, a right R-module M and a left R-module N, the canonical epimorphism

$$\pi : M \otimes_k N \to M \otimes_R N, \qquad m \otimes_k n \mapsto m \otimes_R n$$

is split by

$$\iota : M \otimes_R N \to M \otimes_k N, \qquad m \otimes_R n \mapsto \sum_i m.e_i \otimes_k f_i.n,$$

naturally in M and N. Thus the image of ι is isomorphic to $M \otimes_R N$.

Let R be a k-algebra. A monad A on R in Bim is given by a k-algebra homomorphism $\tilde{\eta} : R \to A$. (Then $\tilde{\eta}$ induces an R-bimodule structure on A; $\tilde{\eta}$ serves as the R-bilinear unit morphism; and the R-bilinear multiplication $\tilde{\mu} : A \otimes_R A \to A$ is the projection of the multiplication $\mu : A \otimes_k A \to A$ of the k-algebra A.) A distributive law in Bim over R is an R-bimodule map $\Phi : A \otimes_R B \to B \otimes_R A$ which is compatible with the units and the multiplications of both R-rings A and B.

Then Φ induces on $B \otimes_R A$ the structure of a monad in Bim over R – that is, an algebra structure $(b' \otimes_R a')(b \otimes_R a) = b'\Phi(a' \otimes_R b)a$ and an algebra homomorphism $\tilde{\eta} \otimes_R \tilde{\eta} : R \to B \otimes_R A$. Moreover,

$$\alpha := \tilde{\eta} \otimes_R A : A \to B \otimes_R A \leftarrow B : B \otimes_R \tilde{\eta} =: \beta$$

are monad morphisms – that is, algebra homomorphisms which are compatible with the homomorphisms $\tilde{\eta}$. Composing $\beta \otimes_k \alpha : B \otimes_k A \to B \otimes_R A \otimes_k B \otimes_R A$ with the multiplication induced by Φ on $B \otimes_R A$, we re-obtain the canonical epimorphism $\pi : B \otimes_k A \to B \otimes_R A$.

Whenever R is a separable Frobenius algebra, π possesses a B-A bimodule section ι above. That is to say, $(\alpha : A \to B \otimes_R A \leftarrow B : \beta, \iota : B \otimes_R A \to B \otimes_k A)$ is a bilinear factorization structure. Hence by Theorem 1.8 there is a weak distributive law of the k-algebra A over B. Explicitly, it comes out as

$$A \otimes_k B \xrightarrow{\ \pi\ } A \otimes_R B \xrightarrow{\ \Phi\ } B \otimes_R A \xrightarrow{\ \iota\ } B \otimes_k A \qquad (17)$$

with corresponding idempotent

$$B \otimes_k A \xrightarrow{\ \pi\ } B \otimes_R A \xrightarrow{\ \iota\ } B \otimes_k A \ .$$

Hence the resulting weak wreath product is isomorphic to the algebra $B \otimes_R A$ with the multiplication induced by Φ.

There is a 1-cell in the bicategory $\mathsf{Wdl}(\mathsf{Bim})$ from the distributive law Φ (on the object R) to the weak distributive law (17) (on the object k) as follows. It is given by the k-R bimodule R and the k-R bimodule maps

$$A \otimes_k R \to R \otimes_R A, \ a \otimes_k r \mapsto 1 \otimes_R a\tilde{\eta}(r) \quad \text{and}$$
$$B \otimes_k R \to R \otimes_R B, \ b \otimes_k r \mapsto 1 \otimes_R b\tilde{\eta}(r).$$

The induced functor $R \otimes_R (-)$ from the category of left R-modules to the category of k-modules lifts to an isomorphism, from the category of left modules over the wreath product R-ring corresponding to Φ, to the category of left modules over the weak wreath product k-algebra corresponding to the weak distributive law (17).

2.5. The direct sum of weak distributive laws. Assume that we have a finite collection $\Phi_i : A_i \otimes B_i \to B_i \otimes A_i$ of distributive laws between algebras over a commutative ring k. Consider the direct sum algebras $A := \oplus_i A_i$ (with multiplication $a_i a'_j = \delta_{i,j} a_i a'_i$ and unit $\sum_i 1_{A_i}$) and $B := \oplus_i B_i$. It is straightforward to see that

$$A \otimes B = \oplus_{i,j}(A_i \otimes B_j) \to \oplus_{i,j}(B_j \otimes A_i) = B \otimes A, \qquad a_i \otimes b_j \mapsto \delta_{i,j} \Phi_i(a_i \otimes b_i) \tag{18}$$

is a weak distributive law.

We claim that it is of the type in Paragraph 2.4. Let R be the algebra $\oplus_i k$ with minimal orthogonal idempotents p_i. Clearly, R is a separable Frobenius algebra via the Frobenius functional $\psi : R \to k$, $p_i \mapsto 1$ and the separable Frobenius basis $\sum_i p_i \otimes p_i \in R \otimes R$. Thus we conclude that $A \otimes_R B$ is isomorphic to $\oplus_i(A_i \otimes B_i)$ and $B \otimes_R A$ is isomorphic to $\oplus_i(B_i \otimes A_i)$. An R-distributive law is given by

$$A \otimes_R B \cong \oplus_i(A_i \otimes B_i) \xrightarrow{\oplus \Phi_i} \oplus_i(B_i \otimes A_i) \cong B \otimes_R A.$$

Applying to it the construction in Paragraph 2.4, we re-obtain the weak distributive law (18).

2.6. Smash product with a weak bialgebra. *Weak bialgebras* are generalizations of bialgebras, see [12] and [6]. A weak bialgebra over a commutative ring k is a k-module H carrying both an (associative and unital) k-algebra structure (μ, η) and a (coassociative and counital) k-coalgebra structure (δ, ε). The comultiplication is required to be multiplicative – equivalently, the multiplication is required to be comultiplicative. However, multiplicativity of the counit, unitality of the comultiplication and unitality of the counit are replaced by the weaker axioms

$$\varepsilon(ab_1)\varepsilon(b_2 c) = \varepsilon(abc) = \varepsilon(ab_2)\varepsilon(b_1 c), \quad \text{for all } a,b,c \in H,$$
$$(\delta(1) \otimes 1)(1 \otimes \delta(1)) = \delta^2(1) = (1 \otimes \delta(1))(\delta(1) \otimes 1),$$

where the usual Sweedler-Heynemann index convention is used for the components of the comultiplication, with implicit summation understood. In particular, we write $\delta(1) = 1_1 \otimes 1_2 = 1_{1'} \otimes 1_{2'}$ – possibly with primed indices if several copies occur.

The category of (say) right modules of a weak bialgebra over k is monoidal though not with the same monoidal structure as the category of k-modules. Indeed, if M and N are right H-modules, then there is a diagonal action $(m \otimes n) \leftharpoonup h := m \leftharpoonup h_1 \otimes n \leftharpoonup h_2$ on the k-module tensor product $M \otimes N$ but it fails to be unital. A unital H-module is obtained by taking the k-module retract

$$M \boxtimes N := \{m \leftharpoonup 1_1 \otimes n \leftharpoonup 1_2 \mid m \in M, n \in N\}.$$

This defines a monoidal product $\boxtimes$ with monoidal unit

$$\{\overline{\sqcap}(h) := \varepsilon(h1_1)1_2 \mid h \in H\},$$

with H-action $\overline{\sqcap}(h) \leftharpoonup h' := \overline{\sqcap}(\overline{\sqcap}(h)h') = \varepsilon(h1_{1'})\varepsilon(1_{2'}h'1_1)1_2 = \varepsilon(hh'1_1)1_2 = \overline{\sqcap}(hh')$. With respect to this monoidal structure, the forgetful functor from the category of right H-modules to the category of k-modules is both monoidal and opmonoidal (hence preserves algebras and coalgebras) but it is not strict monoidal.

A right *module algebra* of a weak bialgebra H is a monoid in the category of right H-modules. That is, a k-algebra A equipped with an (associative and unital) right H-action such that

$$(a \leftharpoonup h_1)(a' \leftharpoonup h_2) = aa' \leftharpoonup h \qquad \text{and} \qquad 1 \leftharpoonup h = 1 \leftharpoonup \overline{\sqcap}(h),$$

for all $a, a' \in A$ and $h \in H$. For any right H-module algebra A, there is a weak distributive law

$$\Psi : A \otimes H \to H \otimes A, \qquad a \otimes h \mapsto h_1 \otimes a \leftharpoonup h_2.$$

It is multiplicative in A by the H-linearity of the multiplication in A:

$$
\begin{aligned}
(H \otimes \mu)(\Psi \otimes A)(A \otimes \Psi)(a' \otimes a \otimes h) &= h_1 \otimes (a' \leftharpoonup h_2)(a \leftharpoonup h_3) \\
&= h_1 \otimes (a'a) \leftharpoonup h_2 \\
&= \Psi(\mu \otimes H)(a' \otimes a \otimes h).
\end{aligned}
$$

Multiplicativity in H follows by multiplicativity of the comultiplication in H:

$$
\begin{aligned}
(\mu \otimes A)(H \otimes \Psi)(\Psi \otimes H)(a \otimes h \otimes h') &= h_1 h_1' \otimes a \leftharpoonup h_2 h_2' = \\
(hh')_1 \otimes a &\leftharpoonup (hh')_2 = \Psi(A \otimes \mu)(a \otimes h \otimes h').
\end{aligned}
$$

In order to check the weak unitality condition, note that for all $a \in A$,

$$
\begin{aligned}
1_1 \otimes a \leftharpoonup 1_2 = 1_1 \otimes (1a) \leftharpoonup 1_2 = 1_1 \otimes (1 \leftharpoonup 1_2)(a \leftharpoonup 1_3) = \\
1_1 \otimes (1 \leftharpoonup 1_2 1_{1'})(a \leftharpoonup 1_{2'}) = 1_1 \otimes (1 \leftharpoonup 1_2)a.
\end{aligned}
$$

Also, for all $h \in H$,

$$
\begin{aligned}
\delta(h1_1) \otimes 1_2 = h_1 1_1 \otimes h_2 1_2 \otimes 1_3 = h_1 1_{1'} \otimes h_2 1_{2'} 1_1 \otimes 1_2 = \\
(h1)_1 \otimes (h1)_2 1_1 \otimes 1_2 = h_1 \otimes h_2 1_1 \otimes 1_2,
\end{aligned}
$$

hence $h1_1 \otimes 1_2 = h_1 \varepsilon(h_2 1_1) \otimes 1_2$. With these identities at hand,

$$
\begin{aligned}
(H \otimes \mu)(\Psi \otimes A)(\eta \otimes H \otimes A)(h \otimes a) &= h_1 \otimes (1 \leftharpoonup h_2)a = h_1 \otimes (1 \leftharpoonup \overline{\sqcap}(h_2))a \\
&= h_1 \varepsilon(h_2 1_1) \otimes (1 \leftharpoonup 1_2)a = h1_1 \otimes a \leftharpoonup 1_2 \\
&= (\mu \otimes A)(H \otimes \Psi)(H \otimes A \otimes \eta)(h \otimes a).
\end{aligned}
$$

The weak wreath product corresponding to Ψ is known as a *weak smash product*, see [13].

In the rest of this paragraph we show that the weak distributive law Ψ above is of the kind discussed in Paragraph 2.4. Let us introduce a further map $\sqcap : H \to H$, $h \mapsto 1_1 \varepsilon(h1_2)$. It is easy to see that for any $h, h' \in H$,

- $\varepsilon \sqcap (h) = \varepsilon(h) = \varepsilon \overline{\sqcap}(h)$;

- $\delta \sqcap (h) = 1_1 \otimes \sqcap(h)1_2$ and $\delta\overline{\sqcap}(h) = 1_1\overline{\sqcap}(h) \otimes 1_2$;

- $\sqcap(\sqcap(h)h') = \sqcap(hh') = \sqcap(\overline{\sqcap}(h)h')$ and $\overline{\sqcap}(\sqcap(h)h') = \overline{\sqcap}(hh') = \overline{\sqcap}(\overline{\sqcap}(h)h')$;

- $\overline{\sqcap}(h) \sqcap (h') = \sqcap(h')\overline{\sqcap}(h)$;

- $\sqcap(h \sqcap (h')) = \sqcap(\sqcap(h) \sqcap (h')) = 1_1\varepsilon(\sqcap(h) \sqcap (h')1_2) = \sqcap(h)_1 \sqcap (h')_1\varepsilon(\sqcap(h)_2 \sqcap (h')_2) = \sqcap(h) \sqcap (h')$ and symmetrically, $\overline{\sqcap}(h\overline{\sqcap}(h')) = \overline{\sqcap}(h)\overline{\sqcap}(h')$.

Note that $\overline{\sqcap}(H)$ possesses a separable Frobenius structure (cf. [18]) with Frobenius functional given by the restriction of ε and Frobenius basis $\overline{\sqcap}(1_2) \otimes \overline{\sqcap}(1_1) = 1_2 \otimes \overline{\sqcap}(1_1)$ (where the equality follows by $1_1 \otimes \overline{\sqcap}(1_2) = 1_1 \otimes \varepsilon(1_2 1_{1'})1_{2'} = 1_1 \otimes \varepsilon(1_2)1_3 = 1_1 \otimes 1_2$):

$$1_2\varepsilon(\overline{\sqcap}(1_1)\overline{\sqcap}(h)) = \varepsilon(1_1\overline{\sqcap}(h))1_2 = \varepsilon(\overline{\sqcap}(h)_1)\overline{\sqcap}(h)_2 =$$
$$\overline{\sqcap}(h) = \overline{\sqcap} \sqcap \overline{\sqcap}(h) = \varepsilon(\overline{\sqcap}(h)1_2)\overline{\sqcap}(1_1).$$

Hence also the opposite algebra $R := \overline{\sqcap}(H)^{op}$ has a separable Frobenius structure with the same Frobenius functional ε and Frobenius basis $\overline{\sqcap}(1_1) \otimes 1_2$. Moreover,

$$\sqcap (\overline{\sqcap}(h)\overline{\sqcap}(h')) = \sqcap(\sqcap(h)\overline{\sqcap}(h')) = \sqcap(\overline{\sqcap}(h') \sqcap (h)) =$$
$$\sqcap (h' \sqcap (h)) = \sqcap(h') \sqcap (h) = \sqcap\overline{\sqcap}(h') \sqcap \overline{\sqcap}(h).$$

That is, the restriction of $\sqcap$ yields an algebra homomorphism $R \to H$. There is an algebra homomorphism $R \to A$, $r \mapsto 1 \leftharpoonup r$ as well. They induce R-actions on A and H. By $\overline{\sqcap}\sqcap = \overline{\sqcap}$ we conclude that, for all $h \in H$, $1 \leftharpoonup \overline{\sqcap}(h) = 1 \leftharpoonup h = 1 \leftharpoonup \sqcap(h)$ and thus

$$a \leftharpoonup \sqcap(h) = (a1) \leftharpoonup \sqcap(h) = (a \leftharpoonup 1_1)(1 \leftharpoonup \sqcap(h)1_2) =$$
$$a(1 \leftharpoonup \sqcap(h)) = a(1 \leftharpoonup \overline{\sqcap}(h)).$$

Consequently,

$$\Psi(a(1 \leftharpoonup \overline{\sqcap}(h')) \otimes h) = \Psi(a \leftharpoonup \sqcap(h') \otimes h) = h_1 \otimes a \leftharpoonup \sqcap(h')h_2 =$$
$$(\sqcap(h')h)_1 \otimes a \leftharpoonup (\sqcap(h')h)_2 = \Psi(a \otimes \sqcap(h')h).$$

This means that Ψ projects to an R-distributive law

$$A \otimes_R H \to H \otimes_R A, \qquad a \otimes_R h \mapsto h_1 \otimes_R a \leftharpoonup h_2.$$

It is evidently a morphism of right R-modules. It is also a morphism of left R-modules as

$$h_1 \otimes ((1 \leftharpoonup \overline{\sqcap}(h'))a) \leftharpoonup h_2 = h_1 \otimes a \leftharpoonup \overline{\sqcap}(h')h_2 = \sqcap\overline{\sqcap}(h')h_1 \otimes a \leftharpoonup h_2,$$

where the last equality follows by $\sqcap\overline{\sqcap}(1_1) \otimes 1_2 = \sqcap(1_1) \otimes 1_2 = 1_1 \otimes 1_2$, and $1_2 \otimes \overline{\sqcap}(1_1)$ being a separability element for R:

$$h_1 \otimes \overline{\sqcap}(h')h_2 = 1_1h_1 \otimes \overline{\sqcap}(h')1_2h_2 = \sqcap\overline{\sqcap}(1_1)h_1 \otimes \overline{\sqcap}(h')1_2h_2$$
$$= \sqcap(\overline{\sqcap}(1_1)\overline{\sqcap}(h'))h_1 \otimes 1_2h_2 = \sqcap\overline{\sqcap}(h') \sqcap \overline{\sqcap}(1_1)h_1 \otimes 1_2h_2 =$$
$$= \sqcap\overline{\sqcap}(h')h_1 \otimes h_2.$$

Multiplicativity in both arguments is obvious. Unitality follows by

$$1_1 \otimes_R a \leftharpoonup 1_2 = 1_1 \otimes_R a \leftharpoonup \overline{\sqcap}(1_2) = 1_1 \otimes_R (1 \leftharpoonup \overline{\sqcap}(1_2))a = 1_1 \sqcap (1_2) \otimes_R a = 1 \otimes_R a$$

and

$$h_1 \otimes_R 1 \leftharpoonup h_2 = h_1 \otimes_R 1 \leftharpoonup \overline{\sqcap}(h_2) = h_1 \sqcap (h_2) \otimes_R 1 = h \otimes_R 1.$$

Applying the construction in Paragraph 2.4 to this R-distributive law, it yields a weak distributive law $A \otimes H \to H \otimes A$,

$$a \otimes h \mapsto h_1 \sqcap \overline{\sqcap}(1_1) \otimes (1 \leftharpoonup \overline{\sqcap}(1_2))(a \leftharpoonup h_2) = h_1 \sqcap (1_1) \otimes a \leftharpoonup h_2 \overline{\sqcap}(1_2).$$

Since $\sqcap(1_1) \otimes 1_2 = 1_1 \otimes 1_2 = 1_1 \otimes \overline{\sqcap}(1_2)$, this is equal to Ψ.

2.7. 2x2 = 3. In this paragraph we present a bilinear factorization of the algebra T of 2×2 upper triangle matrices over a field k of characteristic different from 2, in terms of two copies of the group algebra $k\mathbb{Z}_2$ of the order 2 cyclic group. So the attitudinizing title refers to the vector space dimensions: we obtain a 3 dimensional non-commutative algebra as a weak wreath product of two 2 dimensional commutative algebras. Note that starting from 2 dimensional algebras A and B, none of the constructions in the previous examples of the section would result in a 3 dimensional weak wreath product algebra. Hence the current example does not belong to any of the previously discussed classes.

A k-linear basis of T is given by

$$1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad b := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These basis elements satisfy $ab = a + b - 1$ and $ba = -(a + b + 1)$. Denote the second order generator of the cyclic group $\mathbb{Z}_2$ by g and consider the following algebra homomorphisms.

$$\alpha : k\mathbb{Z}_2 \to T, \qquad g \mapsto a \qquad \text{and} \qquad \beta : k\mathbb{Z}_2 \to T, \qquad g \mapsto b.$$

In terms of α and β, we put

$$\pi := \left(k\mathbb{Z}_2 \otimes k\mathbb{Z}_2 \xrightarrow{\beta \otimes \alpha} T \otimes T \xrightarrow{\mu} T \right),$$

with values

$$\pi(1 \otimes 1) = 1, \quad \pi(1 \otimes g) = a, \quad \pi(g \otimes 1) = b, \quad \pi(g \otimes g) = ba = -(a + b + 1).$$

It is straightforward to check that π has a section $\iota : T \to k\mathbb{Z}_2 \otimes k\mathbb{Z}_2$ with values

$$\iota(1) = \frac{1}{4}(3 \cdot 1 \otimes 1 - 1 \otimes g - g \otimes 1 - g \otimes g),$$

$$\iota(a) = \frac{1}{4}(-1 \otimes 1 + 3 \cdot 1 \otimes g - g \otimes 1 - g \otimes g),$$

$$\iota(b) = \frac{1}{4}(-1 \otimes 1 - 1 \otimes g + 3 \cdot g \otimes 1 - g \otimes g),$$

which is a homomorphism of $k\mathbb{Z}_2$-bimodules, with respect to the action induced by β on the first factor and the action induced by α on the second factor. This shows that T has a bilinear factorization in terms of the algebra homomorphisms α and β.

By Theorem 1.8 there is a corresponding weak distributive law

$$\Psi := \left(k\mathbb{Z}_2 \otimes k\mathbb{Z}_2 \xrightarrow{\alpha \otimes \beta} T \otimes T \xrightarrow{\mu} T \xrightarrow{\iota} k\mathbb{Z}_2 \otimes k\mathbb{Z}_2 \right),$$

with values

$$\Psi(1 \otimes 1) = \frac{1}{4}(3 \cdot 1 \otimes 1 - 1 \otimes g - g \otimes 1 - g \otimes g),$$

$$\Psi(1 \otimes g) = \frac{1}{4}(-1 \otimes 1 - 1 \otimes g + 3 \cdot g \otimes 1 - g \otimes g),$$

$$\Psi(g \otimes 1) = \frac{1}{4}(-1 \otimes 1 + 3 \cdot 1 \otimes g - g \otimes 1 - g \otimes g),$$

$$\Psi(g \otimes g) = \frac{1}{4}(-5 \cdot 1 \otimes 1 + 3 \cdot 1 \otimes g + 3 \cdot g \otimes 1 - g \otimes g),$$

such that $k\mathbb{Z}_2 \otimes_\Psi k\mathbb{Z}_2 \cong T$.

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