Infinitely many homoclinic solutions for the second-order discrete p-Laplacian systems *

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Abstract

By using the Symmetric Mountain Pass Theorem, we establish some existence criteria to guarantee the second-order discrete p-Laplacian systems $\triangle(\varphi_p(\Delta u(n-1))) - a(n)|u(n)|^{p-2}u(n) + \nabla W(n, u(n)) = 0$ has infinitely many homoclinic orbits, where $p > 1$, $n \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $a : \mathbb{Z} \to \mathbb{R}$ and $W : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$ are not periodic in n. Our conditions on the nonlinear term $W(n, u(n))$ are rather relaxed and we generalize some existing results in the literature.

1. Introduction

Consider the second-order discrete p-Laplacian system

$$\triangle(\varphi_p(\Delta u(n-1))) - a(n)|u(n)|^{p-2}u(n) + \nabla W(n, u(n)) = 0, \tag{1.1}$$

where $p > 1, \varphi_p(s) = |s|^{p-2}s$, $n \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $a : \mathbb{Z} \to \mathbb{R}$ and $W : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$, Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) -$

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$u(n),\ \Delta^2 u(n) = \Delta(\Delta u(n))$. As usual, we say that a solution $u(n)$ of (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $\to \pm\infty$. In addition, if $u(n) \not\equiv 0$ then $u(n)$ is called a nontrivial homoclinic solution.

In general, system (1.1) may be regarded as a discrete analogue of the following second order Hamiltonian system

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t,u(t)) = 0,\ t \in \mathbb{R}, u \in \mathbb{R}^N. \quad (1.2)$$

When $p = 2$, system (1.2) reduces to second-order Hamiltonian system

$$\ddot{u}(t) - a(t)u(t) + \nabla W(t,u(t)) = 0. \quad (1.3)$$

In recent years, the existence and multiplicity of homoclinic orbits for Hamiltonian systems (1.2) have been investigated in many papers via variational methods and many results were obtained based on various hypotheses on the potential functions, see, e.g., [3-6, 8-11, 13, 14, 17-19, 26-36].

In some recent papers [7, 12, 15-17, 21, 22], the authors studied the existence of periodic solutions and homoclinic solutions of some special forms of (1.1) by using the critical point theory. These papers show that the critical point method is an effective approach to the study of periodic solutions for difference equations. Ma and Guo [20] (with periodicity assumption) and [21] (without periodicity assumption) applied the critical point theory to prove the existence of homoclinic solutions of the following special form of (1.1) (with $N = 1$)

$$\triangle[p(n)\triangle u(n-1)] - q(n)u(n) + f(n,u(n)) = 0, \quad (1.4)$$

where $n \in \mathbb{Z},\ u \in \mathbb{R}, p,\ q : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$.

Using the original ideas of Omana and Willem [26], Ma and Guo [21] used mountain pass theorems and compact imbedding lemma to prove following theorem.

Theorem A[21]**.** *Assume that $p,\ q$ and f satisfy the following conditions:*

(p) $p(n) > 0$ *for all* $n \in \mathbb{Z}$;

(q) $q(n) > 0$ *for all* $n \in \mathbb{Z}$ *and* $\lim_{|n|\to+\infty} q(n) = +\infty$;

(f1) $f \in C(\mathbb{Z} \times \mathbb{R}, \mathbb{R})$ *and there is a constant* $\mu > 2$ *such that*

$$0 < \mu \int_0^x f(n,s)ds \le xf(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times (\mathbb{R} \setminus \{0\});$$

(f2) $\lim_{x\to 0} f(n,x)/x = 0$ *uniformly with respect to* $n \in \mathbb{Z}$.

(f3) $f(n,-x) = -f(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}$.

Then there exists an unbounded sequence of homoclinic solutions for (1.4).

In the last decade there has been an increasing interest in the study of ordinary differential systems driven by the p-Laplacian (or the generalization of Laplacian) and with periodic boundary conditions, see [2, 7, 23, 24, 35, 37] and the references cited therein. However, as the authors are aware, there are few papers discussing the existence of homoclinic solutions for the discrete p-Laplacian systems. In the present paper, we are interested in the existence of homoclinic solutions for system (1.1), where $a(n)$ and $W(n,x)$ are non periodic in n. The intention of this paper is that, under some relaxed assumptions on $W(n,x)$, we establish some existence criteria to guarantee that system (1.1) has infinitely many homoclinic solutions by using the Symmetric Mountain Pass Theorem. In particular, when $p = 2$, our results generalize Theorems A by relaxing condition (f1) and (f2).

When $W(n,x)$ is an even function on x, however, generalize or improve Theorem A by using the Symmetric Mountain Pass Theorem, there has not been much work done up to now, because it is often very difficult to verify the last condition of the Symmetric Mountain Pass Theorem, different from the Mountain Pass Theorem.

Motivated by the above papers, we will obtain some new criteria for guaranteeing that (1.1) has infinitely many homoclinic orbits without any periodicity and generalize Theorem A. Especially, $W(n,x)$ satisfies a kind of new superquadratic condition which is different from the corresponding condition in the known literature.

Our main results are the following theorems.

Theorem 1.1. *Assume that a and W satisfy the following assumptions:*

(A) $a(n) \to +\infty$ *as* $|n| \to \infty$;

(W1) $W(n,x)$ *is continuously differentiable in* x, *and*

$$\frac{1}{a(n)}|\nabla W(n,x)| = o(|x|^{p-1}) \quad \text{as } x \to 0$$

uniformly in $n \in \mathbb{Z}$;

(W2) *For any* $r > 0$, *there exist* $b = b(r)$, $c = c(r) > 0$ *and* $\nu < p$ *such that*

$$0 \le \left(p + \frac{1}{b + c|x|^\nu}\right) W(n,x) \le (\nabla W(n,x), x), \quad \forall\, (n,x) \in \mathbb{Z} \times \mathbb{R}^N, \ |x| \ge r;$$

(W3) *For any* $n \in \mathbb{Z}$

$$\lim_{s \to +\infty} \left[s^{-p} \min_{|x|=1} W(n, sx) \right] = +\infty;$$

(W4) $W(n,-x) = W(n,x), \quad \forall\, (n,x) \in \mathbb{Z} \times \mathbb{R}^N$.

Then there exists an unbounded sequence of homoclinic solutions for system (1.1).

Theorem 1.2. *Assume that a and W satisfy (A), (W4) and the following assumptions:*

(W1′) $W(n,x) = W_1(n,x) - W_2(n,x)$, W_1 *and* W_2 *are continuously differentiable in* x, *and*

$$\frac{1}{a(n)}|\nabla W(n,x)| = o(|x|^{p-1}) \quad \text{as } x \to 0$$

uniformly in $n \in \mathbb{Z}$;

(W5) *There is a constant* $\mu > p$ *such that*

$$0 < \mu W_1(n,x) \le (\nabla W_1(n,x), x), \quad \forall\, (n,x) \in \mathbb{Z} \times \mathbb{R}^N \setminus \{0\};$$

(W6) $W_2(n,0) \equiv 0$ *and there is a constant* $\varrho \in [p,\mu)$ *such that*

$$W_2(n,x) \ge 0, \quad (\nabla W_2(n,x), x) \le \varrho W_2(n,x), \quad \forall\, (n,x) \in \mathbb{Z} \times \mathbb{R}^N.$$

Then there exists an unbounded sequence of homoclinic solutions for system (1.1).

Theorem 1.3. *Assume that a and W satisfy (A), (W4) and (W5) and the following assumptions:*

(W1″) $W(n,x) = W_1(n,x) - W_2(n,x)$, W_1 *and* W_2 *are continuously differentiable in x, and there is a*
bounded set $J \subset \mathbb{Z}$ *such that*

$$\frac{1}{a(n)} |\nabla W(n,x)| = o(|x|^{p-1}) \quad \text{as } x \to 0$$

uniformly in $n \in \mathbb{Z} \setminus J$;

(W6′) $W_2(n,0) \equiv 0$ *and there is a constant* $\varrho \in (p,\mu)$ *such that*

$$(\nabla W_2(n,x), x) \le \varrho W_2(n,x), \quad \forall\, (n,x) \in \mathbb{Z} \times \mathbb{R}^N.$$

Then there exists an unbounded sequence of homoclinic solutions for system (1.1).

Remark 1.1. *If assumption (AR) holds, that is to say, there exists a constant* $\mu > p$ *such that*

$$0 < \mu W(n,x) \le (\nabla W(n,x), x), \quad \forall\, (n,x) \in \mathbb{Z} \times \mathbb{R}^N \setminus \{0\}.$$

Then (W2) also holds by choosing $b > 1/(\mu - p)$, $c > 0$ *and* $\nu \in (0,p)$. *In addition, by (AR), we have*

$$W(n,sx) \ge s^{\mu} W(n,x) \quad \text{for } (n,x) \in \mathbb{Z} \times \mathbb{R}^N, \ s \ge 1.$$

It follows that for any $n \in \mathbb{Z}$

$$s^{-p} \min_{|x|=1} W(n,sx) \ge s^{\mu-p} \min_{|x|=1} W(n,x) \to +\infty, \quad s \to +\infty.$$

This shows that (AR) implies (W3). Therefore, Theorem 1.1 also generalize Theorem A by relaxing conditions (f1) and (f2).

Remark 1.2. *Obviously, both conditions (W1), (W1′) and (W1″) are weaker than (f2) even if* $N = 1$. *Therefore, both Theorem 1.2 and Theorem 1.3 generalize Theorem A by relaxing conditions (f1) and (f2).*

The rest of the this paper is organized as follows: in Section 2, we introduce some notations and preliminary results. In Section 3, we complete the proofs of Theorems 1.1-1.3. In Section 4, we give some examples to to illustrate our results.

Throughout this paper, we let $q \in (1,\infty)$ such that $1/p + 1/q = 1$.

2. Preliminaries

Let

$$S = \left\{ \{u(n)\}_{n\in\mathbb{Z}} \ : \ u(n) \in \mathbb{R}^N,\ n \in \mathbb{Z} \right\},$$

$$E = \left\{ u \in S \ : \sum_{n\in\mathbb{Z}} [|\Delta u(n-1)|^p + a(n)|u(n)|^p] < +\infty \right\}$$

and for $u \in E$, let

$$\|u\| = \left\{ \sum_{n\in\mathbb{Z}} [|\Delta u(n-1)|^p + a(n)|u(n)|^p] \right\}^{1/p}. \tag{2.1}$$

Then E is a uniform convex Banach space with this norm, then E is a reflexive Banach space with this norm.

As usual, let

$$l^p(\mathbb{Z}, \mathbb{R}) = \left\{ u \in S \ : \ \sum_{n\in\mathbb{Z}} |u(n)|^p < +\infty \right\},$$

and

$$l^\infty(\mathbb{Z}, \mathbb{R}) = \left\{ u \in S \ : \ \sup_{n\in\mathbb{Z}} |u(n)| < +\infty \right\},$$

and their norms are defined by

$$\|u\|_p = \left(\sum_{n\in\mathbb{Z}} |u(n)|^p \right)^{1/p}, \ \ \forall\, u \in l^p(\mathbb{Z}, \mathbb{R}); \quad \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)|, \ \ \forall\, u \in l^\infty(\mathbb{Z}, \mathbb{R}),$$

respectively.

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{p}\|u\|^p - \sum_{n\in\mathbb{Z}} W(n, u(n)). \tag{2.2}$$

If (A) and (W1), (W1′) or (W1″) hold, then $I \in C^1(E, \mathbb{R})$ and one can easily check that

$$\begin{aligned} \langle I'(u), v \rangle \ &= \ \sum_{n\in\mathbb{Z}} \Big[|\Delta u(n-1)|^{p-2} (\Delta u(n-1), \Delta v(n-1)) \\ &\quad + a(n)|u(n)|^{p-2}(u(n), v(n)) - (\nabla W(n, u(n)), v(n)) \Big]. \end{aligned} \tag{2.3}$$

Furthermore, the critical points of I in E are classical solutions of (1.1) with $u(\pm\infty) = 0$.

We will obtain the critical points of I by using the Symmetric Mountain Pass Theorem. We recall it and a minimization theorem as:

Lemma 2.1.[31] *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy (PS)-condition. Suppose that I satisfies the following conditions:*

(i) $I(0) = 0$;

(ii) There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;

(iii) For each finite dimensional subspace $E' \subset E$, there is $r = r(E') > 0$ such that $I(u) \leq 0$ for $u \in E' \setminus B_r(0)$, where $B_r(0)$ is an open ball in E of radius r centered at 0.

Then I possesses an unbounded sequence of critical values.

Remark 2.1. *A deformation lemma can be proved with condition (C) replacing the usual (PS)-condition, and it turns out that Lemma 2.1 hold true under condition (C). We say I satisfies condition (C), i.e., for every sequence $\{u_k\} \subset E$, $\{u_k\}$ has a convergent subsequence if $I(u_k)$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\| \to 0$ as $k \to \infty$.*

Lemma 2.2. *For $u \in E$*

$$\|u\|_\infty \leq a^{-1/p}\|u\| = \lambda\|u\|, \tag{2.4}$$

where $a = \inf_{n\in\mathbb{Z}} a(n)$, $\lambda = a^{-1/p}$.

Proof. Since $u \in E$, it follows that $\lim_{|n|\to\infty} = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that

$$\|u\|_\infty = |u(n^*)| = \max_{n\in\mathbb{Z}} |u(n)|.$$

By (2.1), we have

$$\|u\|^p \geq \sum_{n\in\mathbb{Z}} a(n)|u(n)|^p \geq a \sum_{n\in\mathbb{Z}} |u(n)|^p \geq a\|u\|_\infty^p = a|u(n^*)|^p. \tag{2.5}$$

It follows from (2.1) and (2.5) that (2.4) holds.

Lemma 2.3. *Assume that (W5) and (W6) or (W6') hold. Then for every $(n, x) \in \mathbb{Z} \times \mathbb{R}^N$,*

(i) *$s^{-\mu}W_1(n, sx)$ is nondecreasing on $(0, +\infty)$;*

(ii) *$s^{-\varrho}W_2(n, sx)$ is nonincreasing on $(0, +\infty)$.*

The proof of Lemma 2.3 is routine and so we omit it.

3. Proof of theorems

Proof of Theorem 1.1. We first show that I satisfies condition (C). Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a (C) sequence of I, that is, $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\| \to 0$ as $k \to +\infty$. Then it follows from (2.2) and (2.3) that

$$\begin{aligned} C_1 &\geq pI(u_k) - \langle I'(u_k), u_k\rangle \\ &= \sum_{n\in\mathbb{Z}} \left[(\nabla W(n, u_k(n)), u_k(n)) - pW(n, u_k(n))\right]. \end{aligned} \tag{3.1}$$

By (W1), there exists $\eta \in (0, 1)$ such that

$$|\nabla W(n, x)| \leq \frac{1}{2}a(n)|x|^{p-1} \quad \text{for } n \in \mathbb{Z}, \ |x| \leq \eta. \tag{3.2}$$

Since $W(n,0)=0$, it follows that

$$|W(n,x)| \le \frac{1}{2p}a(n)|x|^p \quad \text{for } n\in\mathbb{Z},\ |x|\le\eta. \tag{3.3}$$

By (W2), we have

$$(\nabla W(n,x),x) \ge pW(n,x) \ge 0 \quad \text{for } (n,x)\in\mathbb{Z}\times\mathbb{R}^N,\ k\in\mathbb{N}, \tag{3.4}$$

and

$$W(n,x) \le (b+c|x|^\nu)[(\nabla W(n,x),x) - pW(n,x)] \quad \text{for } (n,x)\in\mathbb{Z}\times\mathbb{R}^N,\ |x|\ge\eta. \tag{3.5}$$

It follows from (2.2), (2.4), (3.1), (3.2), (3.3), (3.4) and (3.5) that

$$\begin{aligned}
\frac{1}{p}\|u_k\|^p &= I(u_k) + \sum_{n\in\mathbb{Z}} W(n,u_k(n)) \\
&= I(u_k) + \sum_{n\in\mathbb{Z}(|u_k(n)|\le\eta)} W(n,u_k(n)) + \sum_{n\in\mathbb{Z}(|u_k(n)|>\eta)} W(n,u_k(n)) \\
&\le I(u_k) + \frac{1}{2p}\sum_{n\in\mathbb{Z}(|u_k(n)|\le\eta)} a(n)|u_k(n)|^p \\
&\quad + \sum_{n\in\mathbb{Z}(|u_k(n)|>\eta)} (b+c|u_k(n)|^\nu)[(\nabla W(n,u_k(n)),u_k(n)) \\
&\qquad\qquad - pW(n,u_k(n))] \\
&\le C_2 + \frac{1}{2p}\|u_k\|^p + \sum_{n\in\mathbb{Z}} (b+c|u_k(n)|^\nu)[(\nabla W(n,u_k(n)),u_k(n)) \\
&\qquad\qquad - pW(n,u_k(n))] \\
&\le C_2 + \frac{1}{2p}\|u_k\|^p + (b+c\|u_k\|_\infty^\nu)\sum_{n\in\mathbb{Z}} [(\nabla W(n,u_k(n)),u_k(n)) \\
&\qquad\qquad - pW(n,u_k(n))] \\
&\le C_2 + \frac{1}{2p}\|u_k\|^p + C_1(b+c\|u_k\|_\infty^\nu) \\
&\le C_2 + \frac{1}{2p}\|u_k\|^p + C_1\{b+\lambda^\nu c\|u_k\|^\nu\},\ k\in\mathbb{N}.
\end{aligned} \tag{3.6}$$

Since $\nu < p$, it follows that there exists a constant $A>0$ such that

$$\|u_k\| \le A \quad \text{for } k\in\mathbb{N}. \tag{3.7}$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. For any given number $\varepsilon>0$, by (W1), we can choose $\xi>0$ such that

$$|\nabla W(n,x)| \le \varepsilon a(n)|x|^{p-1} \quad \text{for } n\in\mathbb{Z}, \text{ and } |x|\le\xi. \tag{3.8}$$

Since $a(n)\to\infty$, we can also choose an integer $\Pi>0$ such that

$$a(n) \ge \frac{A^p}{\xi^p}, \quad |n|\ge\Pi. \tag{3.9}$$

By (2.1), (3.8) and (3.9), we have

$$\begin{aligned} |u_k(n)|^p &= \frac{1}{a(n)} a(n)|u_k(n)|^p \\ &\leq \frac{\xi^p}{A^p} \sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^p \\ &\leq \frac{\xi^p}{A^p} \|u_k\|^p \\ &\leq \xi^p \quad \text{for } |n| \geq \Pi, \ \ k \in \mathbb{N}. \end{aligned} \tag{3.10}$$

Since $u_k \rightharpoonup u_0$ in E, it is easy to verify that $u_k(t)$ converges to $u_0(t)$ pointwise for all $n \in \mathbb{Z}$, that is

$$\lim_{k\to\infty} u_k(n) = u_0(n), \quad \forall\, n \in \mathbb{Z}. \tag{3.11}$$

Hence, we have by (3.10) and (3.11)

$$|u_0(n)| \leq \zeta \quad \text{for } |n| \geq \Pi. \tag{3.12}$$

It follows from (3.11) and the continuity of $\nabla W(n,x)$ on x that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=-\Pi}^{\Pi} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))||u_k(n) - u_0(n)| < \varepsilon \quad \text{for } k \geq k_0. \tag{3.13}$$

On the other hand, it follows from (3.2), (3.3), (3.5), (3.6), (3.7) and (3.8) that

$$\begin{aligned} &\sum_{|n|>\Pi} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))||u_k(nt) - u_0(n)| \\ &\quad\leq \sum_{|n|>\Pi} (|\nabla W(n,u_k(n))| + |\nabla W(n,u_0(n))|)(|u_k(n)| + |u_0(n)|) \\ &\quad\leq \varepsilon \sum_{|n|>\Pi} a(n)(|u_k(n)|^{p-1} + |u_0(n)|^{p-1})(|u_k(n)| + |u_0(n)|) \\ &\quad\leq 2\varepsilon \sum_{|n|>\Pi} a(n)(|u_k(n)|^p + |u_0(n)|^p) \\ &\quad\leq 2\varepsilon(\|u_k\|^p + \|u_0\|^p) \\ &\quad\leq 2\varepsilon(A^p + \|u_0\|^p), \quad k \in \mathbb{N}. \end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14) we get

$$\sum_{n\in\mathbb{Z}} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))|\, |u_k(n) - u_0(n)| \to 0 \quad \text{as } k \to \infty. \tag{3.15}$$

Using the Hölder's inequality

$$ac + bd \leq (a^p + b^p)^{1/p}(c^q + d^q)^{1/q},$$

where a, b, c, d are nonnegative numbers and $1/p + 1/q = 1, p > 1$, it follows from (2.3) that

$$\begin{aligned}
&\langle I'(u_k) - I'(u_0), u_k - u_0\rangle \\
&\quad = \sum_{n\in\mathbb{Z}} |\Delta u_k(n-1)|^{p-2}(\Delta u_k(n-1), \Delta u_k(n-1) - \Delta u_0(n-1)) \\
&\qquad + \sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^{p-2}(u_k(n), u_k(n) - u_0(n)) \\
&\qquad - \sum_{n\in\mathbb{Z}} |\Delta u_0(n-1)|^{p-2}(\Delta u_0(n-1), \Delta u_k(n-1) - \Delta u_0(n-1)) \\
&\qquad - \sum_{n\in\mathbb{Z}} a(n)|u_0(n)|^{p-2}(u_0(n), u_k(n) - u_0(n)) \\
&\qquad - \sum_{n\in\mathbb{Z}} (\nabla W(n, u_k(n)) - \nabla W(n, u_0(n)), u_k(n) - u_0(n)) \\
&\quad = \|u_k\|^p + \|u_0\|^p - \sum_{n\in\mathbb{Z}} |\Delta u_k(n-1)|^{p-2}(\Delta u_k(n-1), \Delta u_0(n-1)) \\
&\qquad - \sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^{p-2}(u_k(n), u_0(n)) \\
&\qquad - \sum_{n\in\mathbb{Z}} |\Delta u_0(n-1)|^{p-2}(\Delta u_0(n-1), \Delta u_k(n-1)) \\
&\qquad - \sum_{n\in\mathbb{Z}} a(n)|u_0(n)|^{p-2}(u_0(n), u_k(n)) \\
&\qquad - \sum_{n\in\mathbb{Z}} (\nabla W(n, u_k(n)) - \nabla W(n, u_0(n)), u_k(n) - u_0(n)) \\
&\quad \geq \|u_n\|^p + \|u_0\|^p - \left(\sum_{n\in\mathbb{Z}} |\Delta u_0(n-1)|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} |\Delta u_k(n-1)|^p\right)^{1/q} \\
&\qquad - \left(\sum_{n\in\mathbb{Z}} a(n)|u_0(n)|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^p\right)^{1/q} \\
&\qquad - \left(\sum_{n\in\mathbb{Z}} |\Delta u_k(n-1)|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} |\Delta u_0(n-1)|^p\right)^{1/q} \\
&\qquad - \left(\sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} a(n)|u_0(n)|^p\right)^{1/q} \\
&\qquad - \sum_{n\in\mathbb{Z}} (\nabla W(n, u_k(n)) - \nabla W(n, u_0(n)), u_k(n) - u_0(n)) \\
&\quad \geq \|u_k\|^p + \|u_0\|^p \\
&\qquad - \left(\sum_{n\in\mathbb{Z}} [|\Delta u_0(n-1)|^p + a(n)|u_0(n)|^p]\right)^{1/p} \\
&\qquad\qquad \left(\sum_{n\in\mathbb{Z}} [|\Delta u_k(n-1)|^p + a(n)|u_k(n)|^p]\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
& -\left(\sum_{n\in\mathbb{Z}}[|\Delta u_k(n-1)|^p+a(n)|u_k(n)|^p]\right)^{1/p} \\
& \qquad\qquad\qquad \left(\sum_{n\in\mathbb{Z}}[|\Delta u_0(n-1)|^p+a(n)|u_0(n)|^p]\right)^{1/q} \\
& -\sum_{n\in\mathbb{Z}}(\nabla W(n,u_k(n))-\nabla W(n,u_0(n)),u_k(n)-u_0(n)) \\
= & \ \|u_k\|^p+\|u_0\|^p-\|u_0\|\|u_k\|^{p-1}-\|u_k\|\|u_0\|^{p-1} \\
& -\sum_{n\in\mathbb{Z}}(\nabla W(n,u_k(n))-\nabla W(n,u_0(n)),u_k(n)-u_0(n)) \\
= & \left(\|u_k\|^{p-1}-\|u_0\|^{p-1}\right)(\|u_k\|-\|u_0\|) \\
& -\sum_{n\in\mathbb{Z}}(\nabla W(n,u_k(n))-\nabla W(n,u_0(n)),u_k(n)-u_0(n)). \qquad (3.16)
\end{aligned}
$$

Since $I'(u_k)\to 0$ as $k\to+\infty$ and $u_k\rightharpoonup u_0$ in E, it follows from (3.16) that

$$\langle I'(u_k)-I'(u_0),u_k-u_0\rangle\to 0 \quad \text{as } k\to\infty,$$

which, together with (3.15) and (3.16), yields that $\|u_k\|\to\|u\|$ as $k\to+\infty$. By the uniform convexity of E and the fact that $u_k\rightharpoonup u_0$ in E, it follows from the Kadec-Klee property that $u_k\to u_0$ in E. Hence, I satisfies (C)-condition.

We now show that there exist constants $\rho,\alpha>0$ such that I satisfies assumption (ii) of Lemma 2.1 with these constants. Let $\delta\le\eta$, if $\|u\|=\frac{\delta}{\lambda}:=\rho$, then by (2.4), $|u(n)|\le\delta\le\eta<1$ for $n\in\mathbb{Z}$.

Set

$$\alpha=\frac{\delta^p}{2p\lambda^p}.$$

Hence, from (2.2) and (3.3), we have

$$
\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p-\sum_{n\in\mathbb{Z}}W(n,u(n)) \\
&\ge \frac{1}{p}\|u\|^p-\frac{1}{2p}\sum_{n\in\mathbb{Z}}a(n)|u(n)|^p \\
&\ge \frac{1}{2p}\|u\|^p \\
&= \alpha. \qquad (3.17)
\end{aligned}
$$

(3.17) shows that $\|u\|=\rho$ implies that $I(u)\ge\alpha$, i.e., I satisfies assumption (ii) of Lemma 2.1.

Finally, it remains to show that I satisfies assumption (iii) of Lemma 2.1. Let E' be a finite dimensional subspace of E. Since all the norms of a finite dimensional normed space are equivalent, so there exists a constant $d>0$ such that

$$\|u\|\le d\|u\|_\infty \quad \text{for } u\in E'. \qquad (3.18)$$

Assume that $\dim E' = m$ and $u_1, u_2, \ldots, u_m$ is a base of E' such that

$$\|u_i\| = d, \quad i = 1, 2, \ldots, m. \tag{3.19}$$

For any $u \in E'$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ such that

$$u(n) = \sum_{i=1}^{m} \lambda_i u_i(n) \quad \text{for } n \in \mathbb{Z}. \tag{3.20}$$

Let

$$\|u\|_* = \sum_{i=1}^{m} |\lambda_i| \|u_i\|. \tag{3.21}$$

It is easy to verify that $\|\cdot\|_*$ defined by (3.21) is a norm of E'. By a similar way in (3.18), we have that there exists $d' > 0$ such that

$$d'\|u\|_* \leq \|u\|. \tag{3.22}$$

Since $u_i \in E$, we can choose $\Pi_1 > \Pi$ such that

$$|u_i(n)| < \frac{d'\eta}{1+d'}, \quad |n| > \Pi_1, \ i = 1, 2, \ldots, m, \tag{3.23}$$

where η is given in (3.3). Set

$$\Theta = \left\{ \sum_{i=1}^{m} \lambda_i u_i(n) \ : \ \lambda_i \in \mathbb{Z}, \ i = 1, 2, \ldots, m; \ \sum_{i=1}^{m} |\lambda_i| = 1 \right\} = \left\{ u \in E' \ : \ \|u\|_* = d \right\}. \tag{3.24}$$

Hence, for $u \in \Theta$, let $n_0 = n_0(u) \in \mathbb{Z}$ such that

$$|u(n_0)| = \|u\|_\infty. \tag{3.25}$$

Then by (3.19), (3.20), (3.21), (3.23), (3.24) and (3.25), we have

$$\begin{aligned} d'd &= d'd \sum_{i=1}^{m} |\lambda_i| = d' \sum_{i=1}^{m} |\lambda_i| \|u_i\| = d'\|u\|_* \\ &\leq \|u\| \leq d\|u\|_\infty = d|u(n_0)| \\ &\leq d \sum_{i=1}^{m} |\lambda_i| |u_i(n_0)|, \quad u \in \Theta. \end{aligned} \tag{3.26}$$

This shows that

$$|u(n_0)| \geq d' \tag{3.27}$$

and there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $|u_{i_0}(n_0)| \geq d'$. By (W3), there exists $\sigma_0 = \sigma_0(d, \Pi_1) > 1$ such that

$$s^{-p} \min_{|x|=1} W(n, sx) \geq \left(\frac{2d}{d'} \right)^p \quad \text{for } s \geq \frac{d'\sigma_0}{2}, \ n \in \mathbb{Z}(-\Pi_1, \Pi_1). \tag{3.28}$$

It follows from (W2), (W3), (2.1) and (3.28) that

$$
\begin{aligned}
I(\sigma u) &= \frac{\sigma^p}{p}\|u\|^p - \sum_{n\in\mathbb{Z}} W(n,\sigma u(n)) \\
&\le \frac{\sigma^p}{p}\|u\|^p - W(n_0,\sigma u(n_0)) \\
&\le \frac{\sigma^p}{p}\|u\|^p - \min_{|x|=1} W(n_0,\sigma|u(n_0)|x) \\
&\le \frac{(d\sigma)^p}{p} - (d\sigma)^p|u(n_0)|^p \\
&\le \frac{(d\sigma)^p}{p} - (d\sigma)^p \\
&= -\frac{(d\sigma)^p}{q}, \quad u\in\Theta,\ \sigma\ge\sigma_0. \qquad (3.29)
\end{aligned}
$$

We deduce that there is $\sigma_0 = \sigma_0(d,\Pi_1) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } u\in\Theta \text{ and } \sigma\ge\sigma_0.$$

That is

$$I(u) < 0 \quad \text{for } u\in E' \text{ and } \|u\|\ge d\sigma_0.$$

This shows that condition (iii) of Lemma 2.1 holds. By Lemma 2.1, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. If $\{\|u_k\|\}$ is bounded, then there exists $B > 0$ such that

$$\|u_k\| \le B \quad \text{for } k\in\mathbb{N}. \qquad (3.30)$$

By a similar fashion for the proof of (3.3), for the given η in (3.3), there exists $\Pi_1' > 0$ such that

$$|u_k(n)| \le \eta \quad \text{for } |n|\ge\Pi_1',\ k\in\mathbb{N}. \qquad (3.31)$$

Thus, from (2.1), (2.4) and (3.3), we have

$$
\begin{aligned}
\frac{1}{p}\|u_k\|^p &= d_k + \sum_{n\in\mathbb{Z}} W(n,u_k(n)) \\
&= d_k + \sum_{|n|>\Pi_1'} W(n,u_k(n)) + \sum_{|n|\le\Pi_1'} W(n,u_k(n)) \\
&\ge d_k - \frac{1}{2p}\sum_{|n|>\Pi_1'} a(n)|u_k(n)|^p - \sum_{|n|\le\Pi_1'} |W(n,u_k(n))| \\
&\ge d_k - \frac{1}{2p}\|u_k\|^p - \sum_{|n|\le\Pi_1'} \max_{|x|\le\lambda B} |W(n,x)|. \qquad (3.32)
\end{aligned}
$$

It follows that

$$d_k \le \frac{3}{2p}\|u_k\|^p + \sum_{|n|\le\Pi_1'} \max_{|x|\le\lambda B} |W(n,x)| < +\infty.$$

This contradicts to the fact that $\{d_k\}_{k=1}^{\infty}$ is unbounded, and so $\{\|u_k\|\}$ is unbounded. The proof is complete.

Proof of Theorem 1.2. It is clear that $I(0) = 0$. We first show that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $M > 0$ such that

$$|I(u_k)| \le M, \quad \|I'(u_k)\|_{E^*} \le \mu M \quad \text{for } k \in \mathbb{N}. \tag{3.33}$$

From (2.1), (2.2), (3.1), (W5) and (W6), we obtain

$$\begin{aligned} & pc + pc\|u_k\| \\ \ge\; & pI(u_k) - \frac{p}{\mu}\langle I'(u_k), u_k\rangle \\ =\; & \frac{\mu - p}{\mu}\|u_k\|^p + p\sum_{n\in\mathbb{Z}}\left[W_2(n,u_k(n)) - \frac{1}{\mu}(\nabla W_2(n,u_k(n)), u_k(n))\right] \\ & -p\sum_{n\in\mathbb{Z}}\left[W_1(n,u_k(n)) - \frac{1}{\mu}(\nabla W_1(n,u_k(n)), u_k(n))\right] \\ \ge\; & \frac{\mu - p}{\mu}\|u_k\|^p, \quad k \in \mathbb{N}. \end{aligned}$$

It follows that there exists a constant $A > 0$ such that

$$\|u_k\| \le A \quad \text{for } k \in \mathbb{N}. \tag{3.34}$$

Similar to the proof of Theorem 1.1, we can prove that $\{u_k\}$ has a convergent subsequence in E. Hence, I satisfies condition (PS)-condition.

Finally, it remains to show that I satisfies assumption (iii) of Lemma 2.1. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional normed space are equivalent, so there is a constant $d' > 0$ such that (3.22) holds. Let η, Π_1 and Θ be the same as in the proof of Theorem 1.1, then (3.27) holds.

Set

$$\tau = \min\{W_1(n,x) : |n| \le \Pi_1, |x| \le d'\}, \tag{3.35}$$

where d' is given in (3.22).

Since $W_1(n,x) > 0$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}^N \setminus \{0\}$, and $W_1(n,x)$ is continuous in x, so $\tau > 0$. It follows from (3.27), (3.35) and Lemma 2.3 (i) that

$$\begin{aligned} \sum_{n=-\Pi_1}^{\Pi_1} W_1(n,u(n)) \;&\ge\; W_1(n_0,u(n_0)) \\ &\ge\; W_1\left(n_0, \frac{u(n_0)d'}{|u(n_0)|}\right)\left(\frac{|u(n_0)|}{d'}\right)^{\mu} \\ &\ge\; \left[\min_{|x|\le d'} W_1(n_0,x)\right]\left(\frac{|u(n_0)|}{d'}\right)^{\mu} \\ &\ge\; \tau \quad \text{for } u \in \Theta. \end{aligned} \tag{3.36}$$

For any $u \in E$, it follows from (2.4) and Lemma 2.3 (ii) that

$$\begin{aligned}
&\sum_{n=-\Pi_1}^{\Pi_1} W_2(n,u(n)) \\
&= \sum_{n\in\mathbb{Z}(-\Pi_1,\Pi_1),\ |u(n)|>1} W_2(n,u(n)) + \sum_{n\in\mathbb{Z}(-\Pi_1,\Pi_1),\ |u(n)|\le 1} W_2(n,u(n)) \\
&\le \sum_{n\in\mathbb{Z}(-\Pi_1,\Pi_1),\ |u(n)|>1} W_2\left(n,\frac{u(n)}{|u(n)|}\right)|u(n)|^{\varrho} \\
&\quad + \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|\le 1}|W_2(n,x)| \\
&\le \|u\|_\infty^{\varrho} \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|=1}|W_2(n,x)| + \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|\le 1}|W_2(n,x)| \\
&\le \lambda^{\varrho}\|u\|^{\varrho} \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|=1}|W_2(n,x)| + \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|\le 1}|W_2(n,x)| \\
&= M_1\|u\|^{\varrho} + M_2,
\end{aligned} \tag{3.37}$$

where

$$M_1 = \lambda^{\varrho} \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|=1}|W_2(n,x)|, \quad M_2 = \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|\le 1}|W_2(n,x)|.$$

From (3.3), (3.24), (3.36), (3.37) and Lemma 2.3, we have for $u \in \Theta$ and $\sigma > 1$

$$\begin{aligned}
I(\sigma u) &= \frac{\sigma^p}{p}\|u\|^p - \sum_{n\in\mathbb{Z}} W(n,\sigma u(n)) \\
&= \frac{\sigma^p}{p}\|u\|^p + \sum_{n\in\mathbb{Z}} W_2(n,\sigma u(n)) - \sum_{n\in\mathbb{Z}} W_1(n,\sigma u(n)) \\
&\le \frac{\sigma^p}{p}\|u\|^p + \sigma^{\varrho}\sum_{n\in\mathbb{Z}} W_2(n,u(n)) - \sigma^{\mu}\sum_{n\in\mathbb{Z}} W_1(n,u(n)) \\
&= \frac{\sigma^p}{p}\|u\|^p + \sigma^{\varrho}\sum_{|n|>\Pi_1} W_2(n,u(n)) - \sigma^{\mu}\sum_{|n|>\Pi_1} W_1(n,u(n)) \\
&\quad +\sigma^{\varrho}\sum_{n=-\Pi_1}^{\Pi_1} W_2(n,u(n)) - \sigma^{\mu}\sum_{n=-\Pi_1}^{\Pi_1} W_1(n,u(n)) \\
&\le \frac{\sigma^p}{p}\|u\|^p - \sigma^{\varrho}\sum_{|n|>\Pi_1} W(n,u(n)) \\
&\quad +\sigma^{\varrho}\sum_{n=-\Pi_1}^{\Pi_1} W_2(n,u(n)) - \sigma^{\mu}\sum_{n=-\Pi_1}^{\Pi_1} W_1(n,u(n))
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\sigma^p}{p}\|u\|^p + \frac{\sigma^\varrho}{2p}\sum_{|n|>\Pi_1} a(n)|u(n)|^p + \sigma^\varrho(M_1\|u\|^\varrho + M_2) - \tau\sigma^\mu \\
&\leq \frac{\sigma^p}{p}\|u\|^p + \frac{\sigma^\varrho}{2p}\|u\|^p + \sigma^\varrho(M_1\|u\|^\varrho + M_2) - \tau\sigma^\mu \\
&= \frac{(d\sigma)^p}{p} + \frac{d^p\sigma^\varrho}{2p} + M_1(d\sigma)^\varrho + M_2\sigma^\varrho - \tau\sigma^\mu. \qquad (3.38)
\end{aligned}
$$

Since $\mu > \varrho > p$, we deduce that there is $\sigma_0 = \sigma_0(d, M_1, M_2, \tau) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } u \in \Theta \text{ and } \sigma \geq \sigma_0.$$

That is

$$I(u) < 0 \quad \text{for } u \in E' \text{ and } \|u\| \geq d\sigma_0.$$

This shows that (iii) of Lemma 2.1 holds. By Lemma 2.1, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. If $\{\|u_k\|\}_{k\in\mathbb{N}}$ is bounded, then there exists $B > 0$ such that

$$\|u_k\| \leq B \quad \text{for } k \in \mathbb{N}. \qquad (3.39)$$

By a similar fashion for the proof of (3.5) and (3.7), for the given η in (3.13), there exists $\Pi_1'' > 0$ such that

$$|u_k(n)| \leq \eta \quad \text{for } |n| \geq \Pi_1'',\ k \in \mathbb{N}. \qquad (3.40)$$

Thus, from (W1′), (W5), (W6), (2.1), (2.4), (3.3), (3.39) and (3.40), we have

$$
\begin{aligned}
\frac{1}{p}\|u_k\|^p &= d_k + \sum_{n\in\mathbb{Z}} W(n, u_k(n)) \\
&= d_k + \sum_{|n|>\Pi_1''} W(n, u_k(n)) + \sum_{n=-\Pi_1''}^{\Pi_1''} W(n, u_k(n)) \\
&\geq d_k - \frac{1}{2p}\sum_{|n|>\Pi_1''} a(n)|u_k(n)|^p - \sum_{n=-\Pi_1''}^{\Pi_1''} W_2(n, u_k(n)) \\
&\geq d_k - \frac{1}{2p}\|u_k\|^p - \sum_{n=-\Pi_1''}^{\Pi_1''} \max_{|x|\leq\lambda B} |W_2(n, x)|. \qquad (3.41)
\end{aligned}
$$

It follows that

$$d_k \leq \frac{3}{2p}\|u_k\|^p + \sum_{n=-\Pi_1''}^{\Pi_1''} \max_{|x|\leq\lambda B} |W_2(n, x)| < +\infty.$$

This contradicts to the fact that $\{d_k\}_{k\in\mathbb{N}}$ is unbounded, and so $\{\|u_k\|\}_{k\in\mathbb{N}}$ is unbounded.

Proof of Theorem 1.3. In the proof of Theorem 1.2, the condition that $W_2(n,x) \geq 0$ for $(n,x) \in \mathbb{Z} \times \mathbb{R}^N$, $|x| \leq 1$ in (W1′) is only used in the the proofs of assumption (ii) of Lemma 2.1. Therefore, we only prove assumption (ii) of Lemma 2.1 still hold use (W1″) instead of (W1′). By (W1″), there exists $\eta \in (0,1)$ such that

$$|\nabla W(n,x)| \leq \frac{1}{2}a(n)|x|^{p-1} \quad \text{for } n \in \mathbb{Z} \setminus J,\ |x| \leq \eta. \tag{3.42}$$

Since $W(n,0) = 0$, it follows that

$$|W(n,x)| \leq \frac{1}{2p}a(n)|x|^{p} \quad \text{for } n \in \mathbb{Z} \setminus J,\ |x| \leq \eta. \tag{3.43}$$

Set

$$M = \sup\left\{ \frac{W_1(n,x)}{a(n)} \;\middle|\; n \in J,\ x \in \mathbb{R}^N,\ |x| = 1 \right\}. \tag{3.44}$$

Set $\delta = \min\{1/(2pM+1)^{1/(\mu-p)}, \eta\}$. if $\|u\| = \delta/\lambda := \rho$, then by (2.4), $|u(n)| \leq \delta \leq \eta < 1$ for $n \in \mathbb{Z}$. By (3.44) and Lemma 2.4 (i), we have

$$\begin{aligned}
\sum_{n\in J} W_1(n,u(n)) &\leq \sum_{\{n\in J,\ u(n)\neq 0\}} W_1\left(n, \frac{u(n)}{|u(n)|}\right)|u(n)|^{\mu} \\
&\leq M\sum_{n\in J} a(n)|u(n)|^{\mu} \\
&\leq M\delta^{\mu-p}\sum_{n\in J} a(n)|u(n)|^{p} \\
&\leq \frac{1}{2p}\sum_{n\in J} a(n)|u(n)|^{p}.
\end{aligned} \tag{3.45}$$

Set

$$\alpha = \frac{a\delta^p}{2p}.$$

Hence, from (2.1), (3.43), (3.45) and (W1″), we have

$$\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\
&= \frac{1}{p}\|u\|^p - \sum_{n\in\mathbb{Z}\setminus J} W(n,u(n)) - \sum_{n\in J} W(n,u(n)) \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{2p}\sum_{n\in\mathbb{Z}\setminus J} a(n)|u(n)|^p - \sum_{n\in J} W_1(n,u(n)) \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{2p}\sum_{n\in\mathbb{Z}\setminus J} a(n)|u(n)|^p - \frac{1}{2p}\sum_{n\in J} a(n)|u(n)|^p \\
&= \frac{1}{p}\sum_{n\in\mathbb{Z}} |\Delta u(n-1)|^p + \frac{1}{2p}\sum_{n\in\mathbb{Z}} a(n)|u(n)|^p \\
&\geq \frac{1}{2p}\sum_{n\in\mathbb{Z}} [|\Delta u(n-1)|^p + a(n)|u(n)|^p] \\
&= \frac{1}{2p}\|u\|^p \\
&= \alpha.
\end{aligned} \tag{3.46}$$

(3.46) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, i.e., I satisfies assumption (ii) of Lemma 2.1. It is obvious that I is even and $I(0) = 0$ and so assumption (ii) of Lemma 2.1 holds. The proof of assumption (iii) of Lemma 2.1 is the same as in the proof of Theorem 1.2, we omit its details.

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. Consider the second-order discrete p-Laplacian system

$$\Delta(|\Delta u(n-1)|^{-\frac{2}{3}}\Delta u(n-1)) - a(n)|u(n)|u(n) + \nabla W(n, u(n)) = 0, \tag{4.1}$$

where $p = \frac{4}{3}, a : \mathbb{Z} \to (0, \infty)$ such that $a(n) \to +\infty$ as $|n| \to +\infty$, and

$$W(n, x) = a(n)(2 - \cos n)|x|^{\frac{4}{3}} \ln(1 + |x|).$$

Since

$$\begin{aligned}(\nabla W(n, x), x) &= a(n)(2 - \cos n)\left[\frac{4}{3}|x|^{\frac{4}{3}} \ln(1 + |x|) + \frac{|x|^{\frac{7}{3}}}{1 + |x|}\right] \\ &\geq \left(\frac{4}{3} + \frac{1}{1 + |x|}\right) W(n, x) \geq 0, \quad \forall\ (n, x) \in \mathbb{Z} \times \mathbb{R}^N.\end{aligned}$$

This shows that (W3) holds with $b = c = \nu = 1$. In addition, for any $n \in \mathbb{Z}$

$$\begin{aligned}s^{-\frac{4}{3}} \min_{|x|=1} W(n, sx) &= s^{-\frac{4}{3}} \min_{|x|=1} \left[a(n)(2 - \cos n)|sx|^{\frac{4}{3}} \ln(1 + |sx|)\right] \\ &= a(n)(2 - \cos n) \ln(1 + s) \\ &\to +\infty, \quad s \to +\infty.\end{aligned}$$

This shows that (W3) also holds. It is easy to verify that assumptions (A) and (W1) of Theorem 1.1 are satisfied. By Theorem 1.1, system (1.1) has an unbounded sequence of homoclinic solutions.

Example 4.2. Consider the second-order discrete p-Laplacian system

$$\Delta(|\Delta u(n-1)|\Delta u(n-1)) - a(n)|u(n)|u(n) + \nabla W(n, u(n)) = 0, \tag{4.2}$$

where $p = 3, n \in \mathbb{Z}, u \in \mathbb{R}^N, a \in C(\mathbb{Z}, (0, \infty))$ such that $a(n) \to +\infty$ as $|n| \to \infty$. Let

$$W(n, x) = a(n)\left(\sum_{i=1}^{m} a_i|x|^{\mu_i} - \sum_{j=1}^{n} b_j|x|^{\varrho_j}\right),$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_n > 3,\ a_i, b_j > 0$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Let $\mu = \mu_m,\ \varrho = \varrho_1$, and

$$W_1(n, x) = a(n)\sum_{i=1}^{m} a_i|x|^{\mu_i}, \quad W_2(n, x) = a(n)\sum_{j=1}^{n} b_j|x|^{\varrho_j}.$$

Then it is easy to verify that all conditions of Theorem 1.2 are satisfied. By Theorem 1.2, system (1.1) has an unbounded sequence of homoclinic solutions..

Example 4.3. Consider the second-order discrete p-Laplacian system

$$\Delta(|\Delta u(n-1)|^2\Delta u(n-1)) - a(n)|u(n)|^2 u(n) + \nabla W(n,u(n)) = 0, \tag{4.3}$$

where $p = 4$, $n \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $a \in C(\mathbb{Z},(0,\infty))$ such that $a(n) \to +\infty$ as $|n| \to \infty$. Let

$$W(n,x) = a(n)\left[a_1|x|^{\mu_1} + a_2|x|^{\mu_2} - (2-|n|)|x|^{\varrho_1} - (2-|n|)|x|^{\varrho_2}\right],$$

where $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > 4$, $a_1, a_2 > 0$. Let $\mu = \mu_2$, $\varrho = \varrho_1$, $J = \{-2,-1,0,1,2\}$ and

$$W_1(n,x) = a(n)\left(a_1|x|^{\mu_1} + a_2|x|^{\mu_2}\right),$$
$$W_2(n,x) = a(n)\left[(2-|n|)|x|^{\varrho_1} + (2-|n|)|x|^{\varrho_2}\right].$$

Then it is easy to verify that all conditions of Theorem 1.3 are satisfied. By Theorem 1.3, system (1.1) has an unbounded sequence of homoclinic solutions.

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