An application of generalized power increasing sequences on factors theorem

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Abstract

In the present paper, by using a new defined -$|C, \alpha, \sigma; \alpha_n|_k$ summability method and some classes of pairs of sequences, we generalize a result of Bor [5] dealing with $\varphi - |C, \alpha, \sigma; \beta|_k$ summability factors.

1 Introduction

A sequence $\{\lambda_n\}$ is said to be of bounded variation, denote by $\{\lambda_n\} \in \mathbf{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. If the sequence $\{\lambda_n\}$ is a null sequence of bounded variation, we denote that $\{\lambda_n\} \in \mathbf{BV}_0$. A positive sequence $\{b_n\}$ is said to be almost increasing, if there exists a positive increasing sequence $\{c_n\}$ and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A positive sequence $\{X_n\}$ is said to be a quasi-β-power increasing, if there exists a constant $K = K(\beta, X) \geq 1$ such that $Kn^{\beta}X_n \geq m^{\beta}X_m$ holds for all $n \geq m \geq 1$. It has been shown that every almost increasing sequence is a quasi-β-power increasing for any nonnegative β, but the converse is not true (see [11]). Write

$$f := \{f_n\} = \left\{n^{\beta}(\log n)^{\mu}\right\}, \ \mu \in \mathbf{R}, \ 0 < \beta < 1. \tag{1.1}$$

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Recently, Sulaiman [12] further generalized the definition of quasi-β-power increasing sequence by using f defined in (1.1). Namely, a positive sequence $\{X_n\}$ is said to be a quasi-f-power increasing, if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ holds for all $n \geq m \geq 1$.

Let $\sum a_n$ be a given infinite series with partial sums $\{s_n\}$. Denote by $u_n^{\alpha,\sigma}$ and $t_n^{\alpha,\sigma}$ the nth Cesáro mean of order (α, σ), with $\alpha + \sigma > -1$, of the sequence $\{s_n\}$ and $\{na_n\}$, respectively, that is (see [7]),

$$u_n^{\alpha,\sigma} := \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\sigma} s_v, \tag{1.2}$$

$$t_n^{\alpha,\sigma} := \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\sigma} v a_v, \tag{1.3}$$

where

$$A_v^{\sigma} = \binom{v+\sigma}{v}, \; A_n^{\alpha+\sigma} = O\left(n^{\alpha+\sigma}\right), A_0^{\alpha+\sigma} = 1 \text{ and } A_{-n}^{\alpha+\sigma} = 0 \text{ for all } n > 0. \tag{1.4}$$

Let $\varphi := \{\varphi_n\}$ be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - |C, \alpha, \sigma|_k$, $k \geq 1$ and $\alpha + \sigma > -1$, if (see [4])

$$\sum_{n=1}^{\infty} \left|\varphi_n \left(u_n^{\alpha,\sigma} - u_{n-1}^{\alpha,\sigma}\right)\right|^k < \infty. \tag{1.5}$$

But since $t_n^{\alpha,\sigma} = n\left(u_n^{\alpha,\sigma} - u_{n-1}^{\alpha,\sigma}\right)$ (see [7]) condition (1.5) can also written as

$$\sum_{n=1}^{\infty} n^{-k} \left|\varphi_n t_n^{\alpha,\sigma}\right|^k < \infty. \tag{1.6}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}, \varphi - |C, \alpha, \sigma|_k$ summability is the same as $|C, \alpha, \sigma|_k$ summability (see [8]). Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}, \varphi - |C, \alpha, \sigma|_k$ summability reduces to $|C, \alpha, \sigma; \delta|_k$ summability. If we take $\sigma = 0$, then we have $\varphi - |C, \alpha|_k$ summability (see [2]). If we take $\varphi_n = n^{1-\frac{1}{k}}, \sigma = 0$, then we get $|C, \alpha|_k$ summability (see [10]). Finally, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}, \sigma = 0$, then we obtain $|C, \alpha; \delta|_k$ summability (see [9]).

Recently, Bor [5] has proved the following theorem for $\varphi - |C, \alpha, \sigma|_k$ summability factors.

Theorem 1. *Let $\{\lambda_n\} \in BV_0$ and $\{X_n\}$ be a quasi-β−power increasing sequence for some β $(0 < \beta < 1)$. Suppose also that there exists a sequence $\{\delta_n\}$ satisfies the following conditions:*

$$|\Delta\lambda_n| \leq \delta_n, \tag{1.7}$$

$$\delta_n \to 0 \textit{ as } n \to \infty, \tag{1.8}$$

$$\sum_{n=1}^{\infty} n\,|\Delta\delta_n|\, X_n < \infty, \tag{1.9}$$

$$\left|\lambda_n\right| X_n = O\left(1\right) \ \text{as } n \to \infty. \tag{1.10}$$

If there exists an $\epsilon > 0$ *such that the sequence* $\left\{n^{\epsilon-k}\left|\varphi_n\right|^k\right\}$ *is non-increasing and if the sequence* $\left\{\theta_n^{\alpha,\sigma}\right\}$ *is defined by*

$$\theta_n^{\alpha,\sigma} := \left|t_n^{\alpha,\sigma}\right|, \ \alpha = 1, \sigma > -1, \tag{1.11}$$

$$\theta_n^{\alpha,\sigma} := \max_{1\leq v\leq n}\left|t_v^{\alpha,\sigma}\right|, \ 0 < \alpha < 1, \sigma > -1 \tag{1.12}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k}\left(\left|\varphi_n\right|\theta_n^{\alpha,\sigma}\right)^k = O\left(X_m\right) \ \text{as } m \to \infty, \tag{1.13}$$

then the series $\sum a_n\lambda_n$ *is summable* $\varphi - \left|C,\alpha,\sigma\right|_k, k \geq 1, 0 < \alpha \leq 1, \sigma > -1$ *and* $(\alpha+\sigma)k + \epsilon > 1$.

To further generalize Theorem 1, we now introduce the definition of $\left|C,\alpha,\sigma,\alpha_n\right|_k$ summability which is a generalization of $\varphi - \left|C,\alpha,\sigma\right|_k$ summability.

Definition 1. *Let* $\{\alpha_n\}$ *be a given nonnegative sequence. A series* $\sum a_n$ *is said to be summable* $\left|C,\alpha,\sigma;\alpha_n\right|_k$, $k \geq 1$, $\alpha+\sigma > -1$, *if*

$$\sum_{n=1}^{\infty}\alpha_n\left|t_n^{\alpha,\sigma}\right|^k < \infty.$$

Obviously, $\varphi - \left|C,\alpha,\sigma\right|_k$ summability is a special case of $\left|C,\alpha,\sigma;\alpha_n\right|_k$ summability when $\alpha_n = \left(\frac{\left|\varphi_n\right|}{n}\right)^k$.

The following two classes of pairs of sequences were introduced in [6]:

Definition 2. *We say that a pair of sequences* $\lambda := \{\lambda_n\}$ *and* $X := \{X_n\}$ *belongs to the class* $\mathbf{M}(\theta,k)$, *denote by* $(\lambda, X) \in \mathbf{M}(\theta,k)$, *if the following conditions are satisfied:*

$$\{\lambda_n\} \in \mathbf{BV}, \tag{1.14}$$

$$\sum_{n=1}^{\infty} n^{\theta+1}\left|\Delta\left|\Delta\lambda_n\right|\right| X_n < \infty, \tag{1.15}$$

$$\sum_{n=1}^{\infty}\left|\Delta\left(n^{\theta}\left|\lambda_n\right|^k\right)\right| X_n < \infty, \tag{1.16}$$

$$n^{\theta}\left|\lambda_n\right|^k X_n < \infty. \tag{1.17}$$

Also, we say $(\lambda, X) \in \mathbf{M}^*(\theta,k)$, *if only the conditions (1.14), (1.15) and (1.17) are satisfied.*

Definition 3. *Let* $\delta := \{\delta_n\}$ *be a positive sequence. We say that a pair of sequences* $\lambda := \{\lambda_n\}$ *and* $X := \{X_n\}$ *belongs to the class* $\mathbf{N}(\theta,k,\delta)$, *denote by* $(\lambda, X) \in \mathbf{N}(\theta,k,\delta)$, *if* $\lambda \in \mathbf{BV}$, *(1.16),(1.17) and the following conditions are satisfied*

$$\left|\Delta\lambda_n\right| \leq \delta_n \to 0, \ \text{as } n \to \infty, \tag{1.18}$$

$$\sum_{n=1}^{\infty} n^{\theta+1} |\Delta\delta_n| X_n < \infty, \tag{1.19}$$

Also, we say $(\lambda, X) \in \mathbf{N}^*(\theta, k, \delta)$, if only $\lambda \in \mathbf{BV}$ and the conditions (1.17), (1.18), (1.19) are satisfied.

The following properties of $\mathbf{M}(\theta,k)$, $\mathbf{M}^*(\theta,k)$, $\mathbf{N}(\theta,k,\delta)$ and $\mathbf{N}^*(\theta,k,\delta)$ are useful (see Theorem 2 of [6]).

Proposition 1. *(a) Let λ, X and δ satisfy all the conditions on Theorem 1 except (1.13), we have $(\lambda, X) \in \mathbf{N}(0,k,\delta)$.*

(b) Let $\{X_n\}$ be a quasi-f-power increasing sequence, $\lambda \in \mathbf{BV}_0$, $\theta > \beta$, and δ be a positive null sequence. Then $M(\theta,1) \subseteq M(\theta,k)$ and $\mathbf{N}(\theta,1,\delta) \subseteq \mathbf{N}(\theta,k,\delta)$ for $k \geq 1$.

(c) Let $\{X_n\}$ be a quasi-f-power increasing sequence and δ be a positive null sequence. If $\lambda \in \mathbf{BV}_0$ and $\theta > \beta$. Then $\mathbf{M}^*(\theta,k) = \mathbf{M}(\theta,k)$ and $\mathbf{N}(\theta,k,\delta) = \mathbf{N}^*(\theta,k,\delta)$.

2 Main Results

In what follows, β always means the number appearing in (1.1).

Now, we can state our main results as follows:

Theorem 2. *Let $\{X_n\}$ be a quasi-f-power increasing sequence and $(\lambda, X) \in \mathbf{M}(\theta,k)$ with $\theta > \beta - 1$ and $k \geq 1$. If $\{\alpha_n\}$ satisfies the following conditions*

$$\sum_{n=v}^{\infty} \alpha_n n^{-(\alpha+\sigma)k} = O\left(\alpha_v v^{-(\alpha+\sigma)k+1}\right), v = 1, 2, \cdots, \tag{2.1}$$

and

$$\sum_{n=1}^{m} n^{-\theta} \alpha_n \, |t_n^{\alpha,\sigma}|^k = O\left(X_m\right) \text{ as } m \to \infty, \tag{2.2}$$

then the series $\sum a_n \lambda_n$ is $|C, \alpha, \sigma, \alpha_n|_k$ summable for $0 < \alpha \leq 1, \sigma > -1$.

Furthermore, if $\lambda \in \mathbf{BV}_0$ and $\theta > \beta$, then the condition $(\lambda, X) \in \mathbf{M}(\theta,k)$ can be relaxed to $(\lambda, X) \in \mathbf{M}^*(\theta,k)$.

Corollary 1. *Let $\{X_n\}$ be a quasi-f-power increasing sequence and $(\lambda, X) \in \mathbf{M}(\theta,k)$ with $\theta > \beta - 1$ and $k \geq 1$. Suppose that $\{\alpha_n\}$ is quasi-ϵ-decreasing with ϵ satisfying $(\alpha+\sigma)k + \epsilon > 1$ and (2.2) holds. Then, the results of Theorem 2 keep true.*

Similar to Theorem 2 and Corollary 1, we have

Theorem 3. *Let $\{X_n\}$ be a quasi-f-power increasing sequence, δ be a positive sequence, and $(\lambda, X) \in \mathbf{N}(\theta,k,\delta)$ with $\theta > \beta - 1$ and $k \geq 1$. If $\{\alpha_n\}$ satisfies the conditions (2.1) and (2.2), then the series $\sum a_n \lambda_n$ is $|C, \alpha, \sigma, \alpha_n|_k$ summable for $0 < \alpha \leq 1$.*

Furthermore, if $\lambda \in \mathbf{BV}_0$ and $\theta > \beta$, then the condition $(\lambda, X) \in \mathbf{N}(\theta,k,\delta)$ can be relaxed to $(\lambda, X) \in \mathbf{N}^*(\theta,k,\delta)$.

Corollary 2. *Let $\{X_n\}$ be a quasi-f-power increasing sequence, δ be a positive sequence, and $(\lambda, X) \in \mathbf{N}(\theta,k,\delta)$ with $\theta > \beta - 1$ and $k \geq 1$. Suppose that $\{\alpha_n\}$ is quasi-ϵ-decreasing with ϵ satisfying $(\alpha+\sigma)k + \epsilon > 1$ and (2.2) holds. Then, the results of Theorem 3 keep true.*

Taking $\alpha_n = \left(\frac{|\varphi_n|}{n}\right)^k$, in view of (a) in Proposition 1, we see that Corollary 2 implies Theorem 1.

3 Proof of Results

3.1 Some Auxiliary Lemmas

Lemma 1. ([3]) *If* $0 < \alpha \leq 1, \sigma > -1$ *and* $1 \leq v \leq n$, *then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\sigma} a_p\right| \leq \max_{1 \leq m \leq v} \left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\sigma} a_p\right|. \tag{3.1}$$

Lemma 2. ([6])*Let* $\{X_n\}$ *be a quasi-*f*-power increasing sequence ,* $\{X_n\}$ *and* $\{\lambda_n\}$ *satisfy the conditions (1.14) and (1.15) with* $\theta > \beta - 1$. *Then the following inequalities hold:*

$$n^{\theta+1} |\Delta\lambda_n| X_n = O(1) \quad \text{as } n \to \infty, \tag{3.2}$$

$$\sum_{n=1}^{\infty} n^{\theta} |\Delta\lambda_n| X_n < \infty. \tag{3.3}$$

If $\lambda \in \mathbf{BV}_0$ *and* $\theta > \beta$, *then*

$$\sum_{n=1}^{\infty} n^{\theta-1} |\lambda_n| X_n < \infty. \tag{3.4}$$

Lemma 3. ([6]) *Let* $\{X_n\}$ *be a quasi-*f*-power increasing sequence and* δ *be a positive null sequence. If* $\lambda \in \mathbf{BV}$ *and the conditions (1.18) and (1.19) are satisfied, then the following inequalities hold:*

$$n^{\theta+1} \delta_n X_n = O(1) \quad \text{as } n \to \infty, \tag{3.5}$$

$$\sum_{n=1}^{\infty} n^{\theta} \delta_n X_n < \infty. \tag{3.6}$$

If $\lambda \in \mathbf{BV}_0$ *and* $\theta > \beta$, *then*

$$\sum_{n=1}^{\infty} n^{\theta-1} |\lambda_n| X_n < \infty. \tag{3.7}$$

3.2 Proof of theorem 2.

Let $T_n^{\alpha,\sigma}$ be the $n-$th (C, α, σ) mean of the sequence $\{n a_n \lambda_n\}$. Then by means of (1.3) we have

$$T_n^{\alpha,\sigma} = \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\sigma} v a_v \lambda_v.$$

First applying Abel's transformation and then using Lemma 1, we have that

$$T_n^{\alpha,\sigma} = \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{u=1}^{v} A_{n-u}^{\alpha-1} A_u^{\sigma} u a_u + \frac{\lambda_n}{A_n^{\alpha+\sigma}} \sum_{u=1}^{n} A_{n-u}^{\alpha-1} A_u^{\sigma} u a_u$$

$$\begin{aligned}
|T_n^{\alpha,\sigma}| &\leq \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left| \sum_{u=1}^{v} A_{n-u}^{\alpha-1} A_u^{\sigma} u a_u \right| + \frac{|\lambda_n|}{A_n^{\alpha+\sigma}} \left| \sum_{u=1}^{n} A_{n-u}^{\alpha-1} A_u^{\sigma} u a_u \right| \\
&\leq \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\sigma} \theta_v^{\alpha,\sigma} |\Delta\lambda_v| + |\lambda_n| \, \theta_n^{\alpha,\sigma} \\
&=: T_{n,1}^{\alpha,\sigma} + T_{n,2}^{\alpha,\sigma}, \text{ say.}
\end{aligned}$$

Since

$$\left| T_{n,1}^{\alpha,\sigma} + T_{n,2}^{\alpha,\sigma} \right|^k \leq 2^k \left(\left| T_{n,1}^{\alpha,\sigma} \right|^k + \left| T_{n,2}^{\alpha,\sigma} \right|^k \right),$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \alpha_n \left| T_{n,r}^{\alpha,\sigma} \right|^k < \infty, r = 1, 2.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, by noting that $\lambda \in BV$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} \alpha_n \left| T_{n,1}^{\alpha,\sigma} \right|^k &= \sum_{n=2}^{m+1} \alpha_n \left(\frac{1}{A_n^{\alpha+\sigma}} \right)^k \left(\sum_{v=1}^{n-1} A_v^{\alpha+\sigma} |\theta_v^{\alpha,\sigma}| \, |\Delta\lambda_v| \right)^k \\
&\leq \sum_{n=2}^{m+1} \alpha_n \left(\frac{1}{A_n^{\alpha+\sigma}} \right)^k \left(\sum_{v=1}^{n-1} \left(A_v^{\alpha+\sigma} \right)^k |\theta_v^{\alpha,\sigma}|^k \, |\Delta\lambda_v| \right) \left(\sum_{v=1}^{n-1} |\Delta\lambda_v| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \alpha_n n^{-(\alpha+\sigma)k} \left(\sum_{v=1}^{n-1} v^{(\alpha+\sigma)k} |\Delta\lambda_v| \, |\theta_v^{\alpha,\sigma}|^k \right) \\
&= O(1) \sum_{v=1}^{m} |\Delta\lambda_v| \, |\theta_v^{\alpha,\sigma}|^k \, v^{(\alpha+\sigma)k} \sum_{n=v+1}^{m+1} \alpha_n n^{-(\alpha+\sigma)k} \\
&= O(1) \sum_{v=1}^{m} v^{\theta+1} |\Delta\lambda_v| \, \alpha_v \, |\theta_v^{\alpha,\sigma}|^k \, v^{-\theta}.
\end{aligned}$$

Now, by (2.2), we deduce that

$$\begin{aligned}
\sum_{n=2}^{m+1} \alpha_n \left| T_{n,1}^{\alpha,\sigma} \right|^k &= O(1) \left(\sum_{v=1}^{m} \Delta \left(v^{\theta+1} |\Delta\lambda_v| \right) \sum_{r=1}^{v} r^{-\theta} \alpha_r \, |\theta_v^{\alpha,\sigma}|^k + m^{\theta+1} |\Delta\lambda_m| \sum_{r=1}^{m} r^{-\theta} \alpha_r \, |\theta_v^{\alpha,\sigma}|^k \right) \\
&= O(1) \left(\sum_{v=1}^{m} \Delta \left(v^{\theta+1} |\Delta\lambda_v| \right) X_v + m^{\theta+1} |\Delta\lambda_m| \, X_m \right) \\
&= O(1) \left(\sum_{v=1}^{m} \left| (v+1)^{\theta+1} \Delta \left(|\Delta\lambda_v| \right) - \Delta v^{\theta+1} |\Delta\lambda_v| \right| X_v + m^{\theta+1} |\Delta\lambda_m| \, X_m \right)
\end{aligned}$$

$$= O(1)\left(\sum_{v=1}^{m} v^{\theta+1}\left|\Delta\left(\left|\Delta\lambda_v\right|\right)\right| X_v + \sum_{v=1}^{m} v^{\theta}\left|\Delta\lambda_v\right| X_v + m^{\theta+1}\left|\Delta\lambda_m\right| X_m\right)$$
$$= O(1) \quad \text{as } m \to \infty,$$

where in the last inequality, (1.15), (3.2) and (3.3) are used.

By (2.2), (1.16) and (1.17), we have

$$\begin{aligned}
\sum_{n=1}^{m} \alpha_n \left|T_{n,2}^{\alpha,\sigma}\right|^k &= O(1) \sum_{n=1}^{m} |\lambda_n|^k \alpha_n \left|\theta_n^{\alpha,\sigma}\right|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta\left(n^{\theta}|\lambda_n|^k\right) \sum_{v=1}^{n} v^{-\theta}\alpha_v \left|\theta_v^{\alpha,\sigma}\right|^k \\
&+ O(1)\, m^{\theta}|\lambda_m|^k \sum_{v=1}^{m} v^{-\theta}\alpha_v \left|\theta_v^{\alpha,\sigma}\right|^k \\
&= O(1)\left(\sum_{n=1}^{m-1} \Delta\left(n^{\theta}|\lambda_n|^k\right) X_n + m^{\theta}|\lambda_m|^k X_m\right) \\
&= O(1) \ \text{ as } m \to \infty.
\end{aligned}$$

Therefore, we get that

$$\sum_{n=1}^{m} \alpha_n \left|T_{n,r}^{\alpha,\sigma}\right|^k = O(1), \text{ as } m \to \infty \text{ for } r = 1, 2.$$

which implies the first result of Theorem 2.

By (c) of Proposition 1, we have the second result of Theorem 2.

3.3 Proof of Corollary 1.

If $\{\alpha_n\}$ is quasi-ϵ-decreasing with ϵ satisfying $(\alpha+\sigma)k+\epsilon > 1$, then

$$\begin{aligned}
\sum_{n=v}^{\infty} \alpha_n n^{-(\alpha+\sigma)k} &= \sum_{n=v}^{\infty} \alpha_n n^{-(\alpha+\sigma)k+\epsilon} n^{-\epsilon} \\
&= O\left(\alpha_v v^{\epsilon}\right) \sum_{n=v}^{\infty} n^{-(\alpha+\sigma)k-\epsilon} \\
&= O\left(\alpha_v v^{-(\alpha+\sigma)k+1}\right),\ v = 1, 2, \cdots, \qquad (3.8)
\end{aligned}$$

which implies (2.1), and thus the results of Theorem 2 hold.

3.4 Proof of Theorem 3.

It can be proved exactly in a way similar to that of Theorem 2, by using Lemma 3 instead of Lemma 2, and using δ_n to replace $|\Delta\lambda_n|$.

3.5 Proof of Corollary 2.

Corollary 2 follows from (3.8) and Theorem 3.

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