New existence results on periodic solutions of nonautonomous second order Hamiltonian systems with (q, p)-Laplacian

Daniel Paşca Zhiyong Wang*

Abstract

Some new existence theorems are obtained for periodic solutions of nonautonomous second order Hamiltonian systems with (q, p)-Laplacian by using the least action principle and the minimax methods.

1 Introduction

In the last two decades many authors starting with Mawhin and Willem (see [2]) proved the existence of solutions for the Hamiltonian systems:

$$\begin{cases} \frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) = \nabla F(t, u(t)) \text{ a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

with $p = 2$ or more general with $p > 1$, under suitable conditions on the potential F (see [8]-[21] and references therein).

*The second author is supported by National Natural Science Foundation of China (Grant No.11026213, 10871096), Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No.10KJB110006) and Foundation of Nanjing University of Information Science & Technology (Grant No.20110421).

Received by the editors February 2012.

Communicated by J. Mawhin.

2010 *Mathematics Subject Classification :*

Key words and phrases : Control functions; Periodic solutions; Hamiltonian systems with (q, p)-Laplacian; The least action principle; Saddle Point Theorem.

Inspired by some of these papers in [1], [3], [4], [5], [6], the authors have considered the extensions to second-order Hamiltonian systems with (q,p)–Laplacian:

$$\begin{cases} \frac{d}{dt}\left(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)\right) = \nabla_{u_1}F(t,u_1(t),u_2(t)), & \text{a.e. } t\in[0,T],\\ \frac{d}{dt}\left(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t)\right) = \nabla_{u_2}F(t,u_1(t),u_2(t)), & \text{a.e. } t\in[0,T],\\ u_1(0)-u_1(T)=\dot{u}_1(0)-\dot{u}_1(T)=0, \\ u_2(0)-u_2(T)=\dot{u}_2(0)-\dot{u}_2(T)=0, \end{cases} \tag{1.2}$$

where $1<p,q<+\infty$, $T>0$, and $F:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ satisfy the following assumption (A):

- F is measurable in t for each $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$;
- F is continuously differentiable in (x_1,x_2) for a.e. $t\in[0,T]$;
- there exist $a_1,a_2\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b\in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x_1,x_2)|,\ |\nabla_{x_1}F(t,x_1,x_2)|,\ |\nabla_{x_2}F(t,x_1,x_2)| \le \left[a_1(|x_1|)+a_2(|x_2|)\right]b(t)$$

for all $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and a.e. $t\in[0,T]$.

The aim of this paper is to obtain new existence results for system 1.2 by imposing a more general growth conditions on the potential F. More precisely we assume that there exist constants $C_i^*>0$ and two positive *control functions* $h_i\in C(\mathbb{R}^+,\mathbb{R}^+)$, $i=1,2$, which satisfied the following restrictions:

(i) $h_i(s)\le h_i(t)$ for all $s\le t$, $s,t\in\mathbb{R}^+$,

(ii) $h_i(s+t)\le C_i^*(h(s)+h(t))$ for all $s,t\in\mathbb{R}^+$,

(iii) $th_1(t)-qH_1(t)\to-\infty$ as $t\to\infty$, where $H_1(t)=\int_0^t h_1(s)ds$,

(iv) $th_2(t)-pH_2(t)\to-\infty$ as $t\to\infty$, where $H_2(t)=\int_0^t h_2(s)ds$,

(v) $\frac{H_1(t)}{t^q}\to 0$ as $t\to+\infty$,

(vi) $\frac{H_2(t)}{t^p}\to 0$ as $t\to+\infty$.

The main results are the following theorems.

Theorem 1.1. *Suppose that F satisfies assumption (A) and the following conditions:*

(H_1) *There exist two positive control functions $h_i\in C(\mathbb{R}^+,\mathbb{R}^+)$ with the properties* (i)-(vi). *Moreover, there exist $f_i,g_i\in L^1(0,T;\mathbb{R}^+)$, $i=1,2$, such that*

$$|\nabla_{x_1}F(t,x_1,x_2)|\le f_1(t)h_1(|x_1|)+g_1(t),$$
$$|\nabla_{x_2}F(t,x_1,x_2)|\le f_2(t)h_2(|x_2|)+g_2(t)$$

for all $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and a.e. $t\in[0,T]$;

(H_2) *There exist two positive control functions $h_i \in C(\mathbb{R}^+, \mathbb{R}^+), i = 1, 2$, which satisfy the conditions* (i)-(vi)*, and assume that*

$$\frac{1}{H_1(|x_1|) + H_2(|x_2|)} \int_0^T F(t, x_1, x_2)dt > 0 \quad \text{as } |x| := \sqrt{|x_1|^2 + |x_2|^2} \to +\infty$$

for a.e. $t \in [0, T]$.

Then problem (1.2) *has at least one solution which minimizes the function φ given by*

$$\varphi(u_1, u_2) := \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T F(t, u_1(t), u_2(t))dt$$

On the Banach space $W := W_T^{1,q} \times W_T^{1,p}$ (details see Section 2).

Remark 1.1. Theorem 1 in [4] are obtained under the following conditions:

$(H_1)'$ There exist $f_i, g_i \in L^1(0, T; \mathbb{R}^+), i = 1, 2$ and $\alpha_1 \in [0, q-1), \alpha_2 \in [0, p-1)$ such that

$$|\nabla_{x_1} F(t, x_1, x_2)| \le f_1(t)|x_1|^{\alpha_1} + g_1(t),$$
$$|\nabla_{x_2} F(t, x_1, x_2)| \le f_2(t)|x_2|^{\alpha_2} + g_2(t)$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$;

$(H_2)'$ $\frac{1}{|x_1|^{q'\alpha_1} + |x_2|^{p'\alpha_2}} \int_0^T F(t, x_1, x_2)dt \to +\infty$ as $|x| = \sqrt{|x_1|^2 + |x_2|^2} \to +\infty$, where q' and p' be such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1.1 generalizes Theorem 1 in [4] partly. Indeed, if we replace $(H_1)'$ with the following more stronger assumption

$(H_1)^*$ There exist $f_i, g_i \in L^1(0, T; \mathbb{R}^+), i = 1, 2$ and $\alpha_1 \in [1/q', q-1), \alpha_2 \in [1/p', p-1)$ such that

$$|\nabla_{x_1} F(t, x_1, x_2)| \le f_1(t)|x_1|^{q'\alpha_1 - 1} + g_1(t),$$
$$|\nabla_{x_2} F(t, x_1, x_2)| \le f_2(t)|x_2|^{p'\alpha_2 - 1} + g_2(t)$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$,

and take the control functions $h_1(t) = t^{q'\alpha_1 - 1}, h_2(t) = t^{p'\alpha_2 - 1}$, then we see that condition (H_2) is much weaker than $(H_2)'$. Theorem 1 in [4] it follows from Theorem 1.1 under assumptions $(H_1)^*$ and (H_2). Moreover, if $q = p = 2$, $F(t, x_1, x_2) = F_1(t, x_1)$, $h_1(t) = t^{q'\alpha_1 - 1}$ with $\alpha_1 \in [1/2, 1)$ and

$(H_1)^{**}$ $|\nabla_{x_1} F(t, x_1)| \le f_1(t)|x_1|^{2\alpha_1 - 1} + g_1(t)$ for all $x_1 \in \mathbb{R}^N$ and a.e. $t \in [0, T]$,

Theorem 1 in [10] it follows also from Theorem 1.1 under assumptions $(H_1)^{**}$ and (H_2). There are functions F satisfying our Theorem 1.1 and not satisfying the results in [4, 10]. For example let

$$F(t, x_1, x_2) = |f(t)| \frac{|x_1|^q + |x_2|^p}{\ln\left(e + |x_1|^2 + |x_2|^2\right)},$$

where $f \in L^1(0,T;\mathbb{R}^+)$. Then, for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and $t \in [0,T]$, one has

$$|\nabla_{x_1} F(t,x_1,x_2)| \le (2+q)|f(t)|\frac{|x_1|^{q-1}}{\ln\left(e+|x_1|^2\right)},$$

$$|\nabla_{x_2} F(t,x_1,x_2)| \le (2+p)|f(t)|\frac{|x_2|^{p-1}}{\ln\left(e+|x_2|^2\right)},$$

which implies that we cannot apply Theorem 1 in [4]. On the other hand, if we take

$$h_1(t) = \frac{t^{q-1}}{\ln\left(e+t^2\right)} \quad \text{and} \quad h_2(t) = \frac{t^{p-1}}{\ln\left(e+t^2\right)},$$

we can see that conditions (H_1) and (H_2) are satisfied. Therefore Theorem 1.1 is a new result.

Theorem 1.2. *Suppose that (H_1) and assumption (A) hold. Assume that*

$$(H_3) \quad \frac{1}{H_1(|x_1|)+H_2(|x_2|)}\int_0^T F(t,x_1,x_2)dt < 0 \quad \text{as } |x| = \sqrt{|x_1|^2+|x_2|^2} \to +\infty$$

for a.e. $t \in [0,T]$.
Then problem (1.2) *has at least one solution in W.*

Remark 1.2. Theorem 1.2 is also a new result. What's more, there are functions F satisfying our Theorem 1.2 and not satisfying the results in [4, 10]. For example let

$$F(t,x_1,x_2) = -|f(t)|\frac{|x_1|^q+|x_2|^p}{\ln\left(e+|x_1|^2+|x_2|^2\right)},$$

where $f \in L^1(0,T;\mathbb{R}^+)$.

2 Preliminaries

For the sake of convenience, in the following we will denote various positive constants as $c_i, i = 1,2,3,\ldots$. Firstly, we introduce some functional spaces. Let $T > 0, 1 < q,p < +\infty$ and use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^N$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0,T;\mathbb{R}^N)$ having a weak derivative $\dot{u} \in L^p(0,T;\mathbb{R}^N)$. The norm in $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} := \left(\int_0^T (|u(t)|^p + |\dot{u}(t)|^p)dt\right)^{\frac{1}{p}}.$$

Furthermore, we use the space W defined by

$$W := W_T^{1,q} \times W_T^{1,p}$$

with the norm $\|(u_1,u_2)\|_W := \|u_1\|_{W_T^{1,q}} + \|u_2\|_{W_T^{1,p}}$. It is clear that W is a reflexive Banach space.

For $u \in W_T^{1,p}$, let $\bar{u} := \frac{1}{T}\int_0^T u(t)dt$ and $\tilde{u}(t) := u(t) - \bar{u}$, then one has

$$\|\tilde{u}\|_\infty \le c_1\|\dot{u}\|_p, \quad \|\tilde{v}\|_\infty \le c_1\|\dot{v}\|_q, \quad \text{(Sobolev's inequality)}$$

$$\|\tilde{u}\|_p \le c_2\|\dot{u}\|_p dt, \quad \|\tilde{v}\|_q \le c_2\|\dot{v}\|_q \quad \text{(Wirtinger's inequality)}$$

for each $u \in W_T^{1,p}, v \in W_T^{1,q}$, where $\|u\|_p := (\int_0^T |u(t)|^p dt)^{\frac{1}{p}}$ and $\|\tilde{u}\|_\infty := \max_{0\le t\le T} |\tilde{u}(t)|$.

It follows from assumption (A) that functional φ on W given by

$$\varphi(u_1, u_2) = \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T F(t, u_1(t), u_2(t))dt$$

is continuously differentiable and weakly lower semicontinuous on W (see [4]). Moreover, one has

$$\begin{aligned}(\varphi'(u_1, u_2), (v_1, v_2)) = \; & \int_0^T (|\dot{u}_1|^{q-2}\dot{u}_1, \dot{v}_1)dt + \int_0^T (|\dot{u}_2|^{p-2}\dot{u}_2, \dot{v}_2)dt \\ & + \int_0^T (\nabla_{(u_1,u_2)}F(t, u_1, u_2), (v_1, v_2))dt\end{aligned}$$

for all $u_i \in W_T^{1,q}, v_i \in W_T^{1,p}, i = 1, 2$. It is well known that the solutions of problem (1.2) correspond to the critical points of the functional φ.

To proof of our main theorems, we need the following auxiliary result.

Proposition 2.1. *Let q' and p' be such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose that there exist two positive functions $h_i(t), i = 1, 2$, which satisfy the conditions* (i), (iii)-(vi) *of* (H_1)*, then we have the following estimates:*

(a) $0 < h_1(t) \le \epsilon_1 t^{q-1} + c_3$ *for any* $\epsilon_1 > 0, t \in \mathbb{R}^+$,
(b) $0 < h_2(t) \le \epsilon_2 t^{p-1} + c_4$ *for any* $\epsilon_2 > 0, t \in \mathbb{R}^+$,
(c) $\frac{h_1^{q'}(t)}{H_1(t)} \to 0$ *as* $t \to +\infty$,
(d) $\frac{h_2^{p'}(t)}{H_2(t)} \to 0$ *as* $t \to +\infty$,
(e) $H_1(t) \to +\infty$ *as* $t \to +\infty$,
(f) $H_2(t) \to +\infty$ *as* $t \to +\infty$.

Proof. We only need to proof the estimates (a), (c), (e), the others are similar. It follows from (v) of (H_1) that, for any $\epsilon_1 > 0$, there exists $M_1 > 0$ such that

$$H_1(t) \le \varepsilon_1 t^q \qquad \forall t \ge M_1.$$

Observe that (iii) of (H_1), there exists $M_2 > 0$ such that

$$th_1(t) - qH_1(t) \le 0 \qquad \forall t \ge M_2,$$

which implies that

$$h_1(t) \le \frac{qH_1(t)}{t} \le q\epsilon_1 t^{q-1} \qquad \forall t \ge M,$$

where $M := \max\{M_1, M_2\}$. Hence we obtain

$$h_1(t) \le q\epsilon_1 t^{q-1} + h_1(M)$$

for all $t > 0$ by (i) of (H_1). Obviously, $h_1(t)$ satisfies (a) due to the definition of $h_1(t)$ and the above inequality.

Next, we turn to (b). Recalling the property (v) of (H_1) and the fact $\frac{1}{q} + \frac{1}{q'} = 1$, we get

$$0 < \frac{{h_1}^{q'}(t)}{H_1(t)} = \frac{{h_1}^{q'}(t)}{{H_1}^{q'}(t)} \cdot {H_1}^{q'-1}(t) \le \left(\frac{q}{t}\right)^{q'} \cdot {H_1}^{q'-1}(t)$$
$$= q^{q'} \cdot \frac{{H_1}^{q'-1}(t)}{t^{q'}} = q^{q'} \left(\frac{H_1(t)}{t^q}\right)^{\frac{1}{q-1}} \to 0 \quad \text{as } t \to +\infty.$$

Therefore, estimate (c) holds.

Finally, we show that (e) is also true. By (iii) of (H_1), one arrives at, for every $L > 0$, there exists $M_3 > 0$ such that

$$th_1(t) - qH_1(t) \le -L \qquad \forall t \ge M_3.$$

So, one has

$$\theta t h_1(\theta t) - qH_1(\theta t) \le -L$$

for all $|\theta t| \ge M_3$. Then we have

$$\frac{d}{d\theta}\left[\frac{H_1(\theta t)}{\theta^q}\right] = \frac{\theta t \cdot h_1(\theta t) - qH_1(\theta t)}{\theta^{q+1}} \le -\frac{L}{\theta^{q+1}} = \frac{d}{d\theta}\left(\frac{L}{q\theta^q}\right).$$

Let $\theta > 1$, integrating both sides of the above inequality from 1 to θ, we obtain

$$\frac{H_1(\theta t)}{\theta^q} - H_1(t) \le \frac{L}{q\theta^q} - \frac{L}{q} = \frac{L}{q}\left(\frac{1}{\theta^q - 1}\right).$$

Let $\theta \to +\infty$ in the above inequality, and by (v) of (H_1), one has

$$H_1(t) \ge \frac{L}{q}$$

for all $t \ge M_3$. By the arbitrariness of L, we have

$$H_1(t) \to +\infty \qquad \text{as } t \to +\infty,$$

which completes the proof. ∎

3 Proof of main results

Now, we are ready to proof our main results.
Proof of Theorem 1.1. It follows from (H_1) and Sobolev's inequality that

$$\begin{aligned}
&\left|\int_0^T [F(t,u_1(t),u_2(t) - F(t,\bar{u}_1,\bar{u}_2)]dt\right| \\
&\le \left|\int_0^T [F(t,u_1(t),u_2(t) - F(t,u_1(t),\bar{u}_2)]dt\right| + \left|\int_0^T F(t,u_1(t),\bar{u}_2) - F(t,\bar{u}_1,\bar{u}_2)]dt\right| \\
&= \left|\int_0^T \int_0^1 (\nabla_{x_2}F(t,u_1(t),\bar{u}_2 + s_2\tilde{u}_2(t)),\tilde{u}_2(t))dsdt\right| \\
&\quad + \left|\int_0^T \int_0^1 (\nabla_{x_1}F(t,\bar{u}_1 + s_1\tilde{u}_1(t),\bar{u}_2),\tilde{u}_1(t))dsdt\right| \\
&\le \int_0^T f_2(t)h_2(|\bar{u}_2| + |\tilde{u}_2(t)|)|\tilde{u}_2(t)|dt + \int_0^T g_2(t)|\tilde{u}_2(t)|dt + \int_0^T g_1(t)|\tilde{u}_1(t)|dt \\
&\quad + \int_0^T f_1(t)h_1(|\bar{u}_1| + |\tilde{u}_1(t)|)|\tilde{u}_1(t)|dt \\
&\le \int_0^T f_2(t)[C_2^*(h_2(|\bar{u}_1|) + h_2(|\tilde{u}_2(t)|)]|\tilde{u}_2(t)|dt + \|\tilde{u}_2\|_\infty \int_0^T g_2(t)dt \\
&\quad + \int_0^T f_1(t)[C_1^*(h_1(|\bar{u}_1|) + h_2(|\tilde{u}_2(t)|)]|\tilde{u}_1(t)|dt + \|\tilde{u}_1\|_\infty \int_0^T g_1(t)dt \\
&\le C_2^*[h_2(|\bar{u}_2|) + h_2(|\tilde{u}_2(t)|)]\|\tilde{u}_2\|_\infty \int_0^T f_2(t)dt + \|\tilde{u}_2\|_\infty \int_0^T g_2(t)dt \\
&\quad + C_1^*[h_1(|\bar{u}_1|) + h_1(|\tilde{u}_1(t)|)]\|\tilde{u}_1\|_\infty \int_0^T f_1(t)dt + \|\tilde{u}_1\|_\infty \int_0^T g_1(t)dt \\
&\le C_2^*\left[\frac{1}{2pC_2^*c_1^p}\|\tilde{u}_2\|_\infty^p + 2pC_2^*c_1^p h_2{}^{p'}(|\bar{u}_2|)\left(\int_0^T f_2(t)dt\right)^{p'}\right] + \|\tilde{u}_2\|_\infty \int_0^T g_2(t)dt \\
&\quad C_2^*h_2(\|\tilde{u}_2\|_\infty)\|\tilde{u}_2\|_\infty \int_0^T f_2(t)dt + \|\tilde{u}_1\|_\infty \int_0^T g_1(t)dt + C_1^*h_1(\|\tilde{u}_1\|_\infty)\|\tilde{u}_1\|_\infty \int_0^T f_1(t)dt \\
&\quad + C_1^*\left[\frac{1}{2qC_1^*c_1^q}\|\tilde{u}_1\|_\infty^q + 2qC_1^*c_1^q h_1{}^{q'}(|\bar{u}_1|)\left(\int_0^T f_1(t)dt\right)^{q'}\right] \\
&\le \frac{1}{2p}\|\dot{u}_2\|_p^p + c_5h_2{}^{p'}(|\bar{u}_2|) + c_6\|\dot{u}_2\|_p + C_2^*[\epsilon_2\|\tilde{u}_2\|_\infty^{p-1} + c_4]\|\tilde{u}_2\|_\infty \int_0^T f_2(t)dt \\
&\qquad + \frac{1}{2q}\|\dot{u}_1\|_q^q \quad + c_7h_1{}^{q'}(|\bar{u}_1|) + c_8\|\dot{u}_1\|_q + C_1^*[\epsilon_1\|\tilde{u}_1\|_\infty^{q-1} + c_3]\|\tilde{u}_1\|_\infty \int_0^T f_1(t)dt \\
&\le \left(\frac{1}{2q} + \epsilon_1 c_{10}\right)\|\dot{u}_1\|_q^q + c_7h_1{}^{q'}(|\bar{u}_1|) + c_9\|\dot{u}_1\|_q + \left(\frac{1}{2p} + \epsilon_2 c_{12}\right)\|\dot{u}_2\|_p^p \\
&\quad + c_5h_2{}^{p'}(|\bar{u}_2|) + c_{11}\|\dot{u}_2\|_p. \qquad (3.1)
\end{aligned}$$

Hence, we see that

$$\begin{aligned}\varphi(u_1,u_2) &= \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T [F(t,u_1(t),u_2(t)) \\ &\quad - F(t,\bar{u}_1,\bar{u}_2)]dt + \int_0^T F(t,\bar{u}_1,\bar{u}_2)dt \\ &\geq \left(\frac{1}{2q} - \epsilon_1 c_{10}\right)\|\dot{u}_1\|_q^q - c_9\|\dot{u}_1\|_q + \left(\frac{1}{2p} - c_{12}\epsilon_2 c_{12}\right)\|\dot{u}_2\|_p^p - c_{11}\|\dot{u}_2\|_p \\ &\quad + (H_1(|\bar{u}_1|) + H_2(|\bar{u}_2|))\left[\frac{1}{H_1(|\bar{u}_1|) + H_2(|\bar{u}_2|)}\int_0^T F(t,\bar{u}_1,\bar{u}_2)dt \right. \\ &\quad \left. - c_7\frac{{h_1}^{q'}(|\bar{u}_1|)}{H_1(|\bar{u}_1|) + H_2(|\bar{u}_2|)} - c_5\frac{{h_2}^{p'}(|\bar{u}_2|)}{H_1(|\bar{u}_1|) + H_2(|\bar{u}_2|)}\right]. \end{aligned} \tag{3.2}$$

By Proposition 2.1, we observe that

$$\frac{{h_1}^{q'}(|\bar{u}_1|)}{H_1(|\bar{u}_1|) + H_2(|\bar{u}_2|)} \to 0, \quad \frac{{h_2}^{p'}(|\bar{u}_2|)}{H_1(|\bar{u}_1|) + H_2(|\bar{u}_2|)} \to 0 \quad \text{as } \sqrt{|\bar{u}_1|^2 + |\bar{u}_2|^2} \to +\infty.$$

These together with (3.2), by (H_2) and Proposition 2.1, for ϵ_1, ϵ_2 small enough, one has

$$\varphi(u_1,u_2) \to +\infty \quad \text{as } \|(u_1,u_2)\|_W \to +\infty.$$

Then, by the least action principle,, problem (1.2) has at least one solution which minimizes the function φ. ■

Proof of Theorem 1.2. First we prove that φ satisfies the (PS) condition. Suppose that $\{(u_{1n},u_{2n})\} \subset W$ is a (PS) sequence for φ, that is, $\varphi'(u_{1n},u_{2n}) \to 0$ as $n \to +\infty$ and $\{\varphi(u_{1n},u_{2n})\}$ is bounded. In a way similar to the proof of Theorem 1.1, we have

$$\begin{aligned}&\left|\int_0^T (\nabla_{x_1}F(t,u_{1n}(t),u_{2n}(t)),\tilde{u}_{1n}(t))dt + \int_0^T (\nabla_{x_2}F(t,u_{1n}(t),u_{2n}(t)),\tilde{u}_{2n}(t))dt\right| \\ &\leq \left|\int_0^T (\nabla_{x_1}F(t,u_{1n}(t),u_{2n}(t)),\tilde{u}_{1n}(t))dt\right| + \left|\int_0^T (\nabla_{x_2}F(t,u_{1n}(t),u_{2n}(t)),\tilde{u}_{2n}(t))dt\right| \\ &\leq \left(\frac{1}{2q} + \epsilon_1 c_{10}\right)\|\dot{u}_1\|_q^q + c_7{h_1}^{q'}(|\bar{u}_1|) + c_9\|\dot{u}_1\|_q + \left(\frac{1}{2p} + \epsilon_2 c_{12}\right)\|\dot{u}_2\|_p^p \\ &\quad + c_5{h_2}^{p'}(|\bar{u}_2|) + c_{11}\|\dot{u}_2\|_p\end{aligned}$$

for all n. Hence, we get

$$\begin{aligned}\|(\tilde{u}_{1n},\tilde{u}_{2n})\|_W &\geq (\varphi'(u_{1n},u_{2n}),(\tilde{u}_{1n},\tilde{u}_{2n})) \\ &= \int_0^T \big[(\nabla_{x_1}F(t,u_{1n}(t),u_{2n}(t)),\tilde{u}_{1n}(t)) + (|\dot{u}_{1n}(t)|^{q-2}\dot{u}_{1n}(t),\dot{u}_{1n}(t)) \\ &\quad + (\nabla_{x_2}F(t,u_{1n}(t),u_{2n}(t)),\tilde{u}_{2n}(t)) + (|\dot{u}_{2n}(t)|^{p-2}\dot{u}_{2n}(t),\dot{u}_{2n}(t))\big]dt \\ &\geq \left(1 - \frac{1}{2q} - \epsilon_1 c_{10}\right)\|\dot{u}_{1n}\|_q^q - c_7{h_1}^{q'}(|\bar{u}_1|) - c_9\|\dot{u}_1\|_q \\ &\quad + \left(1 - \frac{1}{2p} - \epsilon_2 c_{12}\right)\|\dot{u}_{2n}\|_p^p - c_5{h_2}^{p'}(|\bar{u}_2|) - c_{11}\|\dot{u}_2\|_p\end{aligned} \tag{3.3}$$

for large n. On the other hand, it follows from Wirtinger's inequality that

$$\|(\tilde{u}_{1n},\tilde{u}_{2n})\|_W = \|\tilde{u}_{1n}\|_{W_T^{1,q}} + \|\tilde{u}_{2n}\|_{W_T^{1,p}} \le (1+c_2^q)^{\frac{1}{q}}\|\dot{u}_{1n}\|_q + (1+c_2^p)^{\frac{1}{p}}\|\dot{u}_{2n}\|_p$$
$$:= c_{13}\|(\dot{u}_{1n},\dot{u}_{2n})\|_{L^q\times L^p} \tag{3.4}$$

for all n. Combing (3.3) with (3.4), we obtain

$$c_{13}\|(\dot{u}_{1n},\dot{u}_{2n})\|_{L^q\times L^p} \ge \left(1-\frac{1}{2q}-\epsilon_1 c_{10}\right)\|\dot{u}_{1n}\|_q^q - c_7 h_1^{q'}(|\bar{u}_1|) - c_9\|\dot{u}_1\|_q$$
$$+\left(1-\frac{1}{2p}-\epsilon_2 c_{12}\right)\|\dot{u}_{2n}\|_p^p - c_5 h_2^{p'}(|\bar{u}_2|) - c_{11}\|\dot{u}_2\|_p,$$

for ϵ_1,ϵ_2 small enough, which implies that

$$c_{14}[h_1^{q'}(|\bar{u}_{1n}|) + h_2^{p'}(|\bar{u}_{2n}|) + 1] \ge \|\dot{u}_{1n}\|_q^q + \|\dot{u}_{2n}\|_p^p \tag{3.5}$$

for all large n. By the proof of (3.1), we also have

$$\int_0^T [F(t,u_{1n}(t),u_{2n}(t)) - F(t,\bar{u}_{1n},\bar{u}_{2n})]dt$$
$$\le \left(\frac{1}{2q}+\epsilon_1 c_{10}\right)\|\dot{u}_1\|_q^q + c_7 h_1^{q'}(|\bar{u}_1|) + c_9\|\dot{u}_1\|_q + \left(\frac{1}{2p}+\epsilon_2 c_{12}\right)\|\dot{u}_2\|_p^p$$
$$+ c_5 h_2^{p'}(|\bar{u}_2|) + c_{11}\|\dot{u}_2\|_p. \tag{3.6}$$

Thus, by (3.5), (3.6), Proposition 2.1 and (H_3), one has

$$\varphi(u_{1n},u_{2n}) = \frac{1}{q}\int_0^T |\dot{u}_{1n}|^q dt + \frac{1}{p}\int_0^T |\dot{u}_{2n}|^p dt$$
$$+ \int_0^T [F(t,u_{1n}(t),u_{2n}(t)) - F(t,\bar{u}_{1n},\bar{u}_{2n})]dt + \int_0^T F(t,\bar{u}_{1n},\bar{u}_{2n})dt$$
$$\le \left(\frac{3}{2q}+\epsilon_1 c_{10}\right)\|\dot{u}_{1n}\|_q^q + \left(\frac{3}{2p}+\epsilon_2 c_{12}\right)\|\dot{u}_{2n}\|_p^p + c_9\|\dot{u}_{1n}\|_q$$
$$+ c_{11}\|\dot{u}_{2n}\|_p + c_7 h_1^{q'}(|\bar{u}_{1n}|) + c_5 h_2^{p'}(|\bar{u}_{2n}|) + \int_0^T F(t,\bar{u}_{1n},\bar{u}_{2n})dt$$
$$\le c_{15}\left[h_1^{q'}(|\bar{u}_{1n}|) + h_2^{p'}(|\bar{u}_{2n}|) + 1\right] + c_{16}\left[h_1^{q'}(|\bar{u}_{1n}|)\right.$$
$$\left.+h_2^{p'}(|\bar{u}_{2n}|) + 1\right]^{\frac{1}{q}} + C_{17}\left[h_1^{q'}(|\bar{u}_{1n}|) + h_2^{p'}(|\bar{u}_{2n}|) + 1\right]^{\frac{1}{p}}$$
$$+ c_7 h_1^{q'}(|\bar{u}_{1n}|) + c_5 h_2^{p'}(|\bar{u}_{2n}|) + \int_0^T F(t,\bar{u}_{1n},\bar{u}_{2n})dt$$
$$\le c_{18} h_1^{q'}(|\bar{u}_{1n}|) + c_{19} h_2^{p'}(|\bar{u}_{2n}|) + c_{20} h_1^{\frac{q'}{q}}(|\bar{u}_{1n}|) + c_{21} h_2^{\frac{p'}{q}}(|\bar{u}_{2n}|)$$
$$+ c_{22} h_1^{\frac{q'}{p}}(|\bar{u}_{1n}|) + c_{23} h_2^{\frac{p'}{p}}(|\bar{u}_{2n}|) + c_{24} + \int_0^T F(t,\bar{u}_{1n},\bar{u}_{2n})dt$$

$$= (H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|))\Big[\frac{c_{18}{h_1}^{q'}(|\bar{u}_{1n}|)}{H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|)} + \frac{c_{19}{h_2}^{p'}(|\bar{u}_{2n}|)}{H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|)}$$

$$+ \frac{c_{20}{h_1}^{\frac{q'}{q}}(|\bar{u}_{1n}|)}{H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|)} + \frac{c_{21}{h_2}^{\frac{p'}{q}}(|\bar{u}_{2n}|)}{H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|)}$$

$$+ \frac{c_{22}{h_1}^{\frac{q'}{p}}(|\bar{u}_{1n}|)}{H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|)} + \frac{c_{23}{h_2}^{\frac{p'}{p}}(|\bar{u}_{2n}|)}{H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|)}$$

$$+ \frac{c_{24}}{H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|)} + \frac{\int_0^T F(t, \bar{u}_{1n}, \bar{u}_{2n})dt}{H_1(|\bar{u}_{1n}|) + H_2(|\bar{u}_{2n}|)}\Big],$$

note $p, q > 1$, which implies that

$$\varphi(u_{1n}, u_{2n}) \to -\infty \quad \text{as } \sqrt{|\bar{u}_{1n}|^2 + |\bar{u}_{2n}|^2} \to +\infty. \tag{3.7}$$

This contradicts the boundedness of $\{\varphi(u_{1n}, u_{2n})\}$. So, $\{|\bar{u}_{1n}|^2 + |\bar{u}_{2n}|^2\}$ is bounded, by (3.5), we know $\{(u_{1n}, u_{2n})\}$ is bounded. Using the same arguments of [4], we conclude that the (PS) condition is satisfied.

Let $\tilde{W} := \tilde{W}_T^{1,q} \times \tilde{W}_T^{1,p}$ be the subspace of W given by

$$\tilde{W} : \{(u_1, u_2) \in W |\ (\bar{u}_1, \bar{u}_2) = (0, 0)\}.$$

Then, for $(u_1, u_2) \in \tilde{W}$, we have

$$\varphi(u_1, u_2) \to +\infty \quad \text{as } \|(u_1, u_2)\|_W \to +\infty. \tag{3.8}$$

Indeed, for $D_1, D_2 > 0$ and ϵ_1, ϵ_2 small enough, by the proof of (3.6), we get

$$\begin{aligned}
\varphi(u_1, u_2) &= \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt \\
&\quad + \int_0^T [F(t, u_1(t), u_2(t)) - F(t, D_1, D_2)]dt + \int_0^T F(t, D_1, D_2)dt \\
&\geq \left(\frac{1}{2q} - \epsilon_1 c_{10}\right)\|\dot{u}_1\|_q^q - c_7{h_1}^{q'}(D_1) - c_9\|\dot{u}_1\|_q + \left(\frac{1}{2p} - \epsilon_2 c_{12}\right)\|\dot{u}_2\|_p^p \\
&\quad - c_5{h_2}^{p'}(D_2) - c_{11}\|\dot{u}_2\|_p + \int_0^T F(t, D_1, D_2)dt \\
&\geq c_{25}\|\dot{u}_1\|_q^q - c_9\|\dot{u}_1\|_q - c_{26}\|\dot{u}_2\|_p^p - c_{11}\|\dot{u}_2\|_p - c_{27}
\end{aligned}$$

for all $(u_1, u_2) \in \tilde{W}$. By the Wirtinger's inequality, the norm

$$\|(u_1, u_2)\| = \|(\dot{u}_1, \dot{u}_2)\|_{L^q \times L^p} = \|\dot{u}_1\|_q + \|\dot{u}_2\|_p$$

is an equivalent norm on $\tilde{W}$. Hence, (3.8) follows from the equivalence and the above inequality.

On the other hand, by (H_3) and Proposition 2.1, we have

$$\varphi(u_1, u_2) \to -\infty \quad \text{as } |(u_1, u_2)| \to +\infty \text{ in } \mathbb{R}^N \times \mathbb{R}^N.$$

Then, by Saddle Point Theorem [7], problem (1.2) has at least one solution in W, and the proof hereby is complete. ■

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Department of Mathematics and Informatics,
University of Oradea,
University Street 1, 410087 Oradea, Romania
email:dpasca@uoradea.ro

Department of Mathematics,
Nanjing University of Information Science & Technology,
Nanjing 210044, Jiangsu, People's Republic of China
email:mathswzhy1979@gmail.com