# Growth of solutions of some higher order linear difference equations*

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**Abstract**

This paper is devoted to studying the growth of solutions of equations of type $f(z+n) + \sum_{j=0}^{n-1}\{P_j(e^z) + Q_j(e^{-z})\}f(z+j) = 0$ and $f(z+n) + \sum_{j=0}^{n-1}\{P_j(e^{A(z)}) + Q_j(e^{-A(z)})\}f(z+j) = 0$, where $P_j(z)$ and $Q_j(z)$ are polynomials in $z$ and $A(z)$ is a transcendental entire function. We prove three theorems of such type, which improve some results in [6, 7].

## 1 Introduction

In this paper, a meromorphic function will mean meromorphic in the whole complex plane, and we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions(e.g. see [12, 24]). Let $\eta$ be a fixed, non-zero complex number, $\Delta f(z) = f(z+\eta) - f(z)$, and $\Delta^n f(z) = \Delta(\Delta^{n-1} f(z))$. In addition, we use $\sigma(f)$ and $\sigma_2(f)$ to denote the order and the hyper-order of a meromorphic function $f(z)$ respectively, and we denote by $\lambda(f)$ and $\lambda(\frac{1}{f})$ the exponent of convergence of zeros and poles of $f(z)$, respectively.

The foundation of the theory of complex difference equations was laid by Batchelder [2], Nörlund [17], and Whittaker [19] in the early twentieth century. Later on, Shimomura [18] and Yanagihara [21, 22, 23] studied nonlinear complex

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difference equations from the viewpoint of Nevanlinna theory. Meromorphic solutions of complex difference equations have become a subject of some interest recently, due to the fact that the existence of finite order solutions is a good detector of integrability of difference equations. In such considerations, Nevanlinna theory appears to be a powerful tool.

Difference counterparts of Nevanlinna theory have been established very recently. The key result is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [10] and Chiang-Feng [7], independently. Halburd and Korhonen [11] also established a version of Nevanlinna theory for difference operators. Bergweiler and Langley [3] considered the value distribution of difference operators of slowly growing meromorphic functions. Some new results can be seen in [5, 13, 14, 20].

In 2008, Chiang and Feng [7] obtained the following results concerning the growth of solutions of linear difference equations.

**Theorem A**. *Let $P_0(z), \ldots, P_n(z)$ be polynomials in $z$ such that there exists an integer $l(0 \le l \le n)$ that satisfies*

$$\deg(P_l) > \max\{\deg(P_j)\}, \quad 0 \le l \le n \quad and \quad j \ne l.$$

*Suppose that $f(z)$ is a meromorphic solution of the difference equation*

$$P_n(z)y(z+n) + \cdots + P_1(z)y(z+1) + P_0(z)y(z) = 0.$$

*Then we have $\sigma(f) \ge 1$.*

**Theorem B**. *Let $A_0(z), \ldots, A_n(z)$ be entire functions such that there exists an integer $l(0 \le l \le n)$ that satisfies*

$$\sigma(A_l) > \max\{\sigma(A_j)\}, \quad 0 \le l \le n \quad and \quad j \ne l. \tag{1.1}$$

*If $f(z)$ is a meromorphic solution of the difference equation*

$$A_n(z)y(z+n) + \cdots + A_1(z)y(z+1) + A_0(z)y(z) = 0,$$

*then $\sigma(f) \ge \sigma(A_l) + 1$.*

**Example**. $f(z) = e^{z^2}$ solves the difference equation

$$f(z+2) + (e^z + e^{-3z})f(z+1) - (e^{4z+4} + e^{3z+1} + e^{-z+1})f(z) = 0. \tag{1.2}$$

Denote $P_0(\zeta) = -e^4\zeta^4 - e\zeta^3 - e\zeta^{-1}$ and $P_1(\zeta) = \zeta + \zeta^{-3}$. It is obvious that the coefficients $P_1(e^z) = e^z + e^{-3z}$ and $P_0(e^z) = -(e^{4z+4} + e^{3z+1} + e^{-z+1})$ of (1.2) are transcendental entire functions which do not satisfy (1.1). By the further computation, we have $\sigma(P_1(e^z)) = \sigma(P_0 e^z) = 1$, $\deg P_0 > \deg P_1$, and $\sigma(f) = \lambda(f-a) = 2$ for every non-zero value $a \in \mathbb{C}$.

The above example suggests us to consider the following difference equation with periodic and transcendental coefficients that do not satisfy the assumption (1.1) of Theorem B

$$f(z+n) + \sum_{j=0}^{n-1}\{P_j(e^z) + Q_j(e^{-z})\}f(z+j) = 0, \tag{1.3}$$

where $P_j(z)$ and $Q_j(z)$ $(j = 0, 1, \ldots, n-1)$ are polynomials in $z$. We obtain the following results.

**Theorem 1**. *Let $P_j(z)$ and $Q_j(z)$ $(j = 0, 1, \ldots, n-1)$ be polynomials that satisfy*

$$\deg(P_0) > \deg(P_j) \quad or \quad \deg(Q_0) > \deg(Q_j), \quad j = 1, \ldots, n-1.$$

*Then, each non-trivial meromorphic solution $f(z)$ of finite order of the difference equation (1.3) satisfies $\sigma(f) = \lambda(f - a) \geq 2$, and so $f$ assumes every non-zero complex value $a \in \mathbb{C}$ infinitely often.*

**Theorem 2**. *Suppose that the assumptions of Theorem 1 are satisfied. If $f(z)$ is a non-trivial entire solution of finite order of the equation (1.3) that satisfies $\lambda(f) < 1$, then $\sigma(f) = 2$.*

**Remark 1.1**. The example mentioned below Theorem B shows that Theorem 1 is sharp. It is also shown that the conclusion both in Theorem 1 and Theorem 2 may occur.

The coefficients of (1.3) are periodic, if the periodic function $e^z$ and $e^{-z}$ are replaced by $e^{A(z)}$ and $e^{-A(z)}$, where $A(z)$ is a transcendental entire function ($e^{A(z)}$ is certainly not periodic in general), then we have the following result.

**Theorem 3**. *Let $P_j(z)$, $Q_j(z)$ $(j = 0, 1, \ldots, n-1)$ be polynomials in $z$ and $A(z)$ be a transcendental entire function. If*

$$\deg(P_0) > \deg(P_j) \quad or \quad \deg(Q_0) > \deg(Q_j), \quad j = 1, \ldots, n-1,$$

*then every solution of the difference equation*

$$f(z+n) + \sum_{j=0}^{n-1} \{P_j(e^{A(z)}) + Q_j(e^{-A(z)})\} f(z+j) = 0 \tag{1.4}$$

*is of infinite order and $\sigma_2(f) \geq \sigma(A)$.*

**Corollary 1**. *Let $P_j(z)$, $Q_j(z)(j = 0, 1, \ldots, n-1)$ be polynomials in $z$. If*

$$\deg(P_0) > \deg(P_j) \quad or \quad \deg(Q_0) > \deg(Q_j), \quad j = 1, \ldots, n-1,$$

*then every finite order solution of the $n$-th difference equation*

$$\Delta^n f + \sum_{j=1}^{n-1} \{P_j(e^z) + Q_j(e^{-z})\} \Delta^j f(z) + \{P_0(e^z) + Q_0(e^{-z})\} f(z) = 0 \tag{1.5}$$

*satisfies $\sigma(f) = \lambda(f - a) \geq 2$, and so $f(z)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

**Corollary 2**. *Let $P_j(z)$, $Q_j(z)(j = 0, 1, \ldots, n-1)$ be polynomials in $z$ and $A(z)$ be a transcendental entire function. If*

$$\deg(P_0) > \deg(P_j) \quad or \quad \deg(Q_0) > \deg(Q_j), \quad j = 1, \ldots, n-1,$$

*then every solution $f(z)$ of the $n$-th difference equation*

$$\Delta^n f + \sum_{j=1}^{n-1} \{P_j(e^{A(z)}) + Q_j(e^{-A(z)})\} \Delta^j f(z) + \{P_0(e^{A(z)}) + Q_0(e^{-A(z)})\} f(z) = 0 \tag{1.6}$$

*is of infinite order and $\sigma_2(f) \geq \sigma(A)$.*

## 2 Some Lemmas

**Lemma 2.1**[7]. *Let $f(z)$ be a meromorphic function, $\eta$ be a non-zero complex number, and let $\tau > 1$, and $\varepsilon > 0$ be given real constants, then there exits a subset $E \subset (1, \infty)$ of finite logarithmic measure,*

(1) *and a constant $A$ depending only on $\tau$ and $\eta$, such that for all $|z| \notin [0,1] \cup E$, we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\tau r, f)}{r} + \frac{n(\tau r)}{r}\log^{\tau} r \log n(\tau r)\right),$$

*where $n(t) = n(t, f) + n(t, \frac{1}{f})$;*

(2) *and if in addition that $f(z)$ is of finite order $\sigma$, and such that for all $|z| = r \notin [0,1] \cup E$, we have*

$$\exp(-r^{\sigma-1+\varepsilon}) \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp(r^{\sigma-1+\varepsilon}).$$

**Lemma 2.2**[1]. *Let $f(z)$ be a holomorphic function in $|z| \leq R$. We use $M(r, f)$ and $D(r, f)$ to denote the the maximum modulus of $f(z)$ and maximum value of $Ref(z)$ on $|z| = r$, respectively. Then*

$$M(r, f) \leq \frac{2r}{R-r}D(R, f) + \frac{R+r}{R-r}|f(0)|,$$

*where $0 < r < R$. In particular, when $R = 2r$, we have*

$$D(R, f) \geq \frac{M(r, f)}{2} - \frac{3}{2}|f(0)|.$$

**Lemma 2.3**[15]. *Let $w(z)$ be a non-constant finite order meromorphic solution of $P(z, w) = 0$, where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r, a) = S(r, w)$, then*

$$m(r, \frac{1}{w-a}) = S(r, w).$$

**Lemma 2.4**[8]. *Let $f(z)$ be a meromorphic function with finite order $\sigma(f)$, $\eta \in C$. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ of $|z| = r$ of finite logarithmic measure, so that*

$$\frac{f(z+\eta)}{f(z)} = \exp\{\eta\frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon})\},$$

*holds for $r \notin [0,1] \cup E$. If $\lambda < 1$, $\beta = \max\{\sigma-2,\ 2\lambda-2\}$; and if $\lambda \geq 1$, $\beta = \max\{\sigma-2,\ \lambda-1\}$, where $\lambda = \max\{\lambda(f), \lambda(\frac{1}{f})\}$.*

**Remark 2.1**. The term $O(r^{\beta+\varepsilon})$ in Lemma 2.5 can be replaced by $o(r^{\sigma-1-\varepsilon})$ provided that $\lambda < 1$ and $\sigma > \lambda + 1$ for $0 < \varepsilon < \frac{1}{2}$. In fact, if $\sigma > \lambda + 1$ and $\lambda < 1$, then $\sigma - 2 > \lambda + 1 - 2 = \lambda - 1 > 2\lambda - 2$. It follows that $\beta = \sigma - 2$. Thus

$$O(r^{\beta+\varepsilon}) = o(r^{\sigma-1-\varepsilon}).$$

**Lemma 2.5** [9]. *Let $f(z)$ be a meromorphic function with $\sigma(f)$, then for any given $\varepsilon > 0$, there exists a set $E \subset (1,\infty)$ of finite linear measure, such that for all $|z| = r \notin [0,1] \cup E$, and $r$ sufficiently large,*

$$\exp\{-r^{\sigma+\varepsilon}\} \le |f(z)| \le \exp\{r^{\sigma+\varepsilon}\}.$$

**Lemma 2.6** [4]. *Suppose that $f(z)$ is a transcendental entire function with finite order $\sigma(f)$, and a set $E \subset (1,\infty)$ has a finite logarithmic measure and $G = \{\varphi_1,\dots,\varphi_n\} \subset [0,2\pi)$. Then there exists a positive number $A \in [\frac{1}{2},1]$, a sequence of points: $z_k = r_k e^{i\theta_k}$ with $|f(z_k)| \ge AM(r_k,f), \theta_k \in [0,2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0,2\pi)/G$, and a sequence of points $r_k \notin E, r_k \to \infty$ such that for any given $\varepsilon > 0$, as $r_k$ sufficiently large, we have*

$$r_k^{\sigma-\varepsilon} < v(r_k,f) < r_k^{\sigma+\varepsilon}.$$

**Lemma 2.7** [16]. *Let*

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

*where $n$ is a positive integer and $a_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0,2\pi)$. For any given $0 < \varepsilon < \frac{\pi}{4n}$, consider $2n$ open angles:*

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon, \quad j = 0,\dots,2n-1.$$

*Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$, when $z \in S_j$ and $j$ is even,*

$$Re\{Q(z)\} > \alpha_n (1-\varepsilon)\sin(n\varepsilon) r^n,$$

*when $z \in S_j$ and $j$ is odd,*

$$Re\{Q(z)\} < -\alpha_n (1-\varepsilon)\sin(n\varepsilon) r^n.$$

## 3 Proof of Theorems

Throughout this section, we assume that

$$P_j(z) = a_{jm_j} z^{m_j} + a_{jm_{j-1}} z^{m_{j-1}} + \cdots + a_{j_1} z + a_{j_0},$$

$$Q_j(z) = b_{jn_j} z^{n_j} + b_{jn_{j-1}} z^{n_{j-1}} + \cdots + b_{j_1} z + b_{j_0},$$

where $a_{jl}, b_{jk} (j = 0,1,\dots,n-1; l = 0,1,\dots,m_j; k = 0,1,\dots,n_j)$ are the constants and $a_{jm_j} b_{jn_j} \neq 0$.

**Proof of Theorem 1**. Assume that $\sigma(f) = \sigma < \infty$. By Lemma 2.1, we know that, for any given $\varepsilon > 0$, there exits a subset $E \subset (1,\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E$, we have

$$\exp(-r^{\sigma-1+\varepsilon}) \le \left|\frac{f(z+j)}{f(z)}\right| \le \exp(r^{\sigma-1+\varepsilon}), \quad j = 1,2,\dots,n-1. \tag{3.1}$$

$$\exp(-r^{\sigma-1+\varepsilon}) \leq \left|\frac{f(z+n)}{f(z)}\right| \leq \exp(r^{\sigma-1+\varepsilon}). \tag{3.2}$$

From the assumption of Theorem 1, we first assume that

$$\deg(P_0) > \deg(P_j)(j = 1, 2, \ldots, n-1)$$

without loss of generality. Let $f \not\equiv 0$ be a solution of the equation (1.3), and let $z = r$. Then, we obtain from (1.3), (3.1) and (3.2), for all sufficiently large $r$ and $r \notin [0,1] \cup E$, that

$$\begin{aligned}
|P_0(e^z) + Q_0(e^{-z})| &= |a_{0m_0}|e^{m_0 r}(1+o(1)) \\
&\leq \left|\frac{f(z+n)}{f(z)}\right| + |P_{n-1}(e^z) + Q_{n-1}(e^{-z})| \left|\frac{f(z+n-1)}{f(z)}\right| + \cdots \\
&\quad + |P_1(e^z) + Q_1(e^{-z})| \left|\frac{f(z+1)}{f(z)}\right| \\
&\leq \exp(r^{\sigma-1+\varepsilon}) + (|a_{n-1m_{n-1}}| + M_{n-1})e^{m_{n-1}r}\exp(r^{\sigma-1+\varepsilon})(1+o(1)) + \cdots \\
&\quad + (|a_{1m_1}| + M_1)e^{m_1 r}\exp(r^{\sigma-1+\varepsilon})(1+o(1)) \\
&\leq \exp(r^{\sigma-1+\varepsilon}) 2nMe^{\max\{m_1,\ldots,m_{n-1}\}r}(1+o(1)),
\end{aligned}$$

and $M = \max\{|a_{n-1m_{n-1}}|, \ldots, |a_{1m_1}|, M_{n-1}, \ldots, M_1\}, 1$, where $M_j$ are real constants, $(j = 1, 2, ..., n-1)$. Since $m_0 > \max\{m_1, \ldots, m_{n-1}\}$, we have

$$\frac{|a_{0m_0}|}{2nM}e^r(1+o(1)) \leq e^{r^{\sigma-1+\varepsilon}}. \tag{3.3}$$

We deduce from (3.3) that $\sigma - 1 + \varepsilon \geq 1$ which implies $\sigma(f) \geq 2$.

If deg $Q_0 >$ deg $Q_j$, by taking a suitable $z = -r$ and using the similar arguments mentioned above, we also get $\sigma(f) \geq 2$.

Now let $a \in \mathbb{C}\backslash\{0\}$, and set

$$P(z,f) = f(z+n) + \sum_{j=1}^{n-1}[P_j(e^z) + Q_j(e^{-z})]f(z+j).$$

It is obvious that

$$P(z,a) = a[1 + P_{n-1}(e^z) + Q_{n-1}(e^{-z}) + \cdots + P_0(e^z) + Q_0(e^{-z})] \not\equiv 0. \tag{3.4}$$

By (3.4) and Lemma 2.3, it follows that

$$m(r, \frac{1}{f-a}) = S(r,f),$$

thus

$$N(r, \frac{1}{f-a}) = T(r,f) + S(r,f),$$

and we deduce that $\lambda(f-a)=\sigma(f)$. Theorem 1 is thus proved.

**Proof of Theorem 2**. Since $\lambda(f)<1$, we know from Theorem 1 that $\sigma(f)>\lambda(f)+1$. By Lemma 2.4 and its Remark 2.1, for any given $0<\varepsilon<\frac{1}{2}$, there exists a set $E_1\in(1,\infty)$ of finite logarithmic measure, such that for all $z$ satisfying $|z|=r\notin[0,1]\cup E_1$ and $r$ sufficiently large, we have

$$\frac{f(z+j)}{f(z)}=\exp\{j\frac{f'(z)}{f(z)}+o(r^{\sigma-1-\varepsilon})\},\quad j=1,\dots,n. \tag{3.5}$$

By Wiman-Valiron theory, there exists a set $E_2\subset(0,\infty)$ of finite logarithmic measure, such that

$$\frac{f'(z)}{f(z)}=(1+o(1))\frac{v(r,f)}{z} \tag{3.6}$$

holds for $z$ that satisfies $|z|=r\notin E_2$ and

$$|f(z)|>M(r,f)v(r,f)^{-\frac{1}{4}+\delta},$$

where $0<\delta<\frac{1}{4}$.

This together with (1.3), (3.5) and (3.6), we obtain

$$\sum_{j=1}^{n}\frac{P_j(e^z)+Q_j(e^{-z})}{P_0(e^z)+Q_0(e^{-z})}\exp\{j\frac{v(r,f)}{z}(1+o(1))+o(r^{\sigma-1-\varepsilon})\}=-1, \tag{3.7}$$

where $P_n(e^z)+Q_n(e^{-z})=1$.

Set $F(z)=\frac{P_j(e^z)+Q_j(e^{-z})}{P_0(e^z)+Q_0(e^{-z})}$, then we get $\sigma(F)=1$. Applying Lemma 2.5 to $F(z)$, there exists a set $E_3\subset(1,\infty)$ of finite linear measure, such that for all $|z|=r\notin[0,1]\cup E_3$ and $r$ sufficiently large

$$\exp\{-2r^{1+\varepsilon}\}\le\left|\frac{P_j(e^z)+Q_j(e^{-z})}{P_0(e^z)+Q_0(e^{-z})}\right|\le\exp\{2r^{1+\varepsilon}\},\quad j=1,\dots,n. \tag{3.8}$$

Set $E=E_1\cup(E_2\cup E_3)$ and $G=\{\frac{\pi}{2},\frac{3\pi}{2}\}$.

For the set $E$ and $G$, by Lemma 2.6, there exists a positive number $A\in[\frac{1}{2},1]$, a sequence of points: $\{z_k=r_ke^{i\theta_k}\}$ with $|f(z_k)|\ge AM(r_k,f)$, $\theta_k\in[0,2\pi)$, $\lim\limits_{k\to\infty}\theta_k=\theta_0\in[0,2\pi)/G$, and a sequence of points $r_k\notin E, r_k\to\infty$, such that for any given $0<\varepsilon<\frac{1}{2}$, as $r_k$ sufficiently large, we have

$$r_k^{\sigma-\varepsilon}<v(r_k,f)<r_k^{\sigma+\varepsilon}. \tag{3.9}$$

Since $\theta_0\notin G$, by Lemma 2.7, for $k$ sufficiently large, we have

$$Rez_k<-\beta_{\theta_k}r_k\quad or\quad Rez_k>\beta_{\theta_k}r_k,$$

where $\beta_{\theta_k} > 0$ is a constant. Note that

$$Re\left\{\frac{v(r_k,f)}{z_k}\right\} = Re\left\{\frac{v(r_k,f)\bar{z_k}}{r_k^2}\right\} = Re\left\{\frac{v(r_k,f)z_k}{r_k^2}\right\}. \tag{3.10}$$

If $Rez_k < -\beta_{\theta_k} r_k$, from (3.8)–(3.10), we get

$$\begin{aligned}
&\left|\frac{P_j(e^{z_k}) + Q_j(e^{-z_k})}{P_0(e^{z_k}) + Q_0(e^{-z_k})}\exp\{j\frac{v(r_k,f)}{z_k}(1+o(1)) + o(r_k^{\sigma-1-\varepsilon})\}\right| \\
&\leq \exp\{-j\beta_{\theta_k} r_k^{\sigma-1+\varepsilon}(1+o(1)) + 2r_k^{1+\varepsilon}\} \\
&\leq \exp\{-\beta_{\theta_k} r_k^{\sigma-1+\varepsilon}(1+o(1)) + 2r_k^{1+\varepsilon}\}.
\end{aligned}$$

Above equation and (3.7) yield

$$\begin{aligned}
1 &= \left|\sum_{j=1}^{n}\frac{P_j(e^{z_k}) + Q_j(e^{-z_k})}{P_0(e^{z_k}) + Q_0(e^{-z_k})}\exp\{j\frac{v(r_k,f)}{z_k}(1+o(1)) + o(r_k^{\sigma-1-\varepsilon})\}\right| \\
&\leq \sum_{j=1}^{n}\left|\frac{P_j(e^{z_k}) + Q_j(e^{-z_k})}{P_0(e^{z_k}) + Q_0(e^{-z_k})}\exp\{j\frac{v(r_k,f)}{z_k}(1+o(1)) + o(r_k^{\sigma-1-\varepsilon})\}\right| \\
&\leq n\exp\{-\beta_{\theta_k} r_k^{\sigma-1+\varepsilon}(1+o(1)) + 2r_k^{1+\varepsilon}\}.
\end{aligned}$$

Hence, we have $\sigma - 1 + \varepsilon \leq 1 + \varepsilon$, which implies $\sigma \leq 2$. By Theorem 1, we have $\sigma(f) = 2$.

If $Rez_k > \beta_{\theta_k} r_k$, we first assume that $\sigma(f) = \sigma > 2$ and take $0 < \varepsilon < \min\{\frac{1}{2}, \frac{\sigma-2}{2}\}$. From (3.8)–(3.10), for $j = 1, \ldots, n-1$, by calculating carefully, we obtain

$$\begin{aligned}
&\left|\frac{P_j(e^{z_k}) + Q_j(e^{-z_k})}{P_0(e^{z_k}) + Q_0(e^{-z_k})}\exp\{j\frac{v(r_k,f)}{z_k}(1+o(1)) + o(r_k^{\sigma-1-\varepsilon})\}\right| \\
&= o\left(\left|\frac{P_n(e^{z_k}) + Q_n(e^{-z_k})}{P_0(e^{z_k}) + Q_0(e^{-z_k})}\exp\{n\frac{v(r_k,f)}{z_k}(1+o(1)) + o(r_k^{\sigma-1-\varepsilon})\}\right|\right).
\end{aligned}$$

It follows from (3.7)–(3.10), that

$$\begin{aligned}
1 &= \left|\sum_{j=1}^{n}\frac{P_j(e^{z_k}) + Q_j(e^{-z_k})}{P_0(e^{z_k}) + Q_0(e^{-z_k})}\exp\{j\frac{v(r_k,f)}{z_k}(1+o(1)) + o(r_k^{\sigma-1-\varepsilon})\}\right| \\
&= \left|\frac{P_n(e^{z_k}) + Q_n(e^{-z_k})}{P_0(e^{z_k}) + Q_0(e^{-z_k})}\exp\{n\frac{v(r_k,f)}{z_k}(1+o(1)) + o(r_k^{\sigma-1-\varepsilon})\}\right|(1+o(1)) \\
&\geq \exp\{n\beta_{\theta_k} r_k^{\sigma-1-\varepsilon}(1+o(1)) - 2r_k^{1+\varepsilon}\}.
\end{aligned}$$

Therefore, $\sigma - 1 - \varepsilon \leq 1 + \varepsilon$, which implies $\sigma \leq 2$ which contradicts the assumption that $\sigma(f) > 2$. Thus $\sigma(f) \leq 2$, by Theorem 1 again, we have $\sigma(f) = 2$. This completes the proof of Theorem 2.

**Proof of Theorem 3**. Suppose that $\deg(P_0) > \deg(P_j)$ for $j = 1, \ldots, n-1$. Then we get $m_0 > m_j (j \neq 0)$. Let $f \not\equiv 0$ be a solution of equation (1.4). Since $P_0(e^{A(z)}) + Q_0(e^{-A(z)}) \not\equiv 0$, comparing the degrees of both side of (1.4), we see that $f(z)$ cannot be a constant. By Lemma 2.1, we see that there exists a subset $E \subset (1, \infty)$ with a finite logarithmic measure and a constant $B > 0$, such that for all $|z| \notin (0,1) \cup E$, by calculating carefully, we have

$$\left|\log\left|\frac{f(z+j)}{f(z)}\right|\right| \leq B(T(2r,f))^2, j = 1, 2, \ldots, n-1. \tag{3.11}$$

It is well known that

$$\max\{|e^{A(z)}|, |z| = r\} = \max\{e^{ReA(z)}, |z| = r\} = e^{D(r,A)}.$$

Because $A(z)$ is a transcendental entire function, from Lemma 2.2, we know that

$$D(r,A) \geq \frac{M(\frac{r}{2}, A)}{2} - \frac{3}{2}|A(0)| \to \infty,$$

as $r \to \infty$. We take suitable $z = re^{i\theta}$ such that $|e^{A(z)}| = e^{D(r,A)}$, then

$$\left|e^{-A(z)}\right| = e^{-D(r,A)} \to 0 \quad \text{and} \quad \left|e^{A(z)}\right| = e^{D(r,A)} \to \infty,$$

as $r \to \infty$. Thus

$$|P_j(e^{A(z)}) + Q_j(e^{-A(z)})| \begin{cases} = |a_{jm_j}|e^{m_j D(r,A)}(1+o(1)), & (m_j \neq 0, r \to \infty) \\ \leq M_j, & (m_j = 0, r \to \infty), \end{cases}$$

where $M_j (> 0)(j = 1, 2, ..., n-1)$ are real constants, and so

$$|P_j(e^{A(z)}) + Q_j(e^{-A(z)})| \leq |a_{jm_j}|e^{m_j D(r,A)}(1+o(1)) + M_j (r \to \infty). \tag{3.12}$$

We obtain from (1.4), (3.11) and (3.12) that

$$\begin{aligned} |P_0(e^A) + Q_0(e^{-A})| &= |a_{0m_0}|e^{m_0 D(r,A)}(1+o(1)) \\ &\leq \left|\frac{f(z+n)}{f(z)}\right| + |P_{n-1}(e^A) + Q_{n-1}(e^{-A})| \left|\frac{f(z+n-1)}{f(z)}\right| + \cdots \\ &\quad + |P_1(e^A) + Q_1(e^{-A})| \left|\frac{f(z+1)}{f(z)}\right| \\ &\leq \left|\frac{f(z+n)}{f(z)}\right| + (|a_{n-1m_{n-1}}|e^{m_{n-1}D(r,A)} + M_{n-1})(1+o(1)) \left|\frac{f(z+n-1)}{f(z)}\right| \\ &\quad + \cdots + (|a_{1m_1}|e^{m_1 D(r,A)} + M_1)(1+o(1)) \left|\frac{f(z+1)}{f(z)}\right| \\ &\leq Me^{\max\{m_1,\ldots,m_{n-1}\}D(r,A)}(1+o(1)) \left[\left|\frac{f(z+n)}{f(z)}\right| + \cdots + \left|\frac{f(z+1)}{f(z)}\right|\right], \end{aligned}$$

where $M = \max\{|a_{n-1m_{n-1}}|, \ldots, |a_{1m_1}|, M_1, \ldots, M_{n-1}\}$. Hence

$$|a_{0m_0}|e^{D(r,A)}(1+o(1)) \leq M \left[\left|\frac{f(z+n)}{f(z)}\right| + \cdots + \left|\frac{f(z+1)}{f(z)}\right|\right] (1+o(1)). \tag{3.13}$$

Suppose that

$$\left|\frac{f(z+l)}{f(z)}\right| = \max\left\{\left|\frac{f(z+j)}{f(z)}\right|\right\}, \quad (j = 1, 2, \cdots, n),$$

then from (3.13) and above equation, we get

$$\begin{aligned}(\log e^{D(r,A)})(1+o(1)) &\leq \log\left(\left|\frac{f(z+n)}{f(z)}\right| + \cdots + \left|\frac{f(z+1)}{f(z)}\right|\right)(1+o(1)) \\ &\leq \log n + \log\left|\frac{f(z+l)}{f(z)}\right|(1+o(1)).\end{aligned}$$

From (3.11) and above equation, we can get

$$D(r, A) \leq B(T(2r, f))^2 \tag{3.14}$$

as $|z| = r$ sufficiently large. By Lemma 2.2, we get

$$\frac{M(\frac{r}{2}, A)}{2} - \frac{3}{2}A(0) \leq B(T(2r, f))^2. \tag{3.15}$$

Note that $A(z)$ is an entire function, therefore

$$T(r, A(z)) = m(r, A(z)) \leq \log^+ M(r, A). \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\frac{T(\frac{r}{2}, A(z))}{\log r} \leq \frac{\log M(\frac{r}{2}, A)}{\log r} \leq \frac{\log T(2r, f)^2}{\log r},$$

as $r$ sufficiently large. From $A(z)$ is transcendental, we have $\sigma(f) = \infty$.

Similarly, from (3.14) and Lemma 2.2, we have

$$\frac{\log T(\frac{r}{2}, A(z))}{\log r} \leq \frac{\log\log M(\frac{r}{2}, A)}{\log r} \leq \frac{\log\log D(r, A)}{\log r} \leq \frac{\log\log B[T(2r, f)]^2}{\log r},$$

as $r$ sufficiently large. Therefore, we have $\sigma_2(f) \geq \sigma(A)$.

If deg $Q_0 >$ deg $Q_j$, by taking a suitable $z$ satisfying

$$\max\{|e^{-A(z)}|, |z| = r\} = \max\{e^{Re-A(z)}, |z| = r\} = e^{D(r,-A)},$$

and using the similar arguments mentioned above. Thus we have completed the proof of Theorem 3.

**Proof of Corollary 1 and Corollary 2**. For the sake of simplicity, We set $P_n(e^z) + Q_n(e^{-z}) = 1$. Since

$$\Delta^n f = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f(z+j),$$

the expression (1.3) then takes the form

$$f(z+n)+[R_{n-1}(e^z)+S_{n-1}(e^{-z})]f(z+n-1)+\cdots+[R_0(e^z)+S_0(e^{-z})]f=0,$$

where

$$R_\ell(e^z)=\sum_{j=\ell}^{n}\binom{j}{\ell}(-1)^{j-\ell}P_j(e^z),\quad S_\ell(e^{-z})=\sum_{j=\ell}^{n}\binom{j}{\ell}(-1)^{j-\ell}Q_j(e^{-z}),$$

where $\ell=1,2,\ldots,n-1$. And

$$R_0(e^z)=\sum_{j=1}^{n}\binom{j}{0}(-1)^{j}P_j(e^z)+P_0(e^z),\quad S_0(e^{-z})=\sum_{j=1}^{n}\binom{j}{0}(-1)^{j}Q_j(e^{-z})+Q_0(e^{-z}).$$

By the assumptions

$$\deg(P_0)>\deg(P_j)\quad\text{or}\quad\deg(Q_0)>\deg(Q_j),\quad j=1,2,\ldots,n-1,$$

of the Corollary 1, we have

$$\deg(R_0)>\deg(R_j)\quad\text{or}\quad\deg(S_0)>\deg(S_j),\quad j=1,2,\ldots,n-1.$$

From Theorem 1, Corollary 1 follows. By Theorem 3 and the similar arguments mentioned above, Corollary 2 holds.

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