Isometric embeddings of pretangent spaces in E^n

Viktoriia Bilet   Oleksiy Dovgoshey

Abstract

We prove some infinitesimal analogs of classical results of Menger, Schoenberg and Blumenthal giving the existence conditions for isometric embeddings of metric spaces in the finite-dimensional Euclidean spaces.

1   Introduction

The definition of pretangent and tangent metric spaces to an arbitrary metric space was introduced in [11] (see also [12]) for studies of generalized differentiation on metric spaces. The development of this theory requires the understanding of interrelations between the infinitesimal properties of initial metric space and geometry of pretangent spaces to this initial.

The necessary and sufficient conditions under which a pretangent space to metric space is unique and a series of interesting examples of metric spaces with unique pretangent spaces were presented in [2]. Some conditions under which pretangent spaces are compact and bounded were found in a recent paper [1]. Criteria of the ultrametricity of pretangent spaces were obtained in [10]. The necessary and sufficient conditions under which subspaces X and Y of metric space Z have the same pretangent spaces in a point of $X \cap Y$ were obtained in [9]. A criterion of the finiteness of pretangent spaces was proved in [11].

Our main goal is to search the criteria of the isometric embeddability of pretangent spaces in the real $n-$dimensional Euclidean space E^n. The second part

of our paper contains the general Transfer Principle, Theorem 2.7, providing, in some cases, the "automatic translation" of global properties of pretangent spaces into the limits relations defined in the initial metric spaces. An immediate consequence of the Transfer Principle is the Conservation Principle describing some properties of metric spaces which are invariant under passage to the pretangent spaces. In the third part of the paper we apply the Transfer Principle to the classical condition of isometric embeddability of metric spaces in E^n obtained by K. Menger and I. Schoenberg. We reformulate their embedding theorems in a suitable form, see Proposition 3.1 and Proposition 3.11 and transfer them to the "infinitesimal" embeddings theorems 3.4 and 3.12. In the fourth part we obtain Theorem 4.2 which gives the infinitesimal form of Blumenthal's embedding theorem. Note that in the last case the Transfer Principle do not seem to be applicable.

2 Pretangent spaces

For convenience we recall the terminology that will be necessary in future.

Let (X, d) be a metric space and let p be a point of X. Fix some sequence $\tilde{r}$ of positive real numbers r_n tending to zero. In what follows $\tilde{r}$ will be called a *normalizing sequence*. Let us denote by $\tilde{X}$ the set of all sequences of points from X.

Definition 2.1. *Two sequences* $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ *and* $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$, $\tilde{x}, \tilde{y} \in \tilde{X}$ *are mutually stable with respect to* $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ *if there is a finite limit*

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}). \tag{2.1}$$

We shall say that a family $\tilde{F} \subseteq \tilde{X}$ is *self-stable* (w.r.t. $\tilde{r}$) if every two $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable. A family $\tilde{F} \subseteq \tilde{X}$ is *maximal self-stable* if $\tilde{F}$ is self-stable and for an arbitrary $\tilde{z} \in \tilde{X}$ either $\tilde{z} \in \tilde{F}$ or there is $\tilde{x} \in \tilde{F}$ such that $\tilde{x}$ and $\tilde{z}$ are not mutually stable.

The standard application of Zorn's Lemma leads to the following

Proposition 2.2. *Let* (X, d) *be a metric space and let* $p \in X$. *Then for every normalizing sequence* $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ *there exists a maximal self-stable family* $\tilde{X}_p = \tilde{X}_{p,\tilde{r}}$ *such that* $\tilde{p} := \{p, p, ...\} \in \tilde{X}_p$.

Note that the condition $\tilde{p} \in \tilde{X}_p$ implies the equality

$$\lim_{n\to\infty} d(x_n, p) = 0 \tag{2.2}$$

for every $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{X}_p$.

Consider a function $\tilde{d} : \tilde{X}_p \times \tilde{X}_p \to \mathbb{R}$ where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y})$ is defined by (2.1). Obviously, $\tilde{d}$ is symmetric and nonnegative. Moreover, the triangle inequality for d implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z}$ from $\tilde{X}_p$. Hence $(\tilde{X}_p, \tilde{d})$ is a pseudometric space.

Definition 2.3. *The pretangent space to the space X (at the point p w.r.t. $\tilde{r}$) is the metric identification of the pseudometric space $(\tilde{X}_{p,\tilde{r}}, \tilde{d})$.*

Since the notion of pretangent space is important for the present paper, we remind this metric identification construction.

Define the relation $\sim$ on $\tilde{X}_p$ by $\tilde{x} \sim \tilde{y}$ if and only if $\tilde{d}(\tilde{x}, \tilde{y}) = 0$. Then $\sim$ is an equivalence relation. Let us denote by $\Omega^X_{p,\tilde{r}}$ the set of equivalence classes in $\tilde{X}_p$ under the equivalence relation $\sim$. It follows from general properties of pseudometric spaces, see for example, [14], that if ρ is defined on $\Omega^X_{p,\tilde{r}}$ by

$$\rho(\alpha, \beta) := \tilde{d}(\tilde{x}, \tilde{y}) \tag{2.3}$$

for $\tilde{x} \in \alpha$ and $\tilde{y} \in \beta$, then ρ is a well-defined metric on $\Omega^X_{p,\tilde{r}}$. By definition the metric identification of $(\tilde{X}_p, \tilde{d})$ is the metric space $(\Omega^X_{p,\tilde{r}}, \rho)$.

Remark that $\Omega^X_{p,\tilde{r}} \neq \varnothing$ because the constant sequence $\tilde{p}$ belongs to $\tilde{X}_{p,\tilde{r}}$, see Proposition 2.2.

Let $\{n_k\}_{k \in \mathbb{N}}$ be an infinite, strictly increasing sequence of natural numbers. Let us denote by $\tilde{r}'$ the subsequence $\{r_{n_k}\}_{k \in \mathbb{N}}$ of the normalizing sequence $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ and let $\tilde{x}' := \{x_{n_k}\}_{k \in \mathbb{N}}$ for every $\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}$. It is clear that if $\tilde{x}$ and $\tilde{y}$ are mutually stable w.r.t. $\tilde{r}$, then $\tilde{x}'$ and $\tilde{y}'$ are mutually stable w.r.t. $\tilde{r}'$ and

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}').$$

If $\tilde{X}_{p,\tilde{r}}$ is a maximal self-stable (w.r.t $\tilde{r}$) family, then, by Zorn's Lemma, there exists a maximal self-stable (w.r.t $\tilde{r}'$) family $\tilde{X}_{p,\tilde{r}'}$ such that

$$\{\tilde{x}' : \tilde{x} \in \tilde{X}_{p,\tilde{r}}\} \subseteq \tilde{X}_{p,\tilde{r}'}.$$

Denote by $in_{\tilde{r}'}$ the map from $\tilde{X}_{p,\tilde{r}}$ to $\tilde{X}_{p,\tilde{r}'}$ with $in_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{p,\tilde{r}}$. It follows from (2.3) that after metric identifications $in_{\tilde{r}'}$ pass to an isometric embedding $em' : \Omega^X_{p,\tilde{r}} \to \Omega^X_{p,\tilde{r}'}$ under which the diagram

$$\begin{array}{ccc} \tilde{X}_{p,\tilde{r}} & \xrightarrow{in_{\tilde{r}'}} & \tilde{X}_{p,\tilde{r}'} \\ \pi \downarrow & & \downarrow \pi' \\ \Omega^X_{p,\tilde{r}} & \xrightarrow{em'} & \Omega^X_{p,\tilde{r}'} \end{array} \tag{2.4}$$

is commutative. Here π and π' are the natural projections, $\pi(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{p,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\}$ and $\pi'(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{p,\tilde{r}'} : \tilde{d}_{\tilde{r}'}(\tilde{x}, \tilde{y}) = 0\}$.

Let X and Y be two metric spaces. Recall that the map $f : X \to Y$ is called an *isometry* if f is distance-preserving and onto.

Definition 2.4. *A pretangent $\Omega^X_{p,\tilde{r}}$ is tangent if $em' : \Omega^X_{p,\tilde{r}} \to \Omega^X_{p,\tilde{r}'}$ is an isometry for every $\tilde{r}'$.*

Remark 2.5. Let $\tilde{X}_{p,\tilde{r}}$ be a maximal self-stable family with corresponding pre-tangent space $\Omega_{p,\tilde{r}}^X$. Then $\Omega_{p,\tilde{r}}^X$ is tangent if and only if for every subsequence $\tilde{r}' = \{r_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $\tilde{r}$ the family $\tilde{X}_{p,\tilde{r}'} := \{\tilde{x}' : \tilde{x} \in \tilde{X}_{p,\tilde{r}}\}$ is maximal self-stable w.r.t. $\tilde{r}'$.

For every natural $k \geq 1$ write X^{k+1} for the set of all $k+1-$tuples $x = (x_0, x_1, ..., x_k)$ with terms $x_n \in X$ for $n = 0, 1, ..., k$.

Denote by $M_n, n \in \mathbb{N}$, the topological space of all real, $n \times n-$matrices t with the topology of pointwise convergence. Let $\mathfrak{M}$ be a class of nonvoid metric spaces and let $\mathfrak{F}$ be a family of continuous functions $f : M_n \to \mathbb{R}, n = n(f)$ which are homogeneous of degree $s_0 = s_0(f) > 0$, i.e.,

$$f(\delta t) = \delta^{s_0}(f(t)) \tag{2.5}$$

for all $\delta \in [0, \infty)$ and all $t \in Dom(f)$. We shall say that $\mathfrak{M}$ is determined by $\mathfrak{F}$ if the following two conditions are equivalent for every metric space (X, d) : $(X, d) \in \mathfrak{M}$; the inequality $f(m) \geq 0$ holds for each $f \in \mathfrak{F}$ and all $m \in Dom(f)$ having the form

$$m = m(x_1, x_2, ..., x_n) = \begin{pmatrix} d(x_1, x_1) & d(x_1, x_2) & ... & d(x_1, x_n) \\ d(x_2, x_1) & d(x_2, x_2) & ... & d(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ d(x_n, x_1) & d(x_n, x_2) & ... & d(x_n, x_n) \end{pmatrix},$$

$$(x_1, x_2, ..., x_n) \in X^n. \tag{2.6}$$

Remark 2.6. Equality (2.5) and the inequality $s_0(f) > 0$ imply that $f(0) = 0$ for every $f \in \mathfrak{F}$ where 0 is the zero $n \times n-$matrix belonging to $Dom(f)$. It is clear that each matrix from (2.6) is equal to 0 for one-point metric spaces. Consequently each one-point metric space belongs to every $\mathfrak{M}$ determinated by some $\mathfrak{F}$.

For example, the class of all ultrametric spaces is determinated by the family $\mathfrak{F}$ with the unique element $f : M_3 \to \mathbb{R}$,

$$f(t) = (t_{1,3} \vee t_{3,2}) - t_{1,2}.$$

Indeed, if t has form (2.6), then the inequality $f(t) \geq 0$ can be written as the ultra-triangle inequality $d(x_1, x_2) \leq d(x_1, x_3) \vee d(x_3, x_2)$.

Let (X, d) be a metric space with marked point p and let $f \in \mathfrak{F}$. We set

$$\delta(x_1, ..., x_n) := \bigvee_{i=1}^{n} d(x_i, p) \tag{2.7}$$

for $(x_1, ..., x_n) \in X^n$ and define the function $f^* : X^n \to \mathbb{R}$ as

$$f^*(x_1, x_2, ..., x_n) := \begin{cases} f\left(\frac{m(x_1, x_2, ..., x_n)}{\delta(x_1, x_2, ..., x_n)}\right) & \text{if } (x_1, x_2, ..., x_n) \neq (p, p, ..., p) \\ 0 & \text{if } (x_1, x_2, ..., x_n) = (p, p, ..., p). \end{cases} \tag{2.8}$$

Theorem 2.7. *(Transfer Principle) Let (X, d) be a metric space with marked point p and let $\mathfrak{M}$ be a family of metric spaces determined by a family $\mathfrak{F}$. The following two statements are equivalent.*

(i) *Each pretangent space $\Omega^X_{p,\tilde{r}}$ belongs to $\mathfrak{M}$.*

(ii) *The inequality*

$$\liminf_{x_1,x_2,...,x_n\to p} f^*(x_1, x_2, ..., x_n) \geq 0 \tag{2.9}$$

holds for each $f : M_n \to \mathbb{R}$ belonging to $\mathfrak{F}$.

Proof. Suppose that (i) holds. Let us prove inequality (2.9) for each $f \in \mathfrak{F}$. Let $f : M_n \to \mathbb{R}$ belong to $\mathfrak{F}$ and let $\tilde{x}_i = \{x_{i,m}\}_{m\in\mathbb{N}} \in \tilde{X}$, $i = 1, 2, ..., n$, be some sequences such that

$$\lim_{m\to\infty} f^*(x_{1,m}, ..., x_{n,m}) = \liminf_{x_1,...,x_n\to p} f^*(x_1, ..., x_n) \tag{2.10}$$

and

$$p = \lim_{m\to\infty} x_{1,m} = \lim_{m\to\infty} x_{2,m} = ... = \lim_{m\to\infty} x_{n,m}. \tag{2.11}$$

Limit relations (2.11) implies

$$\lim_{m\to\infty} \delta(x_{1,m}, ..., x_{n,m}) = 0$$

where δ is defined by (2.7). If for all sufficiently large m we have $\delta(x_{1,m}, ..., x_{n,m}) = 0$, then the limit in (2.10) vanishes, so that (2.9) holds. Consequently we may suppose, going to a subsequence, that

$$\delta(x_{1,m}, ..., x_{n,m}) > 0$$

for all $m \in \mathbb{N}$. Define a normalizing sequence $\tilde{r} = \{r_m\}_{m\in\mathbb{N}}$ as

$$r_m := \delta(x_{1,m}, ..., x_{n,m}), \ m \in \mathbb{N}.$$

All elements of the matrix $\frac{m(x_1,...,x_n)}{\delta(x_1,...,x_n)}$, see (2.6), are bounded because

$$0 \leq \frac{\bigvee_{i,j=1}^{n} d(x_{i,m}, x_{j,m})}{r_m} \leq \frac{2 \bigvee_{i=1}^{n} d(x_{i,m}, p)}{r_m} = 2. \tag{2.12}$$

Hence going to a subsequence once again we can assume that all $\tilde{x}_i$, $i = 1, ..., n$, and $\tilde{p}$ are pairwise mutually stable. The functions $f \in \mathfrak{F}$ are continuous. Hence using (2.8) we obtain

$$\lim_{m\to\infty} f^*(x_{1,m}, ..., x_{n,m}) = f(t), \tag{2.13}$$

where

$$t = \begin{pmatrix} \tilde{d}(\tilde{x}_1, \tilde{x}_1) & \tilde{d}(\tilde{x}_1, \tilde{x}_2) & ... & \tilde{d}(\tilde{x}_1, \tilde{x}_n) \\ \tilde{d}(\tilde{x}_2, \tilde{x}_1) & \tilde{d}(\tilde{x}_2, \tilde{x}_2) & ... & \tilde{d}(\tilde{x}_2, \tilde{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}(\tilde{x}_n, \tilde{x}_1) & \tilde{d}(\tilde{x}_n, \tilde{x}_2) & ... & \tilde{d}(\tilde{x}_n, \tilde{x}_n) \end{pmatrix}.$$

If $\tilde{X}_{p,\tilde{r}}$ is a maximal self-stable family such that $\tilde{x}_i \in \tilde{X}_{p,\tilde{r}}$, $i = 1, ..., n$ and $\Omega_{p,\tilde{r}}^X$ is the metric identification of $\tilde{X}_{p,\tilde{r}}$, then $\Omega_{p,\tilde{r}}^X \in \mathfrak{M}$. Since the family $\mathfrak{M}$ is determined by $\mathfrak{F}$ and

$$t = \begin{pmatrix} \rho(\alpha_1, \alpha_1) & \rho(\alpha_1, \alpha_2) & ... & \rho(\alpha_1, \alpha_n) \\ \rho(\alpha_2, \alpha_1) & \rho(\alpha_2, \alpha_2) & ... & \rho(\alpha_2, \alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(\alpha_n, \alpha_1) & \rho(\alpha_n, \alpha_2) & ... & \rho(\alpha_n, \alpha_n) \end{pmatrix}$$

where $\alpha_i = \pi(\tilde{x}_i)$, see (2.4), we obtain the inequality

$$f(t) \geq 0.$$

This inequality, (2.13) and (2.10) imply (2.9).

Assume now that (2.9) holds for all $f \in \mathfrak{F}$. We must prove that each $\Omega_{p,\tilde{r}}^X$ belongs to $\mathfrak{M}$. Let $\Omega_{p,\tilde{r}}^X$ be a pretangent space with corresponding maximal self-stable family $\tilde{X}_{p,\tilde{r}}$. The relation $\Omega_{p,\tilde{r}}^X \in \mathfrak{M}$ means that for every $f : M_n \to \mathbb{R}$ the inequality

$$f(m(\alpha_1, ..., \alpha_n)) \geq 0 \tag{2.14}$$

holds for all $\alpha_1, ..., \alpha_n \in \Omega_{p,\tilde{r}}^X$ where

$$m(\alpha_1, ..., \alpha_n) = \begin{pmatrix} \rho(\alpha_1, \alpha_1) & \rho(\alpha_1, \alpha_2) & ... & \rho(\alpha_1, \alpha_n) \\ \rho(\alpha_2, \alpha_1) & \rho(\alpha_2, \alpha_2) & ... & \rho(\alpha_2, \alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(\alpha_n, \alpha_1) & \rho(\alpha_n, \alpha_2) & ... & \rho(\alpha_n, \alpha_n) \end{pmatrix}.$$

Inequality (2.14) holds automatically if

$$\bigvee_{i,j=1}^{n} \rho(\alpha_i, \alpha_j) = 0,$$

see Remark 2.6. Hence we may suppose that

$$\bigvee_{i,j=1}^{n} \rho(\alpha_i, \alpha_j) > 0.$$

If $\alpha = \pi(\tilde{p})$, then the last inequality implies

$$\bigvee_{i=1}^{n} \rho(\alpha, \alpha_i) > 0. \tag{2.15}$$

Let $\tilde{x}_i = \{x_{i,m}\}_{m \in \mathbb{N}}$, $i = 1, ..., n$ be elements of $\tilde{X}_{p,\tilde{r}}$ such that $\alpha_i = \pi(\tilde{x}_i)$. Using inequality (2.15) we can write

$$\rho(\alpha_i, \alpha_j) = \lim_{m \to \infty} \frac{d(x_{i,m}, x_{j,m})}{r_m} =$$

$$\lim_{m \to \infty} \frac{\delta(x_{1,m}, x_{2,m}, ..., x_{n,m})}{r_m} \frac{d(x_{i,m}, x_{j,m})}{\delta(x_{1,m}, x_{2,m}, ..., x_{n,m})}$$

$$= \bigvee_{i=1}^{n} \rho(\alpha, \alpha_i) \lim_{m \to \infty} \frac{d(x_{i,m}, x_{j,m})}{\delta(x_{1,m}, x_{2,m}, ..., x_{n,m})} \tag{2.16}$$

for $i, j = 1, ..., n$. From (2.5), (2.8) and (2.16) we obtain

$$f\left(\frac{\boldsymbol{m}(\alpha_1, ..., \alpha_n)}{\overset{n}{\underset{i=1}{\vee}} \rho(\alpha, \alpha_i)}\right) = \lim_{m \to \infty} f^*(x_{1,m}, x_{2,m}, ..., x_{n,m}),$$

$$f(\boldsymbol{m}(\alpha_1, ..., \alpha_n)) = \left(\overset{n}{\underset{i=1}{\vee}} \rho(\alpha, \alpha_i)\right)^{s_0} \lim_{m \to \infty} f^*(x_{1,m}, x_{2,m}, ..., x_{n,m}) \quad (2.17)$$

where $s_0 > 0$ is the degree of homogeneity of f. Since

$$\lim_{m \to \infty} f^*(x_{1,m}, x_{2,m}, ..., x_{n,m}) \geq \liminf_{x_1, x_2, ..., x_n \to p} f^*(x_1, ..., x_n) \geq 0$$

and

$$\left(\overset{n}{\underset{i=1}{\vee}} \rho(\alpha, \alpha_i)\right)^{s_0} > 0,$$

equality (2.17) implies (2.14). ∎

Let $f : \boldsymbol{M}_n \to \mathbb{R}$ be a continuous homogeneous function with the degree of homogeneity $s_0 = s_0(f) > 0$, let (X, d) be a metric space with a marked point p and let $f^* : X^n \to \mathbb{R}$ be the function given by (2.8). Define the family $\mathfrak{U}$ of metric space (X, d) by the rule

$$(X, d) \in \mathfrak{U} \Leftrightarrow f(\boldsymbol{m}(x_1, ..., x_n)) = 0 \quad (2.18)$$

where $\boldsymbol{m}(x_1, ..., x_n)$ is the matrix of form (2.6).

Corollary 2.8. *Let (X, d) be a metric space with a marked point p. All pretangent spaces $\Omega^X_{p,\tilde{r}}$ belong to $\mathfrak{U}$ if and only if*

$$\lim_{x_1, ..., x_n \to p} f^*(x_1, ..., x_n) = 0. \quad (2.19)$$

Proof. Let us consider the two-point set $\mathfrak{F} = \{f, -f\}$. Note that $(-f)$ is also continuous homogeneous function of degree s_0. The family $\mathfrak{U}$ is determined by $\mathfrak{F}$ because $f(\boldsymbol{m}(x_1, ...x_n)) = 0$ if and only if $f(\boldsymbol{m}(x_1, ...x_n)) \geq 0$ and $-f(\boldsymbol{m}(x_1, ...x_n)) \geq 0$. Hence, by Theorem 2.7, all pretangent spaces $\Omega^X_{p,\tilde{r}}$ belong to $\mathfrak{U}$ if and only if

$$\liminf_{x_1, ..., x_n \to p} f^*(x_1, ...x_n) \geq 0 \quad \text{and} \quad \liminf_{x_1, ..., x_n \to p} (-f^*(x_1, ...x_n)) \geq 0. \quad (2.20)$$

The last inequality is the equivalent of

$$\limsup_{x_1, ..., x_n \to p} f^*(x_1, ...x_n) \leq 0.$$

This inequality and the first inequality in (2.20) give (2.19). ∎

Remark 2.9. The proof of Theorem 2.7 is a generalization of the proof of Theorem 3.1 from [5] which gives the necessary and sufficient conditions under which all pretangent spaces $\Omega^X_{p,\tilde{r}}$ are *ptolemaic*. These conditions lead to a criterion of isometric embeddability of pretangent spaces in E^1.

The following corollary is of interest in its own right.

Corollary 2.10. *(Conservation Principle) Let $\mathfrak{M}$ be a class of nonvoid metric spaces determined by a family $\mathfrak{F}$. Then for every metric space $X \in \mathfrak{M}$ all pretangent spaces $\Omega_{p,\tilde{r}}^{X}$ belong to $\mathfrak{M}$ for each $p \in X$.*

Remark 2.11. It is plain to prove that in the Transfer Principle instead of the function

$$\delta(x_1, ..., x_n) = \bigvee_{i=1}^{n} d(x_i, p)$$

we can use an arbitrary function $\varepsilon : X^n \to [0, \infty)$ fulfilling the restrictions

$$\varepsilon(x_1, ..., x_n) = 0 \Leftrightarrow x_1 = ... = x_n = p$$

and

$$\frac{1}{c} \leq \liminf_{x_1, ..., x_n \to p} \frac{\varepsilon(x_1, ..., x_n)}{\delta(x_1, ..., x_n)} \leq \limsup_{x_1, ..., x_n \to p} \frac{\varepsilon(x_1, ..., x_n)}{\delta(x_1, ..., x_n)} \leq c$$

with some constant $c \in [1, \infty)$. Here we put

$$\frac{\varepsilon(p, ..., p)}{\delta(p, ..., p)} = 1.$$

For example we can take

$$\varepsilon(x_1, ..., x_n) = \left(\sum_{i=1}^{n} d^s(p, x_i) \right)^{\frac{1}{s}}$$

with $s > 0$.

3   Infinitesimal versions of Menger's and Schoenberg's embedding theorems

In this section we start with the reformulation of the Menger Embedding Theorem in a suitable form for application of the Transfer Principle. Recall that the Cayley-Menger determinant is the next determinant

$$D_k(x_0, x_1, ..., x_k) = \begin{vmatrix} 0 & 1 & 1 & ... & 1 \\ 1 & 0 & d^2(x_0, x_1) & ... & d^2(x_0, x_k) \\ 1 & d^2(x_1, x_0) & 0 & ... & d^2(x_1, x_k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d^2(x_k, x_0) & d^2(x_k, x_1) & ... & 0 \end{vmatrix}$$

where $(x_0, x_1, ..., x_k) \in X^{k+1}$.

Proposition 3.1. *Let $n \in \mathbb{N}$. A metric space X is isometrically embeddable in E^n if and only if*

$$(-1)^{k+1} D_k(x_0, x_1, ..., x_k) \geq 0 \tag{3.1}$$

for every $(x_0, x_1, ..., x_k) \in X^{k+1}$ with $k \leq n$ and

$$D_k(x_0, x_1, ..., x_k) = 0 \tag{3.2}$$

for every $(x_0, x_1, ..., x_k) \in X^{k+1}$ with $k = n + 1$ and $k = n + 2$.

To prove Proposition 3.1 we shall use some known results of K. Menger and L. Blumenthal. Our first lemma is the simplest form of the Menger Embedding Theorem.

Lemma 3.2. *A metric space X is isometrically embeddable in E^n if and only if each set $A \subseteq X$ with $\operatorname{card} A \leq n + 3$ is isometrically embeddable in E^n.*

The clear proof of it can be found in [[6], p.95].

The following lemma is a corollary of Blumenthal's solution of the problem of isometric embedding of semimetric spaces in the Euclidean spaces, see [[6], p.105].

Lemma 3.3. *Let X be a finite metric space with $\operatorname{card} X = n + 1$. Then X is isometrically embeddable in E^n if and only if the Cayley-Menger determinant $D_k(x_0, x_1, ..., x_k)$ has the sign of $(-1)^{k+1}$ or vanishes for every $(x_0, x_1, ..., x_k) \in X^{k+1}$, $k = 1, 2, ..., n$.*

Proof of Proposition 3.1. Suppose that X is isometrically embeddable in E^n. Let $(x_0, x_1, ..., x_k) \in X^{k+1}$. If $k \leq n$, then inequality (3.1) follows directly from Lemma 3.3. Let $k = n + 1$ or $k = n + 2$. We can consider E^n as a subspace of the Euclidean space E^k.

Let F be an isometric embedding of X in E^k.

Write $x_0^* := F(x_0), x_1^* := F(x_1), ..., x_k^* := F(x_k)$ and denote by $V(x_0^*, x_1^*, ..., x_k^*)$ the volume of the simplex with vertices $x_0^*, x_1^*, ..., x_k^*$. This simplex lies in the subspace E^n of the space E^k. Thus, we have

$$V(x_0^*, x_1^*, ..., x_k^*) = 0. \tag{3.3}$$

Since

$$V^2(x_0^*, x_1^*, ..., x_k^*) = \frac{(-1)^{k+1}}{2^k (k!)^2} D_k(x_0^*, x_1^*, ..., x_k^*) =$$

$$\frac{(-1)^{k+1}}{2^k (k!)^2} D_k(x_0, x_1, ..., x_k), \tag{3.4}$$

see, for example, [[6], p.98], these equalities and (3.3) imply (3.2).

Conversely, suppose for every $(x_0, x_1, ..., x_k) \in X^{k+1}$ we have (3.1) if $k \leq n$ or (3.2) if $k = n + 1$, $k = n + 2$. We must show that X is isometrically embeddable in E^n. By Lemma 3.2 it is sufficient to prove that every $A \subseteq X$ with $\operatorname{card} A \leq n + 3$ has this property. Note that it follows directly from Lemma 3.3 if $\operatorname{card} A \leq n + 1$.

Let us consider the case where

$$A = \{x_0, x_1, ..., x_n, x_{n+1}, x_{n+2}\}$$

(the case $A = \{x_0, ..., x_n, x_{n+1}\}$ is more simple and can be considered similarly). By Lemma 3.3, there is an isometric embedding $F : A \to E^{n+2}$, $F(x_0) = x_0^*, ...,$

$F(x_{n+2}) = x_{n+2}^*$. We may assume, without loss of generality, that $x_0^* = 0$. Denote by L the linear subspace of E^{n+2} generated by the vectors $x_1^*, ..., x_n^*, x_{n+1}^*, x_{n+2}^*$. It is clear that A is isometrically embeddable in E^n if $dim L \leq n$. If the last inequality does not hold, then the set $\{x_1^*, ..., x_n^*, x_{n+1}^*, x_{n+2}^*\}$ contains some linear independent vectors $x_1^{**}, ..., x_n^{**}, x_{n+1}^{**}$.

Let $x_1', ..., x_n', x_{n+1}'$ be elements of the set $\{x_1, ..., x_n, x_{n+1}, x_{n+2}\}$ such that

$$x_1^{**} = F(x_1'), \ x_2^{**} = F(x_2'), ..., x_{n+1}^{**} = F(x_{n+1}').$$

Since $x_1^{**}, x_2^{**}, ..., x_{n+1}^{**}$ are linear independent, we have

$$V(x_0^*, x_1^{**}, ..., x_{n+1}^{**}) > 0.$$

Using the last inequality and (3.4) we obtain

$$D_k(x_0, x_1', ..., x_n', x_{n+1}') \neq 0,$$

contrary to equality (3.2).   ∎

Let (X, d) be a metric space with a marked point p. Similarly (2.8) define the functions $\Theta_{k+1} : X^{k+1} \to \mathbb{R}$ by the rule

$$\Theta_{k+1}(x_0, x_1, ..., x_k) := \begin{cases} \dfrac{(-1)^{k+1} D_k(x_0, x_1, ..., x_k)}{\left(\bigvee\limits_{n=0}^{k} d(x_n, p) \right)^{2k}}, & \text{if } (x_0, x_1, ..., x_k) \neq (p, p, ..., p) \\ 0, & \text{if } (x_0, x_1, ..., x_k) = (p, p, ..., p) \end{cases} \tag{3.5}$$

where $\bigvee\limits_{n=0}^{k} d(x_n, p) := \max\limits_{0 \leq n \leq k} d(x_n, p)$.

The following theorem gives necessary and sufficient conditions under which all pretangent spaces have isometric embeddings in E^n.

Theorem 3.4. *Let (X, d) be a metric space with a marked point p and let $n \in \mathbb{N}$. Every $\Omega_{p, \tilde{r}}^X$ is isometrically embeddable in E^n if and only if inequality*

$$\liminf\limits_{x_0, x_1, ..., x_k \to p} \Theta_{k+1}(x_0, x_1, ..., x_k) \geq 0 \tag{3.6}$$

holds for all $k \leq n$ and the equality

$$\lim\limits_{x_0, x_1, ..., x_k \to p} \Theta_{k+1}(x_0, x_1, ..., x_k) = 0 \tag{3.7}$$

holds for $k = n + 1$ and $k = n + 2$.

The theorem can be proved by application of Proposition 3.1, Theorem 2.7 and Corollary 2.8 with $\mathfrak{M}$ equals the class of all metric spaces which are embeddable in E^n and

$$\mathfrak{F} = \{D_1, D_2, ..., D_n\} \cup \{D_{n+1}, -D_{n+1}, D_{n+2}, -D_{n+2}\}.$$

Note only that all Cayley-Menger determinants $D_1, ..., D_n, D_{n+1}$ and D_{n+2} are continuous functions on $M_2, ..., M_{n+1}, M_{n+2}$ and M_{n+3} and with degrees of homogeneity equal $2, ..., 2n, 2(n+1), 2(n+2)$ respectively.

Remark 3.5. The main information about Cayley-Menger determinants can be found in the books of M. Berger [3] and L. Blumenthal [6]. These determinants play an important role in some questions of metric geometry. In 1928 Menger used them to characterize the Euclidean spaces solely in metric terms. They also participate in metric characterization of Riemann's manifolds of the constant sectional curvature, obtained by Berger [4]. In a recent paper [8], it was proved that the Cayley-Menger determinant of an $n-$dimensional simplex is an absolutely irreducible for $n \geq 3$. The following results, indicated also in [8], are found in using of these determinants: this is a proof of the bellows conjecture, which asserts that all flexible polyhedra keep a constant volume in 3-dimensional Euclidean space (see, [7], [17]); the study of the spatial form of the molecules in the stereochemistry [15].

The following is immediate from the Conservation Principle.

Corollary 3.6. *If X is a subset of E^n and $p \in X$, then all pretangent spaces $\Omega^X_{p,\tilde{r}}$ are isometrically embeddable in E^n.*

Let X be a metric space with a marked point p. Define the second pretangent space to X at the point $p \in X$ as a pretangent space to a pretangent space $\Omega^X_{p,\tilde{r}}$. More generally suppose we have constructed all $n-$th pretangent spaces to X at p. We shall denote such spaces as $\Omega^n = (\Omega^n, \rho_n)$.

Definition 3.7. *A metric space Y is an $(n+1)-$th pretangent space to X at p if there are an $n-$th pretangent space (Ω^n, ρ_n) and a point $p_n \in \Omega^n$ and a normalizing sequence $\tilde{r}^n$ and a maximal self-stable family $\tilde{\Omega}^n_{p_n,\tilde{r}_n} \subseteq \tilde{\Omega}^n$ such that Y is the metric identification of the pseudometric space $\left(\tilde{\Omega}^n_{p_n,\tilde{r}_n}, \tilde{\rho}_n \right)$.*

Corollary 3.8. *Let X be a metric space with a marked point p and let $k \in \mathbb{N}$. If each (first) pretangent space to X at p is isometrically embeddable in E^k, then for every $n \geq 2$ all $n-$th pretangent spaces to X at p are also isometrically embeddable in E^k.*

Let a_{jk}, $0 \leq j, k \leq n$ be real constants such that $a_{jj} = 0$ and $a_{jk} = a_{kj}$ if $k \neq j$. The following is Schoenberg's embedding theorem from [18].

Theorem 3.9. *A necessary and sufficient condition that a_{jk} be the lengths of the edges of an $n-$simplex lying in E^m, but not in E^l with $l < m$ is that the quadratic form*

$$F(y_1, y_2, ..., y_n) = \sum_{j,k=1}^{n} (a_{0j}^2 + a_{0k}^2 - a_{jk}^2) y_j y_k \tag{3.8}$$

be positive semidefinite and of rank m.

For applications of this result to embeddings of pretangent spaces in E^m we introduce the determinant $Sch(x_0, x_1, ..., x_n)$. Let (X, d) be a metric space and let $(x_0, x_1, ..., x_n) \in X^{n+1}$. Write

$$Sch(x_0, x_1, ..., x_n) = \begin{vmatrix} \tau_{11} & \tau_{12} & ... & \tau_{1n} \\ \tau_{21} & \tau_{22} & ... & \tau_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_{n1} & \tau_{n2} & ... & \tau_{nn} \end{vmatrix} \tag{3.9}$$

where

$$\tau_{ij} = d^2(x_0, x_i) + d^2(x_0, x_j) - d^2(x_i, x_j) \tag{3.10}$$

for $1 \leq i, j \leq n$.

Lemma 3.10. *Let X be a finite metric space with cardX=$n + 1$, $n \geq 1$. Then X is isometrically embeddable in E^n if and only if the inequality*

$$Sch(x_0, x_1, ..., x_k) \geq 0 \tag{3.11}$$

holds for every $(x_0, x_1, ..., x_k) \in X^{k+1}, k = 1, 2, ..., n$.

Proof. Suppose that X is isometrically embeddable in E^n. The determinant $Sch(x_0, x_1, ..., x_k)$ vanishes if there is $i_0 \in [1, ..., k]$ with $x_{i_0} = x_0$ or there are distinct $i_0, j_0 \in [1, ..., k]$ such that $x_{i_0} = x_{j_0}$. Hence it is sufficient to show (3.11) if all $x_i, i \in [0, ..., n]$ are pairwise distinct. Since X is isometrically embeddable in E^n, Theorem 3.9 implies that quadratic form (3.8) is positive semidefinite. A well-known criterion states that a quadratic form is positive semidefinite if and only if all principal minors of the matrix of this form are nonnegative, see, for example,[13, p. 272]. Hence (3.11) follows.

Conversely, suppose that inequality (3.11) holds for all $(x_0, x_1, .., x_k) \in X^{k+1}$, $k = 1, ..., n$. The criterion given above, implies that quadratic form (3.8) is positive semidefinite. Let us denote by m the rank of this form. It is clear that $m \leq n$. Consequently there is an isometric embedding of X in E^m and thus in E^n also. ∎

The above lemma is closely related to the main result of [16]. The next proposition is similar to Proposition 3.1.

Proposition 3.11. *Let $n \in \mathbb{N}$ and let (X, d) be a nonvoid metric space. The metric space (X, d) is isometrically embeddable in E^n if and only if the inequality*

$$Sch(x_0, x_1, ..., x_k) \geq 0 \tag{3.12}$$

holds for every $(x_0, x_1, ..., x_k) \in X^{k+1}$ with $k = 1, ..., n$ and the equality

$$Sch(x_0, x_1, ..., x_k) = 0 \tag{3.13}$$

holds for every $(x_0, x_1, ..., x_k) \in X^{k+1}$ with $k = n + 1, n + 2$.

Proof. If (3.12) holds for all $(x_0, x_1, ..., x_k) \in X^{k+1}$, $k = 1, ..., n$ and (3.13) holds for all $(x_0, x_1, ..., x_k) \in X^{k+1}$, $k = n + 1, n + 2$, then quadratic form (3.8) is positive semidefinite and the rank of this form is at most n. (Recall that the rank of quadratic form is the rank of matrix of this form.) Consequently if A is a subspace of X and $cardX \leq n + 2$ then, by Lemma 3.10, A is isometrically embeddable in E^n. Now Lemma 3.2 implies that X is also isometrically embeddable in E^n.

It still remains to note that if X is isometrically embeddable in E^n, then (3.12) and (3.13) follows directly from Theorem 3.9 and Lemma 3.10. ∎

Let (X, d) be a metric space with marked point p. Define the function $S_{k+1} : X^{k+1} \to \mathbb{R}$ by analogy with the function Θ_{k+1}

$$S_{k+1}(x_0, x_1, ..., x_k) := \begin{cases} \frac{Sch(x_0, x_1, ..., x_k)}{(\bigvee\limits_{n=0}^{k} d(x_n, p))^{2k}}, & \text{if } (x_0, x_1, ..., x_k) \neq (p, p, ..., p) \\ 0, & \text{if } (x_0, x_1, ..., x_k) = (p, p, ..., p). \end{cases} \tag{3.14}$$

The next theorem is similar to Theorem 3.4 but it presents some other necessary and sufficient conditions of isometric embeddability of all pretangent spaces to X at the marked point p.

Applying the Transfer Principle and Corollary 2.8 to Proposition 3.11 we obtain the following infinitesimal analog of Schoenberg's Embedding Theorem.

Theorem 3.12. *Let (X, d) be a metric space with a marked point p and let $n \in \mathbb{N}$. Every $\Omega_{p,\tilde{r}}^X$ is isometrically embeddable in E^n if and only if the inequality*

$$\liminf_{x_0, x_1, \dots, x_k \to p} S_{k+1}(x_0, x_1, \dots, x_k) \geq 0 \tag{3.15}$$

holds for all $k \leq n$ and the equality

$$\liminf_{x_0, x_1, \dots, x_k \to p} S_{k+1}(x_0, x_1, \dots, x_k) = 0 \tag{3.16}$$

holds for all $k = n + 1$ and $k = n + 2$.

4 Application of Blumenthal's embedding theorem

Theorem 3.4 and Theorem 3.12 proved in the previous section describe some necessary and sufficient conditions under which **all** pretangent spaces $\Omega_{p,\tilde{r}}^X$ are isometrically embeddable in E^n with given n but it is possible that there exists an isometric embedding of **a fixed** $\Omega_{p,\tilde{r}}^X$ in E^n even if these conditions do not occur. We study this situation in the present section. It turns out that a suitable tool for these studies is an infinitesimal modification of Blumenthal's embedding theorem. We first reformulate this theorem in a appropriate form.

Theorem 4.1. *A metric space (X, d) is isometrically embeddable in $E^n, n \geq 1$, if and only if there are some points $a_0, a_1, \dots, a_n \in X$ such that*

$$(-1)^{k+1} D_k(a_0, a_1, \dots, a_k) > 0 \tag{4.1}$$

for each $k = 1, 2, \dots, n$ and the equalities

$$D_{k+1}(a_0, a_1, \dots, a_n, y) = 0, \quad D_{k+2}(a_0, a_1, \dots, a_n, y, z) = 0 \tag{4.2}$$

hold for all $y, z \in X$. Moreover if (4.1) holds for $k = 1, 2, \dots, n$ and (4.2) holds for all $y, z \in X$, then there are no isometric embeddings of X in E^m with $m < n$.

The proof of Theorem 4.1 is a straightforward application of theorems 41.1 and 42.1 and of Lemma 42.1 from [6] to the standard form of Blumenthal's embedding theorem, see Theorem 38.1 in [6], and we omit it here.

Theorem 4.2. *Let (X, d) be a metric space with a marked point p. If there are a **tangent** space $\Omega_{p,\tilde{r}}^X$ and a natural number n such that $\Omega_{p,\tilde{r}}^X$ is isometrically embeddable in E^n but there are no isometric embeddings of this $\Omega_{p,\tilde{r}}^X$ in E^l with $l < n$, then there exist some sequences*

$$\tilde{x}^i = \{x_m^i\}_{m \in \mathbb{N}} \in \tilde{X}, \quad i = 0, 1, \dots, n,$$

having the following properties:

(i) *The limit relations*

$$\lim_{m \to \infty} x_m^0 = \lim_{m \to \infty} x_m^1 = ... = \lim_{m \to \infty} x_m^n = p \tag{4.3}$$

and

$$\bigwedge_{k=1}^{n} \liminf_{m \to \infty} \Theta_{k+1}(x_m^0, x_m^1, ..., x_m^k) > 0 \tag{4.4}$$

hold;

(ii) *The equalities*

$$\lim_{m \to \infty} \Theta_{n+2}(x_m^0, x_m^1, ..., x_m^n, y_m) = 0 \tag{4.5}$$

and

$$\lim_{m \to \infty} \Theta_{n+3}(x_m^0, x_m^1, ..., x_m^n, y_m, u_m) = 0 \tag{4.6}$$

hold for $\tilde{y} = \{y_m\}_{m \in \mathbb{N}} \in \tilde{X}, \tilde{u} = \{u_m\}_{m \in \mathbb{N}} \in \tilde{X}$ *if*

$$\lim_{m \to \infty} u_m = \lim_{m \to \infty} y_m = p. \tag{4.7}$$

Conversely, suppose that there are $\tilde{x}^0, ..., \tilde{x}^n \in \tilde{X}$ *having properties (i)-(ii), then there is a* **pretangent** *space* $\Omega_{p,\tilde{r}}^X$ *which is isometrically embeddable in* E^n *but there are no isometric embeddings of this* $\Omega_{p,\tilde{r}}^X$ *in* E^l *with* $l < n$.

Recall that the functions Θ_k were defined by (3.5).

Lemma 4.3. *Let* (X, d) *be a metric space with a marked point* p, $\mathfrak{B}$ *a countable subfamily of* $\tilde{X}, \tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ *a normalizing sequence and let* $\tilde{X}_{p,\tilde{r}}$ *be a maximal self-stable family. Suppose that the inequality*

$$\limsup_{n \to \infty} \frac{d(y_n, p)}{r_n} < \infty \tag{4.8}$$

holds for every $\tilde{y} = \{y_n\}_{n \in \mathbb{N}} \in \mathfrak{B}$ *and that a pretangent space* $\Omega_{p,\tilde{r}}^X = \pi(\tilde{X}_{p,\tilde{r}})$ *is separable and tangent. Then there is a strictly increasing, infinite sequence* $\{n_k\}_{k \in \mathbb{N}}$ *of natural numbers such that for every* $\tilde{y} = \{y_n\}_{n \in \mathbb{N}} \in \mathfrak{B}$ *there exists* $\tilde{z} = \{z_n\}_{n \in \mathbb{N}} \in \tilde{X}_{p,\tilde{r}}$ *with* $\tilde{z}' = \tilde{y}'$, *i.e., the equality*

$$z_{n_k} = y_{n_k} \tag{4.9}$$

holds for all $k \in \mathbb{N}$.

For the proof see Proposition 3 in [1].

Proof of Theorem 4.2. Suppose that there are a tangent space $\Omega_{p,\tilde{r}}^X$ and a natural n such that $\Omega_{p,\tilde{r}}^X$ is isometrically embeddable in E^n but there are no isometric embeddings of $\Omega_{p,\tilde{r}}^X$ in E^l with $l < m$. By Theorem 4.1 the metric space $\Omega_{p,\tilde{r}}^X$ contains some points $\beta_0, \beta_1, .., \beta_n$ such that

$$(-1)^{k+1} D_k(\beta_0, ..., \beta_k) > 0 \tag{4.10}$$

for $k = 1, ..., n$ and

$$D_{n+1}(\beta_0, \beta_1, .., \beta_n, \gamma) = D_{n+2}(\beta_0, \beta_1, .., \beta_n, \gamma, v) = 0 \tag{4.11}$$

for all $\gamma, v \in \Omega^X_{p,\tilde{r}}$. Let $\tilde{X}_{p,\tilde{r}}$ be a maximal self-stable family corresponding $\Omega^X_{p,\tilde{r}}$ and let $\tilde{x}^i = \{x^i_m\}_{m \in \mathbb{N}}$, $i = 0, .., n$ be elements of $\tilde{X}_{p,\tilde{r}}$ such that $\pi(\tilde{x}^i) = \beta_i$, $i = 0, ..., n$, where π is the natural projection. We claim that these $\tilde{x}^0, ..., \tilde{x}^n$ have properties (i) and (ii).

To prove it note firstly that (4.3) follows from (2.2). Moreover we have the equality

$$\lim_{m \to \infty} \frac{1}{r_m} \left(\bigvee_{i=0}^{k} d(x^i_m, p) \right) = \bigvee_{i=0}^{k} \rho(\alpha, \beta_i)$$

for $k = 1, ..., n$ where $\alpha = \pi(\tilde{p})$. This equality and (3.5) imply

$$\bigwedge_{k=1}^{n} \liminf_{m \to \infty} \Theta_{k+1}(x^0_m, ..., x^k_m) =$$

$$\bigwedge_{k=1}^{n} \left(\frac{1}{\left(\bigvee_{i=0}^{k} \rho(\alpha, b_i) \right)^{2k}} \liminf_{m \to \infty} \frac{(-1)^{k+1} D_k(x^0_m, ..., x^k_m)}{r_m^{2k}} \right)$$

$$= \bigwedge_{k=1}^{n} \left(\frac{1}{\left(\bigvee_{i=0}^{k} \rho(\alpha, b_i) \right)^{2k}} (-1)^{k+1} D_k(\beta_0, ..., \beta_k) \right). \quad (4.12)$$

It should be pointed here that

$$\bigvee_{i=0}^{k} \rho(\alpha, \beta_i) > 0 \quad (4.13)$$

for $k = 1, .., n$. Indeed in the opposite case we have $\alpha = \beta_0 = \beta_1 = ... = \beta_k$ that implies $D_k(\beta_0, \beta_1, ..., \beta_k) = 0$ for $k = 1, ..., n$, contrary to (4.10). Now using (4.10), (4.12) and (4.13) we obtain (4.4).

Let us prove property (ii). Let $\tilde{y} = \{y_m\}_{m \in \mathbb{N}} \in \tilde{X}$ be a sequence such that

$$\lim_{m \to \infty} y_m = p$$

and let c be a limit point of the sequence $\{\Theta_{n+2}(x^0_m, x^1_m, ..., x^n_m, y_m)\}_{m \in \mathbb{N}}$, i.e.,

$$\lim_{k \to \infty} \Theta_{n+2}(x^0_{m_k}, x^1_{m_k}, ..., x^n_{m_k}, y_{m_k}) = c \quad (4.14)$$

for some sequence $\{m_k\}_{k \in \mathbb{N}}$. We must prove $c = 0$.

Inequalities (2.12) imply that the function Θ_{n+2} is bounded from above and from below. Consequently c is finite. For convenience we write $x^{1,0}_k = x^0_{m_k}, ...,$ $x^{1,n}_k = x^n_{m_k}$, $y^1_k = y_{m_k}$ and $r^1_k = r_{m_k}$ so that we have

$$\lim_{k \to \infty} \Theta_{n+2}(x^{1,0}_k, x^{1,1}_k, ..., x^{1,n}_k, y^1_k) = c. \quad (4.15)$$

Note also that the space $\Omega^X_{p,\tilde{r}}$ is tangent by the condition of the theorem and separable as an isometrically embeddable in E^n space. Furthermore, according to

Remark 2.5 the family $\tilde{X}_{p,\tilde{r}'} = in_{\tilde{r}'}(\tilde{X}_{p,\tilde{r}})$, see (2.4), is maximal self-stable w. r. t. the normalizing sequence $\tilde{r}' = \{r_{m_k}\}_{k\in\mathbb{N}}$, so that we can use Lemma 4.3. If the inequality

$$\limsup_{k\to\infty} \frac{d(y_k^1, p)}{r_k^1} < \infty \tag{4.16}$$

holds, then using this lemma with $\mathfrak{B}$ consisting of the unique element $\{y_k^1\}_{k\in\mathbb{N}}$ we can find $\{z_k^1\}_{k\in\mathbb{N}} \in \tilde{X}_{p,\tilde{r}'}$ and strictly increasing infinite sequence $\{k_j\}_{j\in\mathbb{N}}$ of natural numbers such that

$$y_{k_j}^1 = z_{k_j}^1$$

for all $j \in \mathbb{N}$. Using Remark 2.5 we see that there is $\tilde{z} = \{z_m\}_{m\in\mathbb{N}} \in \tilde{X}_{p,\tilde{r}}$ such that

$$z_{m_k} = z_k^1$$

for all $k \in \mathbb{N}$. Write $\gamma = \pi(\tilde{z})$. Similarly (4.12) we have

$$c = \lim_{k\to\infty} \Theta_{n+2}(x_k^{1,0}, x_k^{1,1}, ..., x_k^{1,n}, y_k^1) = \lim_{j\to\infty} \Theta_{n+2}(x_{k_j}^{1,0}, x_{k_j}^{1,1}, ..., x_{k_j}^{1,n}, z_{k_j}^1)$$

$$= \lim_{m\to\infty} \Theta_{n+2}(x_m^0, ..., x_m^n, z_m) =$$

$$\frac{1}{\left(\left(\bigvee_{i=0}^{n}\rho(\alpha, \beta_i)\right) \vee (\rho(\alpha, \gamma))\right)^{2(n+1)}}(-1)^{n+2}D_{n+1}(\beta_0, ..., \beta_n, \gamma). \tag{4.17}$$

It follows from (4.11) and (4.13) that

$$0 = \text{sign}D_{n+1}(\beta_0, ..., \beta_n, \gamma) = (-1)^{(n+2)}\text{sign}\,c.$$

Thus $c = 0$ if (4.16) holds. Suppose contrary that

$$\limsup_{k\to\infty} \frac{d(y_k^1, p)}{r_k^1} = \infty.$$

Let $\tilde{y}^{1'} = \{y_{k_j}^1\}_{k\in\mathbb{N}}$ be a subsequence of $\{y_k^1\}_{k\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \frac{d(y_{k_j}^1, p)}{r_{k_j}^1} = \infty. \tag{4.18}$$

In this case we have

$$\left(\bigvee_{i=0}^{n} d(x_{k_j}^{1,i}, p)\right) \vee \left(d(y_{k_j}^1, p)\right) = d(y_{k_j}^1, p) \tag{4.19}$$

for all sufficiently large j. In addition, (4.18) and (4.19) imply the limit relations

$$\lim_{j\to\infty} \frac{d(x_{k_j}^{1,s}, x_{k_j}^{1,t})}{\left(\bigvee_{i=0}^{n} d(x_{k_j}^{1,i}, p)\right) \vee \left(d(y_{k_j}^1, p)\right)} = 0, \tag{4.20}$$

$$\lim_{j\to\infty} \frac{d(x_{k_j}^{1,t}, y_{k_j}^1)}{\left(\bigvee_{i=0}^n d(x_{k_j}^{1,i}, p)\right) \vee \left(d(y_{k_j}^1, p)\right)} = 1 \tag{4.21}$$

for all $s, t \in \{1, 2, ..., n\}$. Consequently we have

$$\lim_{j\to\infty} \Theta_{n+2}(x_{k_j}^{1,0}, x_{k_j}^{1,1}, ..., x_{k_j}^{1,n}, y_{k_j}^1) = (-1)^{n+2} \begin{vmatrix} 0 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 0 & ... & 0 & 1 \\ 1 & 0 & 0 & ... & 0 & 1 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 1 & 0 & 0 & ... & 0 & 1 \\ 1 & 1 & 1 & ... & 1 & 0 \end{vmatrix}. \tag{4.22}$$

The second row of this determinant coincides with the third one, thus the determinant is zero. Hence in (4.14) we have $c = 0$.

Let us turn to equality (4.6). Consider, as in (4.15), two sequences $\tilde{y} = \{y_m\}_{m\in\mathbb{N}}$ and $\tilde{u} = \{u_m\}_{m\in\mathbb{N}}$ such that

$$p = \lim_{m\to\infty} y_m = \lim_{m\to\infty} u_m$$

and

$$\lim_{k\to\infty} \Theta_{n+3}(x_k^{1,0}, x_k^{1,1}, ..., x_k^{1,n}, y_k^1, u_k^1) = c \tag{4.23}$$

where the constant c is an arbitrary limit number of the sequence

$$\{\Theta_{n+3}(x_m^0, x_m^1, ..., x_m^n, y_m, u_m)\}_{m\in\mathbb{N}}.$$

As in the proof of equality (4.5) we want to use Lemma 4.3 for the demonstration of equality $c = 0$. In accordance with this lemma it is relevant to consider three possible cases:

(i_1) $\quad \limsup\limits_{k\to\infty} \dfrac{d(y_k^1, p)}{r_k^1} < \infty \quad$ and $\quad \limsup\limits_{k\to\infty} \dfrac{d(u_k^1, p)}{r_k^1} < \infty;$

(i_2) $\quad \limsup\limits_{k\to\infty} \dfrac{d(y_k^1, p)}{r_k^1} < \infty \quad$ and $\quad \limsup\limits_{k\to\infty} \dfrac{d(u_k^1, p)}{r_k^1} = \infty$

$\qquad$ or

$\qquad \limsup\limits_{k\to\infty} \dfrac{d(y_k^1, p)}{r_k^1} = \infty \quad$ and $\quad \limsup\limits_{k\to\infty} \dfrac{d(u_k^1, p)}{r_k^1} < \infty;$

(i_3) $\quad \limsup\limits_{k\to\infty} \dfrac{d(y_k^1, p)}{r_k^1} = \infty \quad$ and $\quad \limsup\limits_{k\to\infty} \dfrac{d(u_k^1, p)}{r_k^1} = \infty.$

Reasoning as in the proofs of (4.17) and (4.22) we can show that $c = 0$ if (i_1) or (i_2) holds. Thus it is sufficient to consider only case (i_3). Passing to the subsequence we may suppose that

$$\lim_{j\to\infty} \frac{d(y_{k_j}^1, p)}{r_{k_j}^1} = \lim_{j\to\infty} \frac{d(u_{k_j}^1, p)}{r_{k_j}^1} = \infty. \tag{4.24}$$

Indeed, if there are no subsequences for which (4.24) holds, then we can reduce the situation to cases (i_1) or (i_2) which were considered above. In addition to (4.24) we may assume that there are $k_1, k_2 \in (0, \infty)$ such that

$$\lim_{j \to \infty} \frac{d(y^1_{k_j}, p)}{d(u^1_{k_j}, p)} = k_1 \tag{4.25}$$

and

$$\lim_{j \to \infty} \frac{d(y^1_{k_j}, u^1_{k_j})}{d(u^1_{k_j}, p)} = k_2 \tag{4.26}$$

because if

$$\lim_{j \to \infty} \frac{d(y^1_{k_j}, p)}{d(u^1_{k_j}, p)} = 0 \quad \text{or} \quad \lim_{j \to \infty} \frac{d(y^1_{k_j}, p)}{d(u^1_{k_j}, p)} = \infty, \tag{4.27}$$

then, similarly (4.22), we find

$$\lim_{k \to \infty} \Theta_{n+3}(x^{1,0}_k, x^{1,1}_k, ..., x^{1,n}_k, y^1_k, u^1_k) = (-1)^{n+3} \begin{vmatrix} 0 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 0 & ... & 0 & 1 \\ 1 & 0 & 0 & ... & 0 & 1 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 1 & 0 & 0 & ... & 0 & 1 \\ 1 & 1 & 1 & ... & 1 & 0 \end{vmatrix} = 0$$

or, respectively,

$$\lim_{k \to \infty} \Theta_{n+3}(x^{1,0}_k, x^{1,1}_k, ..., x^{1,n}_k, y^1_k, u^1_k) = (-1)^{n+3} \begin{vmatrix} 0 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 0 & ... & 0 & 1 \\ 1 & 0 & 0 & ... & 0 & 1 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 1 & 1 & 1 & ... & 1 & 0 \\ 1 & 0 & 0 & ... & 0 & 1 \end{vmatrix} = 0.$$

Limit relations (4.24), (4.25) and (4.26) imply that the quantity

$$(-1)^{n+3} \lim_{k \to \infty} \Theta_{n+3}(x^{1,0}_k, x^{1,1}_k, ..., x^{1,n}_k, y^1_k, u^1_k)$$

equals the determinant of the matrix with the second and third rows of the form

$$(1, 0, 0, ..., 0, k_1, 1) \quad \text{if} \quad k_1 < 1 \quad \text{or} \quad (1, 0, 0, ..., 0, 1, k_1) \quad \text{if} \quad k_1 \geq 1.$$

Consequently we have

$$\lim_{k \to \infty} \Theta_{n+3}(x^{1,0}_k, x^{1,1}_k, ..., x^{1,n}_k, y^1_k, u^1_k) = c = 0$$

in all possible cases and (4.6) follows.

Suppose now that there exist some sequences $\tilde{x}^i$, $i = 0, ..., n$, with the properties (i) and (ii). Limit relations (4.3) imply that the quantities

$$r_m := \bigvee_{i=0}^{n} d(x_m^i, p)$$

become vanishingly small with $m \to \infty$. Consequently we can consider $\tilde{r} = \{r_m\}_{m \in \mathbb{N}}$ as a normalizing sequence. As in the proof of the Theorem 3.4, going to subsequence we can assume all $\tilde{x}^i$ and $\tilde{p}$ to be mutually stable. Let $\tilde{X}_{p,\tilde{r}}$ be a maximal self-stable family such that $\tilde{x}^i \in \tilde{X}_{p,\tilde{r}}$ for $i = 0, ..., n$ and let $\Omega_{p,\tilde{r}}^X$ be the metric identification of $\tilde{X}_{p,\tilde{r}}$. Write

$$\alpha_0 = \pi(\tilde{x}^0), \alpha_1 = \pi(\tilde{x}^1), ..., \alpha_n = \pi(\tilde{x}^n)$$

where π is the natural projection of $\tilde{X}_{p,\tilde{r}}$ on $\Omega_{p,\tilde{r}}^X$. Going to the limit under $m \to \infty$ and using (4.4) we obtain

$$\bigwedge_{k=1}^{n} \liminf_{m \to \infty} \Theta_{k+1}(x_m^0, x_m^1, ..., x_m^k) = \bigwedge_{k=1}^{n} (-1)^{k+1} D_k(\alpha_0, \alpha_1, ..., \alpha_k) > 0. \qquad (4.28)$$

Similarly for all $\beta, \gamma \in \Omega_{p,\tilde{r}}^X$ property (ii) implies that

$$\lim_{m \to \infty} \Theta_{n+2}(x_m^0, x_m^1, ..., x_m^n, y_m) = \lim_{m \to \infty} \Theta_{n+3}(x_m^0, x_m^1, ..., x_m^n, y_m, u_m)$$

$$= (-1)^{n+2} D(\alpha_0, \alpha_1, ..., \alpha_n, \beta) = (-1)^{n+3} D_{n+2}(\alpha_0, \alpha_1, ..., \alpha_n, \beta, \gamma) = 0, \qquad (4.29)$$

where $\{y_m\}_{m \in \mathbb{N}} \in \tilde{X}_{p,\tilde{r}}$ and $\{u_m\}_{m \in \mathbb{N}} \in \tilde{X}_{p,\tilde{r}}$ such that $\pi(\{y_m\}_{m \in \mathbb{N}}) = \beta$ and $\pi(\{u_m\}_{m \in \mathbb{N}}) = \gamma$. Hence by Theorem 4.1 the pretangent space $\Omega_{p,\tilde{r}}^X$ has an isometric embedding in E^n but there are no isometric embeddings of $\Omega_{p,\tilde{r}}^X$ in E^l with $l < n$, as required. $\blacksquare$

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Institute of Applied Mathematics and Mechanics of NASU,
R. Luxemburg str. 74, Donetsk 83114, Ukraine.
E-mail: biletvictoriya@mail.ru, aleksdov@mail.ru