An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces

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Abstract

In this paper we prove an existence result for some class of variational boundary value problems for quasilinear elliptic equations in the Musielak-Orlicz spaces $W^m L_\varphi(\Omega)$, under the assumption that the conjugate function of φ satisfies the Δ_2 condition. An imbedding theorem has also been provided without assuming this condition.

1 Introduction

This paper is concerned with the existence of solutions for variational boundary value problems for quasi-linear elliptic equations of the form

$$A(u) = f,$$

where the operator A is in the form:

$$A(u) \equiv \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla u, ..., \nabla^m u) \tag{1}$$

on an open subset Ω of R^n. Existence theorems for problems of this type were first obtained by Višik [23, 24] using compactness arguments and a priori estimates on $(m+1)st$ derivatives. Since 1963, these problems have been extensively studied by Browder and others in the context of the theory of mappings of monotone type

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from a reflexive Banach space to its dual and in the case where the coefficients A_α have polynomial growth in u and its derivatives [2] , [3] , [20]. From 1970 these results have been extended by Donaldson [6], Gossez [15] , [16] and Gossez and Mustonen in [17] to the case where the coefficients A_α do not necessarily have polynomial growth in u and its derivatives. The Banach spaces in which the problems are formulated (the Orlicz-Sobolev spaces) are not reflexive and the corresponding mappings of monotone type are not bounded nor everywhere defined and do not generally satisfy a global a priori bounded(and consequently are not generally coercive).

In the last decade several works have been concerned to extend the classical polynomial growth to the non-standard growth case in the so-called variable exponent Sobolev spaces (see [14] and references within), and also [25] .

Recently Mihăilescu and Rădulescu in [21] and Fan and Guan in [9], [10] have obtained new results which improved the already known existence results for the p(x)-Laplacian operator in the Musielak-Orlicz-Sobolev spaces $W^1L_\varphi(\Omega)$ under some assumptions such as the condition Δ_2 on φ and also the uniform convexity of φ which assure that the space $L_\varphi(\Omega)$ is reflexive.

Our purpose in this paper is to initiate a study of these problems in the general case when the Musielak-Orlicz-Sobolev spaces $W^mL_\varphi(\Omega)$ are not reflexive. The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field [18].

The main difficulty encountered is the construction of a suitable complementary system to formulate the problems. Our existence result is only obtained with the condition that the conjugate function ψ of φ has the Δ_2 property. It is a generalization of the result in [6].

Note that the Δ_2 condition on ψ in this paper is only used for building the suitable complementary system with non-attendance of the analogous of [15, Theorem 1.3] in the context of Musielak-Orlicz-Sobolev spaces.

This result can for example be applied for finding a weak solution for the φ-Laplacian equation

$$\Delta_\varphi u(= div(\frac{a(x,|\nabla u|)}{|\nabla u|}.\nabla u)) + f = 0$$

where a is the derivative of φ with respect to t.

One of the main results of this paper is to give some imbedding theorems in $W^mL_\varphi(\Omega)$ for a general Musielak-Orlicz function φ. These theorems, which are very useful in the literature of the PDE and the Banach spaces, generalize the imbedding results in [7], [1] and [15].

In the particular case when $\varphi(x,t) = t^{p(x)}$, our results give essential improvements of some imbedding theorems that were already published e.g. [8], [11] and [13]. They also improve the existence result for (1) in the statement of the variable exponent Sobolev spaces $W^{m,p(x)}$ by avoiding the condition of continuity or log-Holder continuity of $p(.)$ and also the condition that $p^+ = \text{ess sup}_{x\in\Omega}\, p(x)$ is finite, see Corollary 2 below.

Section 2 contains some preliminaries. In Section 3 we introduce our main results, the compact imbedding (subsection 3.1) and the existence results (subsection 3.2).

2 Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [22]. We also include the definition of complementary system, an abstract result and some preliminaries Lemmas to be used later.

2.1 Musielak-Orlicz-Sobolev spaces

Let Ω be an open subset of R^n and let φ be a real-valued function defined in $\Omega \times R_+$ and satisfying the following conditions :
a) $\varphi(x,.)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$, and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0,$$

$$\lim_{t\to \infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty,$$

b) $\varphi(.,t)$ is a measurable function.
A function $\varphi(x,t)$, which satisfies the conditions a) and b) is called a Musielak-Orlicz function. For a Musielak-Orlicz function $\varphi(x,t)$ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function with respect to t, φ_x^{-1} i.e.

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t$$

For any two Musielak-Orlicz functions φ and γ we introduce the following ordering :
c) if there exists two positives constants c and T such that for almost everywhere $x \in \Omega$:

$$\varphi(x,t) \leq \gamma(x,ct) \text{ for } t \geq T$$

we write $\varphi \prec \gamma$ and we say that γ dominate φ globally if $T = 0$ and near infinity if $T > 0$.

d) if for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t\to 0}(\sup_{x\in\Omega} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0 \text{ or } \lim_{t\to \infty}(\sup_{x\in\Omega} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0$$

we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near infinity respectively.

In the following the measurability of a function $u : \Omega \mapsto R$ means the Lebesgue measurability.

We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx$$

where $u : \Omega \mapsto R$ is a measurable function.
The set

$$K_\varphi(\Omega) = \left\{u : \Omega \to R \text{ mesurable } / \varrho_{\varphi,\Omega}(u) < +\infty\right\}.$$

is called the Musielak-Orlicz class (the generalized Orlicz class).

The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.
Equivalently:

$$L_\varphi(\Omega) = \left\{u : \Omega \to R \text{ mesurable } / \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \text{ for some } \lambda > 0\right\}$$

Let

$$\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\},$$

that is, ψ is the Musielak-Orlicz function complementary to (or conjugate of) $\varphi(x,t)$ in the sense of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0/ \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx, \leq 1\}.$$

which is called the Luxemburg norm and the so-called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \leq 1} \int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak-Orlicz function complementary (or conjugate) to φ. These two norms are equivalent [22].

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$ [22].

We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ has the Δ_2 property for large values of t, or for all values of t, according to whether Ω has finite measure or not, i.e., there exists $k > 0$ independent of $x \in \Omega$ and a nonnegative function h , integrable in Ω such that $\varphi(x, 2t) \leq k\varphi(x,t) + h(x)$ for large values of t, or for all values of t.

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m\ D^\alpha u \in L_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with nonnegative integers α_i $|\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.

Let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } ||u||^m_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$$

for $u \in W^m L_\varphi(\Omega)$. These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\langle W^m L_\varphi(\Omega), ||u||^m_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [22]:

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c. \tag{2}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \prod L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed. Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and $W_0^m E_\varphi(\Omega)$ is the (norm) closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\}$$

$$W^{-m} E_\psi(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega)\}$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For two complementary Musielak-Orlicz functions φ and ψ the following inequality is called the young inequality [22]:

$$t.s \leq \varphi(x,t) + \psi(x,s) \text{ for } t, s \geq 0, \ x \in \Omega$$

This inequality implies that

$$|||u|||_{\varphi,\Omega} \leq \varrho_{\varphi,\Omega}(u) + 1.$$

We have also for two complementary Musielak-Orlicz functions φ and ψ if $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$ the Hölder inequality [22]:

$$|\int_\Omega u(x)v(x)\, dx| \leq ||u||_{\varphi,\Omega} |||v|||_{\psi,\Omega}.$$

In the particular case when $\varphi(x,t) = t^{p}(x)$ we use the notations $L^{p(x)}(\Omega) = L_\varphi(\Omega)$, and $W^{m,p(x)}(\Omega) = W^m L_\varphi(\Omega)$. These spaces are called Variable exponent Lebesgue and Sobolev spaces.

We recall that a family $\mathfrak{R}$ of functions $u(x)$ has equi-absolutely continuous integrals if for arbitrary $\varepsilon > 0$ an $h > 0$ can be found such that for all functions in the family $\mathfrak{R}$ we have

$$\int_E u(x)dx < \varepsilon$$

provided $|E| < h$. Where $|E|$ is the measure of the set E.

2.2 Complementary system

Definition 1. *Let Y and Z be two real Banach spaces in duality with respect to a continuous pairing $<,>$ and let Y_0 and Z_0 be subspaces of Y and Z respectively. Then $(Y, Y_0; Z, Z_0)$ is called a complementary system if, by means of $<,>$, Y_0^* can be identified (i.e.,is linearly homeomorphic) to Z and Z_0^* to Y.*

Let φ and ψ be two complementary Musielak-Orlicz functions then $(L_\varphi(\Omega), E_\varphi(\Omega); L_\psi(\Omega), E_\psi(\Omega))$ is a complementary system. Other examples are $(X^{**}, X; X^*, X^*)$ and $(X^*, X^*; X^{**}, X)$ where X is Banach space. Note that in a complementary system, Y_0 is $\sigma(Y,Z)$ dense in Y. Note also that if cl Y_0 [cl Z_0] denotes the (norm) closure of Y_0 [Z_0] in Y [Z], then $(Y, clY_0; Z, clZ_0)$ is a complementary system.

The following lemma gives an important method by which from a complementary system $(Y, Y_0; Z, Z_0)$ and a closed subspace E of Y, one can construct a new complementary system $(E, E_0; F, F_0)$. some restriction must be imposed on E. Define $E_0 = E \cap Y_0$, $F = Z/E_0^\perp$ and $F_0 = \{z + E_0^\perp; z \in Z_0\} \subset F$, where $\perp$ denotes the orthogonal in the duality (Y, Z), i.e. $E_0^\perp = \{z \in Z; < y, z >= 0 \text{ for all } y \in E_0\}$.

Lemma 1. *[15] The pairing $<,>$ between Y and Z induces a pairing between E and F if and only if E_0 is $\sigma(Y,Z)$ dense in E. In this case, $(E, E_0; F, F_0)$ is a complementary system if E is $\sigma(Y, Z_0)$ closed, and conversely, when Z_0 is complete, E is $\sigma(Y, Z_0)$ closed if $(E, E_0; F, F_0)$ is a complementary system.*

Corollary 1. *Let φ and ψ be two complementary Musielak-Orlicz functions , we assume that ψ has the Δ_2 property. Then $W_0^m L_\varphi(\Omega)$ generates a complementary system in $(\Pi L_\varphi(\Omega), \Pi E_\varphi(\Omega); \Pi L_\psi(\Omega), \Pi L_\psi(\Omega))$*

Indeed, by definition $D(\Omega)$ is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ dense in $W_0^m L_\varphi(\Omega)$ and the fact that ψ has the Δ_2 property implies that $\sigma(\Pi L_\varphi, \Pi E_\psi) \equiv \sigma(\Pi L_\varphi, \Pi L_\psi)$. Hence $D(\Omega)$ is $\sigma(\Pi L_\varphi, \Pi L_\psi)$ dense in $W_0^m L_\varphi(\Omega)$ and applying Lemma 1 we obtain that $(W_0^m L_\varphi(\Omega), W_0^m E_\varphi(\Omega), W^{-m} L_\psi(\Omega), W^{-m} E_\psi(\Omega))$ is a complementary system.

2.3 An Abstract Results

Let $(Y, Y_0; Z, Z_0)$ be a complementary system and T be a mappings from the domain $D(T)$ in Y to Z which satisfy the following conditions, with respect to some element $\bar{y} \in Y_0$ and $f \in Z_0$:

(i) (finite continuity) $D(T) \supset Y_0$ and T is continuous from each finite dimensional subspaces of Y_0 to the $\sigma(Z, Y_0)$ topology of Z,

(ii) (sequential pseudo-monotonicity) for any sequence $\{y_i\}$ with $y_i \to y \in Y$ for $\sigma(Y, Z_0)$, $T(y_i) \to z \in Z$ for $\sigma(Z, Y_0)$ and lim sup $< T(y_i), y_i > \leq < z, y >$, it follows that $T(y) = z$ and $< T(y_i), y_i > \to < z, y >$,

(iii) $T(y)$ remains bounded in Z whenever $y \in D(T)$ remains bounded in Y and $< y - \bar{y}, Tu >$ remains bounded from above,

(iv) $< y - \bar{y}, t(y) - f >$ is > 0 when $y \in D(T)$ has sufficiently large norm in Y.

It is of importance to note that the condition (iii) is weaker than the condition that T transforms each bounded set of Y into a bounded set of Z, and also that the condition (iv) is weaker than the assumption of coercivity, because in our applications, the mapping T will generally not transform a bounded set into a bounded set nor be coercive.

Theorem 1. *[17] Let $(Y, Y_0; Z, Z_0)$ be a complementary system and let $T : D(T) \subset Y \to\to Z$ satisfy (i)...(iv). Then Z_0 is contained in the range of T.*

2.4 Preliminary lemmas

Lemma 2. *if a sequence $g_n \in L_\varphi(\Omega)$ converges in measure to a measurable function g and if g_n remains bounded in $L_\varphi(\Omega)$, then $g \in L_\varphi(\Omega)$ and $g_n \to g$ for $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$.*

Proof. In virtue of the fact that every sequence of functions in $L_\varphi(\Omega)$ which are bounded in norm contains an $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$ convergent subsequence. It is therefore sufficient in our case to show that for any subsequence $g_{n_k}(x)$ which converges in $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$ to $g_0(x)$, we have $g_0(x) = g(x)$.

We denote by $K_m(x)$ the characteristic function of some fixed set of points on which $|g(x) - g_0(x)| \leq m$, and the function sgn $[g(x) - g_0(x)]$ by $f_0(x)$.

Suppose $\varepsilon > 0$ is prescribed. Since the functions $g_0(x)$, $g_{n_k}(x)$ have equi-absolutely continuous integrals [22], a $\delta > 0$ can be found such that

$$\int_D |g_0(x)| dx < \frac{\varepsilon}{5}, \int_D |g_{n_k}(x)| dx < \frac{\varepsilon}{5}$$

provided $|D| < \delta(D \subset \Omega)$. We shall assume that $\delta < \frac{\varepsilon}{5m}$. It follows from the convergence in measure of the subsequence $g_{n_k}(x)$ to the function $g(x)$ and the convergence of this sequence to the function $g_0(x)$ in $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$ that there exists a k_0 such that , for$k > k_0$,

$$\int_\Omega [g_{n_k}(x) - g_0(x)] f_0(x) K_m(x) dx < \frac{\varepsilon}{5}$$

and $|\Omega_k| < \delta$, where

$$\Omega_k = \{|g_{n_k}(x) - g(x)| \geq \frac{\varepsilon}{5|\Omega|}\}.$$

Then, for $k > k_0$, we have that

$$\begin{aligned}\int_\Omega |g(x) - g_0(x)| K_m(x) dx \leq & \mid \int_\Omega [g_{n_k}(x) - g_0(x)] f_0(x) K_m(x) dx \mid \\ & + \int_{\Omega \backslash \Omega_k} |g(x) - g_{n_k}(x)| dx + \int_{\Omega_k} |g_{n_k}(x)| dx \\ & + \int_{\Omega_k} |g_0(x)| dx + \int_{\Omega_k} |g(x) - g_0(x)| K_m(x) dx \\ & < \frac{\varepsilon}{5} + \frac{\varepsilon}{5|\Omega|} \, |\Omega \backslash \Omega_k| + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + m \, |\Omega_k| < \varepsilon\end{aligned}$$

Since ε is arbitrary, we have that

$$\int_\Omega |g(x) - g_0(x)| K_m(x) dx = 0,$$

i.e.$g_0(x) = g(x)$ almost everywhere.

Lemma 3. *[17] Let the functions A_α satisfy the conditions (A_1) and (A_3) below. if for the sequences $\eta_k \subset R^{n_1}$, $\zeta_k \subset R^{n_2}$, and $\xi_k \subset R^{n_2}$ we have $\eta_k \to \eta$, $\zeta_k \to \zeta$, and*

$$\Sigma_{|\alpha|=m} (A_\alpha(x, \eta_k, \zeta_k) - A_\alpha(x, \eta_k, \xi_k))(\zeta_{\alpha k} - \xi_{\alpha k}) \to 0$$

as $k \to \infty$, then ξ_k is bounded in R^{n_2} and $\xi_k \to \zeta$ as $k \to \infty$.

Lemma 4. *[1] Let $u \in W^{1,1}_{loc}(\Omega)$ and let f satisfy a Lipschitz condition in R. If $g(x) = f(|u(x)|)$, then $g \in W^{1,1}_{loc}(\Omega)$ and*

$$D^\alpha g(x) = f'(|u(x)|) \; sgn \; u(x) . D^\alpha u(x).$$

3 Main results

3.1 Some imbedding results

Theorem 2. *Let Ω have finite measure and let φ and ϕ two Msuielak-Orlicz functions such that $\phi(., t)$ is integrable on Ω and increasing essentially more slowly than φ near infinity. If the sequence $\{u_j\}$ is bounded in $L_\varphi(\Omega)$ and convergent in measure on Ω, then it is convergent in norm in $L_\phi(\Omega)$.*

Proof. Fix ε and let $v_{j,k} = \frac{u_j(x)-u_k(x)}{\varepsilon}$. Clearly $\{v_{j,k}\}$ is bounded in $L_\varphi(\Omega)$; say $||v_{j,k}||_{\varphi,\Omega} < K$. Now there exists a positive number t_0 such that if $t > t_0$, then

$$\phi(x,t) \leq \frac{1}{4}\varphi(x, \frac{t}{K}).$$

Let $\delta > 0$ such that

$$\int_D \phi(x,t_0)dx \leq \frac{1}{4}$$

provided $|D| < \delta$.
Set

$$\Omega_{j,k} = \{x \in \Omega : |v_{j,k}(x)| \geq \phi_x^{-1}(\frac{1}{2|\Omega|})\}.$$

Since $\{u_j\}$ converges in measure, there exists an integer N such that if $j,\ k > N$, then $|\Omega_{j,k}| \leq \delta$. Set

$$\Omega'_{j,k} = \{x \in \Omega_{j,k} : |v_{j,k}(x)| \geq t_0\}, \quad \Omega''_{j,k} = \Omega_{j,k} \setminus \Omega'_{j,k}$$

For $j,k \geq N$ we have

$$\begin{aligned}\int_\Omega \phi(x,|v_{j,k}(x)|)dx &= \int_{\Omega\setminus\Omega_{j,k}} \phi(x,|v_{j,k}(x)|)dx + \int_{\Omega'_{j,k}} \phi(x,|v_{j,k}(x)|)dx + \\ &\qquad \int_{\Omega''_{j,k}} \phi(x,|v_{j,k}(x)|)dx \\ &\leq \frac{|\Omega|}{2|\Omega|} + \frac{1}{4}\int_{\Omega'_{j,k}} \varphi(x, \frac{|v_{j,k}(x)|}{K})dx + \int_{\Omega_{j,k}} \phi(x,t_0)dx \leq 1.\end{aligned}$$

Hence $||u_j - u_k||_{\phi,\Omega} \leq \varepsilon$ and so $\{u_j\}$ converges in $L_\phi(\Omega)$.

Theorem 3. *Let Ω have finite measure and let φ and ϕ as in the Theorem 2. Then any bounded subset S of $L_\varphi(\Omega)$ which is precompact in $L^1(\Omega)$ is also precompact in $L_\phi(\Omega)$.*

Proof. Evidently $L_\varphi(\Omega) \hookrightarrow L^1(\Omega)$ since Ω has finite volume. If $\{u_j^*\}$ is a sequence in S , then it has a subsequence $\{u_j\}$ that converges in $L^1(\Omega)$; say $u_j \to u$ in $L^1(\Omega)$. Thus $\{u_j\}$ converges to u in measure on Ω and hence by Theorem 2. it converges also in $L_\phi(\Omega)$.

Theorem 4. *Let Ω be an open subset of R^n. If an Musielak-Orlicz function φ satisfy the following conditions*

$$\int_1^\infty \frac{\varphi_x^{-1}(t)}{t^{\frac{n+1}{n}}}dt = \infty, \quad \int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{n+1}{n}}}dt < \infty. \tag{3}$$

Let $f(x,t) = \int_0^t \frac{\varphi_x^{-1}(\tau)}{\tau^{\frac{n+1}{n}}}d\tau,\ t \geq 0$. The Sobolev conjugate φ_* of φ is the reciprocal function of f with respect to t

Then $W_0^1L_\varphi(\Omega) \hookrightarrow L_{\varphi_*}(\Omega)$. Moreover, if D is bounded subdomain of Ω, then the following imbeddings $W_0^1L_\varphi(\Omega) \hookrightarrow L_\phi(D)$ exist and are compact for any Msuielak-Orlicz function ϕ increasing essentially more slowly than φ_* near infinity such that $\phi(.,t)$ is integrable on Ω.

Proof. Evidently the function $s = \varphi_*(x,t)$ as defined above is an Msuielak-Orlicz function and satisfies the differential equation

$$\varphi_x^{-1}(s)\frac{ds}{dt} = s^{\frac{n+1}{n}}, \tag{4}$$

and hence, since $s < \varphi_x^{-1}(s)\psi_x^{-1}(s)$,

$$\frac{ds}{dt} \leq s^{\frac{1}{n}}\psi_x^{-1}(s).$$

Therefore $\nu(t) = [\varphi_*(x,t)]^{\frac{n-1}{n}}$ satisfies the differential inequality

$$\frac{d\nu}{dt} \leq \frac{n-1}{n}\psi_x^{-1}((\nu(t))^{\frac{n-1}{n}}). \tag{5}$$

Let $u \in W_0^1L_\varphi(\Omega)$ and suppose, for the moment, that u is bounded on Ω and is not zero in $L_\varphi(\Omega)$. Then $\int_\Omega \varphi_*(x, \frac{|u(x)|}{\lambda})dx$ decreases continuously from infinity to zero as λ increases from zero to infinity, and accordingly assumes the value unity for some positive value of λ. Thus

$$\int_\Omega \varphi_*(x, \frac{|u(x)|}{K})dx = 1, \quad K = ||u||_{\varphi_*}. \tag{6}$$

Let $f(x) = \nu(\frac{|u(x)|}{K})$. Evidently $u \in W_0^{1,1}(\Omega)$ and ν is Lipschitz on the range of $\frac{|u(x)|}{K}$ so that, by Lemma 4, $f \in W_0^{1,1}(\Omega)$. by Sobolev inequality we have

$$||f||_{0,\frac{n}{n-1}} \leq K_1\sum_1^n ||D^jf||_{0,1} = K_1\sum_1^n \frac{1}{K}\int_\Omega \nu'(\frac{|u(x)|}{K})|D^ju(x)|dx. \tag{7}$$

By (6) and Hölder's inequality, we obtain

$$1 = \{\int_\Omega \varphi_*(x, \frac{|u(x)|}{K})dx\}^{\frac{n-1}{n}} = ||f||_{0,\frac{n}{n-1}} \leq \frac{cK_1}{K}\sum_1^n ||\nu'(\frac{|u|}{K})||_\psi ||D^ju||_\varphi. \tag{8}$$

Making use of (5), we have

$$\begin{aligned} ||\nu'(\frac{|u|}{K})||_\psi &\leq \frac{n-1}{n}||\psi_x^{-1}((\nu(\frac{|u|}{K}))^{\frac{n-1}{n}})||_\psi \\ &= \frac{n-1}{n}\inf\{\lambda > 0 : \int_\Omega \psi(x, \frac{\psi_x^{-1}(\varphi_*(x, \frac{|u(x)|}{K}))}{\lambda})dx \leq 1\}. \end{aligned}$$

Suppose $\lambda > 1$. then

$$\int_\Omega \psi(x, \frac{\psi_x^{-1}(\varphi_*(x, \frac{|u(x)|}{K}))}{\lambda})dx \leq \frac{1}{\lambda}\int_\Omega \varphi_*(x, \frac{|u(x)|}{K})dx = \frac{1}{\lambda} < 1.$$

Thus

$$||\nu'(\frac{|u|}{K})||_\psi \leq \frac{n-1}{n}. \tag{9}$$

Hence,

$$1 \leq \frac{K_3}{K}||u||_{\varphi}^1$$

so that

$$||u||_{\varphi_*} = K \leq K_3||u||_{\varphi}^1 \tag{10}$$

To extend (10) to arbitrary $u \in W^1L_{\varphi}(\Omega)$ let

$$u_k(x) = \begin{cases} |u(x)| & \text{if } |u(x)| \leq k \\ k \text{ sgn } u(x) & \text{if } |u(x)| > k \end{cases}$$

Clearly u_k is bounded and it belongs to $W_0^1L_{\varphi}(\Omega)$ by Lemma 5. Moreover, $||u_k||_{\varphi_*}$ increases with k but is bounded by $K_4||u||_{\varphi}$. Therefore, $\lim_{k\to\infty}||u_k||_{\varphi_*} = K$ exists and $K \leq K_4||u||_{\varphi}^1$. By Fatou's Lemma

$$\int_{\Omega} \varphi_*(x, \frac{|u(x)|}{K})dx \leq \lim_{k\to\infty} \int_{\Omega} \varphi_*(x, \frac{|u_k(x)|}{K})dx \leq 1$$

whence $u \in L_{\varphi_*}(\Omega)$ and (10) holds.

If D is a bounded subdomain of Ω, we have

$$W_0^1L_{\varphi}(\Omega) \hookrightarrow W_0^{1,1}(\Omega) \hookrightarrow L^1(\Omega),$$

the latter imbedding being compact a bounded subset of $W_0^1L_{\varphi}(D)$ is bounded in $L_{\varphi_*}(D)$ and precompact in $L^1(D)$, and hence precompact in $L_{\phi}(D)$ by Theorem 3. whenever ϕ increases essentially more slowly than φ_* near infinity.

3.2 Existence Results

Let φ and ψ be two complementary Musielak-Orlicz functions. We assume that $\varphi(.;t)$ is locally integrable and that ψ satisfies the Δ_2 condition.

We are interested here in the Dirichlet problem for the operator

$$A(u) \equiv \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \nabla u, ..., \nabla^m u) \tag{11}$$

on Ω.

The following notations will be used. If $\xi = \{\xi_{\alpha}; |\alpha| \leq m\} \in R^n$ is an m-jet, with $\alpha = (\alpha_1, ..., \alpha_n)$ a multi-index of integers and $|\alpha| = \alpha_l + ... + \alpha_n$, then $\zeta = \{\xi_{\alpha}; |\alpha| = m\} \in R^{n_2}$ denotes its top order part and $\eta = \{\xi_{\alpha}; |\alpha| < m\} \in R^{n_1}$ its lower order part. For u a derivable function, $\xi(u)$ denotes $\{D^{\alpha}u; |\alpha| \leq m\} \in R^n$.

The basic conditions imposed on the coefficients A_{α} of (11) are the followings:

(A_1) Each $A_{\alpha}(x, \xi)$ is a real valued function defined on $\Omega \times R^{n_0}$ is measurable in x for fixed ξ and continuous in ξ for fixed x.

(A_2) There exist a Musielak-Orlicz function γ with $\gamma \prec\prec \varphi$, functions a_α in $E_\psi(\Omega)$, constants c_1 and c_2 such that for all x in Ω and ξ in R^{n_0},if

$$|\alpha| = m : |A_\alpha(x,\xi)| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} \psi_x^{-1}(\varphi(x, c_2\xi_\beta)) + c_1 \sum_{|\beta|<m} \psi_x^{-1}(\gamma(x, c_2\xi_\beta)),$$

if

$$|\alpha| < m : |A_\alpha(x,\xi)| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} \psi_x^{-1}(\gamma(x, c_2\xi_\beta)) + c_1 \sum_{|\beta|<m} \psi_x^{-1}(\varphi(x, c_2\xi_\beta)).$$

(A_3) For each $x \in \Omega$, $\eta \in R^{n_1}$, ξ, and ξ' in R^{n_2} with $\xi \neq \xi'$,

$$\sum_{|\alpha|=m} (A_\alpha(x,\xi,\eta) - A_\alpha(x,\xi',\eta))(\xi_\alpha - \xi'_\alpha) > 0.$$

(A_4) There exist functions $b_\alpha(x)$ in $E_\psi(\Omega)$, $b(x)$ in $L^1(\Omega)$, positive constants d_1 and d_2 such that, for some fixed element v in $W_0^m E_\varphi(\Omega)$,

$$\sum_{|\alpha|\leq m} A_\alpha(x,\xi)(\xi_\alpha - v) \geq d_1 \sum_{|\alpha|\leq m} \varphi(x, d_2\xi_\alpha) - \sum_{|\alpha|\leq m} b_\alpha(x)\xi_\alpha - b(x)$$

for all x in Ω and ξ in R^{n_0}.

Associated to the differential operator (11) we define a mapping T from

$$D(T) = \{u \in W_0^m L_\varphi(\Omega); A_\alpha(\xi(u)) \in L_\psi(\Omega) \text{ for all } |\alpha| \leq m\} \subset W_0^m L_\varphi(\Omega)$$

into $W^{-m}L_\psi(\Omega)$ by the formula

$$< v, Tu >= \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(\xi(u)) D^\alpha v dx$$

for $v \in W_0^m L_\varphi(\Omega)$.

Now we are ready to present our main existence result.

Theorem 5. *Let Ω be an open subset of R^n. Assume that the coefficients of (11) satisfy $(A_1), ..., (A_4)$. Then for any $f \in W^{-m}E_\psi(\Omega)$, the Dirichlet problem for $A(u) = f$ has at least one solution.*

Proof. we consider the complementary system $(W_0^m L_\varphi(\Omega), W_0^m E_\varphi(\Omega), W^{-m}L_\psi(\Omega), W^{-m}E_\psi(\Omega))$ and for simplicity we use the notation (Y, Y_0, Z, Z_0). We should show that the mapping T satisfies the condition (i),...,(iv) of Theorem 1.

To show that (i) holds we introduce the following lemma. It is a generalization of lemma 4.3 of [15].

Lemma 5. *Suppose that A_1 and A_2 hold (with $a(x) \in L_\psi(\Omega)$). Then the mapping $\omega = (\omega_\beta)_{|\beta|\leq m} \mapsto (A_\alpha(\omega))_{|\alpha|\leq m}$ sends $\Pi E_\varphi(\Omega)$ into $\Pi L_\psi(\Omega)$ and is finitely continuous from $\Pi E_\varphi(\Omega)$ to the $\sigma(\Pi L_\psi(\Omega), \Pi E_\varphi(\Omega))$ topology of $\Pi L_\psi(\Omega)$.*

Proof. By (A_2) we can conclude immediately that for all $|\alpha| \leq m$, $A_\alpha(\omega) \in L_\psi(\Omega)$ if $\omega \in \Pi E_\varphi(\Omega)$. We will show that the mapping is continuous from each simplex in $\Pi E_\varphi(\Omega)$ to the $\sigma(\Pi L_\psi(\Omega), \Pi E_\varphi(\Omega))$ topology of $\Pi L_\psi(\Omega)$. Let $S = conv\{\omega^1, ..., \omega^r\}$ be a simplex in $\Pi E_\varphi(\Omega)$ and write $\omega = \Sigma_{i=1}^r \lambda_i \omega^i \in S$ with $\lambda_i \geq 0$ and $\Sigma_{i=1}^r \lambda_i = 1$. We have for some $c_3 > 0$,

$$\psi_x^{-1}(\varphi(x, c_3\omega_\beta)) = \psi_x^{-1}(\varphi(x, \sum_{i=1}^r \lambda_i c_3 \omega_\beta^i)) \leq \psi_x^{-1}(\sum_{i=1}^r \lambda_i \varphi(x, c_3\omega_\beta^i)),$$

which implies that each $A_\alpha(\omega)$ remains bounded il $L_\psi(\Omega)$ when ω runs over S. It is then easy to complete the proof by the lemma 2.

In order to verify the condition (ii) we let y_i be a sequence in Y with the properties $y_i \to y \in Y$ for $\sigma(Y, Z_0)$, $T(y_i) \to z \in Z$ for $\sigma(Z, Y_0)$ and $\limsup < T(y_i), y_i > \leq < z, y >$. We must show that $y \in D(T)$, $T(y) = z$ and $< T(y_i), y_i > \to < z, y >$. Obviously it is sufficient to prove the last convergence for an infinite subsequence. The proof will be done by the following steps.

1. The functions $A_\alpha(, \xi(y_i))$ remains bounded in $L_\psi(\Omega)$ for all $|\alpha| \leq m$. Indeed, for $|\alpha| < m$ we use the fact that $\gamma \prec\prec \varphi$, which implies that for any $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ such that $\gamma(x, t) \leq k(\varepsilon)\varphi(x, \varepsilon t)$ for all $t > 0$. Therefore

$$|A_\alpha(x, \xi(y_i))| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} \psi_x^{-1}(k(\varepsilon)\varphi(x, \varepsilon c_2 D^\beta(y_i))) + c_1 \sum_{|\beta|<m} \psi_x^{-1}(x, \varphi(x, c_2 D^\beta(y_i))).$$

When ε is sufficiently small, $||\varepsilon c_2 D^\beta(y_i)||_\varphi \leq 1$ uniformly for all $|\beta| \leq m$.

$$||\psi_x^{-1}(k(\varepsilon)\varphi(x, \varepsilon c_2 D^\beta(y_i)))||_\psi \leq 1 + k(\varepsilon) \int_\Omega \varphi(x, \varepsilon c_2 D^\beta(y_i))$$

and

$$||\psi_x^{-1}(\varphi(x, c_2 D^\beta(y_i)))||_\psi \leq 1 + \int_\Omega \varphi(x, c_2 D^\beta(y_i))$$

we can conclude

$$||A_\alpha(x, \xi(y_i))||_\psi \leq ||a_\alpha||_\psi + c_1 \sum_{|\beta|=m} (1 + k(\varepsilon)||\varepsilon c_2 D^\beta(y_i)||_\varphi) + c_1 \sum_{|\beta|<m} (1 + \int_\Omega \varphi(x, c_2 D^\beta(y_i))) \leq const.$$

To show the same property for $|\alpha| = m$ let $\omega = (\omega_\alpha) \in \Pi_{|\alpha|=m} E_\varphi(\Omega)$. By A_3 we have

$$\sum_{|\alpha|=m} (A_\alpha(x, \xi(y_i)) - A_\alpha(x, \eta(y_i), \zeta(y_i)))(D^\alpha(y_i) - \omega_\alpha) \geq 0.$$

for all $x \in \Omega$ and hence

$$\begin{aligned}\int_\Omega \sum_{|\alpha|=m} A_\alpha(x, \xi(y_i))\omega_\alpha \quad &\leq \quad \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x, \xi(y_i))D^\alpha(y_i) - \int_\Omega \sum_{|\alpha|<m} A_\alpha(x, \xi(y_i))D^\alpha(y_i) \\ &- \quad \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x, \eta(y_i), \zeta(y_i))(D^\alpha(y_i) - \omega_\alpha)\end{aligned}$$

the first integral in the right hand side is $\leq cst$ by assumption and the second one remains bounded by the previous discussion. The third integral remains bounded by Hölder's inequality provided $||A_\alpha(.,\eta(y_i),\zeta(y_i))||_\psi$ is bounded. To show this we use A_2 to get

$$\begin{aligned} ||A_\alpha(.,\eta(y_i),\zeta(y_i))||_\psi &\leq ||a_\alpha||_\psi + c_1 \sum_{|\beta|=m} ||\psi_x^{-1}(\varphi(x,c_2\omega_\beta))||_\psi \\ &+ c_1 \sum_{|\beta|<m} ||\psi_x^{-1}(k(\varepsilon)\varphi(x,\varepsilon c_2 D^\beta(y_i)))||_\psi, \end{aligned}$$

where

$$||\psi_x^{-1}(\varphi(x,c_2\omega_\beta))||_\psi \leq 1 + \int_\Omega \varphi(x,c_2\omega_\beta) \leq const$$

for all $|\beta| = m$, since $\omega_\beta \in E_\varphi(\Omega)$. moreover,

$$||\psi_x^{-1}(k(\varepsilon)\varphi(x,\varepsilon c_2 D^\beta(y_i)))||_\psi \leq 1 + k(\varepsilon)\int_\Omega \varphi(x,\varepsilon c_2 D^\beta(y_i)) \leq const,$$

when ε is made sufficiently small. Thus we have shown that $A_\alpha(,\xi(y_i))$ remains bounded in $L_\psi(\Omega)$ for all $|\alpha| = m$ for $\sigma(L_\psi(\Omega), E_\varphi(\Omega))$, which implies the boundedness in norm.

2.We may assume, by passing to a subsequence if necessary, that $A_\alpha(,\xi(y_i)) \to h_\alpha$ for $\sigma(L_\psi(\Omega), E_\varphi(\Omega))$ with some $h_\alpha \in L_\psi(\Omega)$ for each $|\alpha| \leq m$. Hence the linear form $z \in Z = Y_0^*$ can be identified to $(h_\alpha) \in \Pi L_\psi(\Omega)$, i.e.,

$$(z,v) = \int_\Omega \sum_{|\alpha|\leq m} h_\alpha D^\alpha v dx \tag{12}$$

holds for all v in Y.

3. We are aiming to show that $D^\alpha y_i(x) \to D^\alpha y$ a.e. in Ω for all $|\alpha| \leq m$. By Theorem 4. the imbedding of $W_0^m L_\varphi(\Omega)$ to $W_0^{m-1} L_\phi(K)$ is compact for any subdomain K with compact closure in Ω and any Musielak-Orlicz function ϕ which is integrable on Ω with respect to x and increases essentially more slowly than φ_* near infinity, hence we may assume that $D^\alpha y_i(x) \to D^\alpha y$ a.e. in Ω for all $|\alpha| \leq m-1$. in order to get the a.e. convergence also for $|\alpha| = m$ we invoke the Lemma 3. with the specialization $\eta_k = \eta(y_i)$, $\xi_k = \xi(y_i)$ and $\zeta_k = \zeta(y)$ for each $x \in \Omega$. In view of Lemma 3 it suffice to show that $q_i(x) \to 0$ a.e. in Ω with

$$q_i(x) = \Sigma_{|\alpha|=m}(A_\alpha(x,\eta(y_i),\xi(y_i)) - A_\alpha(x,\eta(y_i),\zeta(y)))(D^\alpha(y_i) - D^\alpha(y)).$$

In fact, as $q_i(x) \geq 0$ for all $x \in \Omega$, it will be enough to show that

$$\limsup_i \int_{\Omega_k} q_i(x)dx \leq \varepsilon_k, \tag{13}$$

where $\varepsilon_k \to 0$ as $k \to \infty$ and $\Omega_k = \{x \in \Omega; |x| < k, |D^\alpha y(x)| \leq k \text{ for all } |\alpha| \leq m\}$ for any $k \in \mathbb{N}$. Obviously $\Omega_k \subset \Omega_{k+1}$ and mes $(\Omega \setminus \cup_{k=1}^\infty \Omega_k) = 0$. We denote further

$$\begin{aligned} p_i(x) &= \sum_{|\alpha|\leq m} A_\alpha(x,\eta(y_i),\xi(y_i))(D^\alpha(y_i)-D^\alpha(y)), \\ r_i(x) &= \sum_{|\alpha|=m} A_\alpha(x,\eta(y_i),\zeta(y))(D^\alpha(y)-D^\alpha(y_i)), \\ s_i(x) &= \sum_{|\alpha|\leq m-1} A_\alpha(x,\eta(y_i),\xi(y_i))(D^\alpha(y)-D^\alpha(y_i)). \end{aligned}$$

then $q_i(x) = p_i(x) + r_i(x) + s_i(x)$ and the assertion (13) will be shown when we prove that

$$\limsup_i \int_{\Omega_k} p_i(x)dx \leq \varepsilon_k, \tag{14}$$

$$\lim_{i\to\infty} \int_{\Omega_k} r_i(x)dx = 0, \tag{15}$$

$$\lim_{i\to\infty} \int_{\Omega_k} s_i(x)dx = 0 \tag{16}$$

for any $k \in \mathbb{N}$ and $\varepsilon_k \to 0$ as $k \to \infty$.

4. We show the assertion (14). To this end we write

$$\begin{aligned} \int_{\Omega_k} p_i(x)dx &= \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y_i))D^\alpha(y_i)dx - \int_{\Omega\backslash\Omega_k} \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y_i))D^\alpha(y_i)dx \\ &- \int_{\Omega_k} \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y_i))D^\alpha(y) := H_1(i) + H_2(i,k) + H_3(i,k) \end{aligned}$$

by assumption and (12),

$$\limsup_i H_1(i) \leq (z,y) = \int_\Omega \sum_{|\alpha|\leq m} h_\alpha D^\alpha y dx.$$

By (A_4) we have further

$$H_2(i,k) \leq \int_{\Omega\backslash\Omega_k} \sum_{|\alpha|\leq m} b_\alpha D^\alpha(y_i) + \int_{\Omega\backslash\Omega_k} b,$$

where $b \in L^1(\Omega)$ and $b_\alpha \in E_\psi(\Omega)$ for all $|\alpha| \leq m$. By Hölder's inequality

$$\int_{\Omega\backslash\Omega_k} \sum_{|\alpha|\leq m} b_\alpha D^\alpha(y_i) \leq 2 \sum_{|\alpha|\leq m} ||D^\alpha(y_i)||_\varphi ||(1-\chi_k)b_\alpha||_\psi \leq c \sum_{|\alpha|\leq m} ||(1-\chi_k)b_\alpha||_\psi,$$

with c some positive constant and χ_k the characteristic function of the set Ω_k. By the dominate convergence theorem we conclude that $||(1-\chi_k)b_\alpha||_\psi \to 0$ as $k \to \infty$.

Finally, as $\chi_k D^\alpha y \in E_\varphi(\Omega)$, we have

$$\lim_{n\to\infty} H_3(i,k) = \int_{\Omega_k} \sum_{|\alpha|\leq m} h_\alpha D^\alpha(y).$$

Consequently we obtain

$$\limsup_i \int_{\Omega_k} p_i(x)dx \le \int_{\Omega\setminus\Omega_k} \sum_{|\alpha|\le m} h_\alpha D^\alpha y dx + c\sum_{|\alpha|\le m} ||(1-\chi_k)b_\alpha||_\psi + \int_{\Omega\setminus\Omega_k} bdx := \varepsilon_k,$$

where $\varepsilon_k \to \infty$ as $k \to \infty$.

5. We show that (15) hold for any fixed k. As $D^\alpha(y_i) \to D^\alpha y$ for $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$, it suffices to prove that $\chi_k A_\alpha(x, \eta(y_i), \zeta(y)) \to \chi_k A_\alpha(x, \eta(y), \zeta(y))$ in norm in $E_\psi(\Omega)$ for all $|\alpha| = m$. From (A_1) and (A_2) it follows that $\chi_k A_\alpha(x, \eta(y_i), \zeta(y)) \in E_\psi(\Omega)$ and that the a.e. convergence holds. So the norm convergence follows by Vitali's Theorem using the dominated convergence theorem in the right hand side of (A_2).

6.We prove (16) for any fixed k. For all $|\alpha| \le m-1$ we may assume by the previous argument that $\chi_k D^\alpha(y_i) \to \chi_k D^\alpha(y)$ in norm in $L_\varphi(\Omega)$ and hence (16) is obtained immediately by Hölder's inequality.

7. We have shown that (13) holds implying that $D^\alpha(y_i)(x) \to D^\alpha(y)(x)$ a.e. in Ω for all $|\alpha| \le m$, at least for a subsequence. By (A_1) we can conclude that $A_\alpha(x, \zeta(y_i)) \to A_\alpha(x, \zeta(y))$ a.e. in Ω for all $|\alpha| \le m$. On the other hand, $A_\alpha(x, \zeta(y_i)) \to h_\alpha$ for $\sigma(L_\psi(\Omega), E_\varphi(\Omega))$, so that By Lemma 2. $A_\alpha(x, \zeta(y)) = h_\alpha$ for each $|\alpha| \le m$. Hence $y \in D(T)$ and $T(y) = z$.

8. to complete the proof of (ii) it remains to show that $(T(y_i), y_i) \to (z, y) = (T(y), y)$. Bearing in mind the assumption that lim sup $< T(y_i), y_i > \le < z, y >$ it will be sufficient to prove that

$$\liminf \int_\Omega \sum_{|\alpha|\le m} A_\alpha(x, \zeta(y_i))D^\alpha(y_i) \ge \int_\Omega \sum_{|\alpha|\le m} A_\alpha(x, \zeta(y))D^\alpha(y). \tag{17}$$

By (A_3) we have, for all $x \in \Omega$,

$$\sum_{|\alpha|=m} A_\alpha(x, \zeta(y_i))D^\alpha(y_i) \ge \sum_{|\alpha|=m} A_\alpha(x, \eta(y_i), \xi(y))(D^\alpha(y_i) - D^\alpha y) + \sum_{|\alpha|=m} A_\alpha(x, \zeta(y_i))D^\alpha(y).$$

As a consequence we obtain, by (A_4),

$$\begin{aligned}
&\int_\Omega \sum_{|\alpha|\le m} A_\alpha(x, \zeta(y_i))D^\alpha(y_i) \\
= &\int_{\Omega_k} \sum_{|\alpha|=m} A_\alpha(x, \zeta(y_i))D^\alpha(y_i) \\
+ &\int_{\Omega_k} \sum_{|\alpha|\le m-1} A_\alpha(x, \zeta(y_i))D^\alpha(y_i) + \int_{\Omega\setminus\Omega_k} \sum_{|\alpha|\le m} A_\alpha(x, \zeta(y_i))D^\alpha(y_i) \\
\ge &\int_{\Omega_k} \sum_{|\alpha|=m} A_\alpha(x, \eta(y_i), \xi(y))(D^\alpha(y_i) - D^\alpha(y)) + \int_{\Omega_k} \sum_{|\alpha|=m} A_\alpha(x, \zeta(y_i))D^\alpha(y) \\
+ &\int_{\Omega_k} \sum_{|\alpha|\le m-1} A_\alpha(x, \zeta(y_i))D^\alpha(y_i) - \int_{\Omega\setminus\Omega_k} \sum_{|\alpha|\le m} b_\alpha D^\alpha(y_i) - \int_{\Omega\setminus\Omega_k} b.
\end{aligned}$$

Using the same argument as above, we get

$$\liminf_i \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y_i))D^\alpha(y_i) \geq$$
$$\int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y))D^\alpha(y) - \int_{\Omega\backslash\Omega_k} \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y))D^\alpha(y)$$
$$- c\sum_{|\alpha|\leq m} ||(1-\chi_k)b_\alpha||_\psi - \int_{\Omega\backslash\Omega_k} b = \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y))D^\alpha(y) - \varepsilon_k$$

with $\varepsilon_k \to 0$ as $k \to \infty$ and we obtain (17).

To prove that (iii) holds, we let y varies in $D(T)$ with $||y||_\varphi^m \leq cst$ and $< y-\bar{y}, T(y) > \leq cst$ with respect to $\bar{y} = v$. By a method similar to the first step of proof of (ii) we can conclude that $A_\alpha(x,\xi(y))$ remains bounded in $L_\psi(\Omega)$ for all $|\alpha| \leq m$, which clearly implies that $T(y)$ remains bounded in $W^{-m}L_\psi(\Omega)$.

Let us finally show that the condition (iv) holds with respect to $\bar{y} = v$ and any f in $W^{-m}E_\psi(\Omega)$. Let $f = \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha f_\alpha \in W^{-m}E_\psi(\Omega)$, we claim that

$$\{y \in D(T); < y - v, T(y) - f > \leq 0\} \tag{18}$$

is bounded in $W_0^m L_\varphi(\Omega)$, which clearly yields the conclusion. If u belongs to (18), then

$$\int_\Omega \sum_{|\alpha|\leq m} (A_\alpha(x,\xi(y)) - f_\alpha)(D^\alpha y - D^\alpha v)dx \leq 0$$

and consequently, by (A_4) and young's inequality,

$$d_1 \int_\Omega \sum_{|\alpha|\leq m} \varphi(x, d_2 D^\alpha(y))dx$$
$$\leq \int_\Omega \sum_{|\alpha|\leq m} (b_\alpha + f_\alpha)D^\alpha(y)dx + \int_\Omega b(x)dx - \int_\Omega \sum_{|\alpha|\leq m} f_\alpha D^\alpha(v)dx$$
$$\leq \int_\Omega \sum_{|\alpha|\leq m} \psi(x, r(b_\alpha + f_\alpha))dx + \int_\Omega \sum_{|\alpha|\leq m} \varphi(x, \frac{D^\alpha(y)}{r})dx + cst$$

where $r > 0$ can be taken as large as needed.
for $r \geq \frac{1}{d_2}$, we have

$$d_1 \int_\Omega \sum_{|\alpha|\leq m} \varphi(x, d_2 D^\alpha(y))dx \leq \frac{1}{rd_2} \int_\Omega \sum_{|\alpha|\leq m} \varphi(x, d_2 D^\alpha(y))dx + cst$$

this inequality, for $r > \frac{1}{d_1 d_2}$, provides a bound on each integral $\int_\Omega \varphi(x, d_2 D^\alpha(y))dx$. It then follows that each $D^\alpha(y)$ remains bounded in $L_\varphi(\Omega)$.

Remark 1. *By [17, Proposition 1], same arguments as above give an existence result for the inequality associate to the above Dirichlet problem.*

The following Corollary improves the known existence results for the Dirichlet problem in the variable exponent Sobolev spaces :

Corollary 2. *Let* Ω *be an open subset of* R^n*. Let* $p : \Omega \to]1, +\infty]$ *be a locally integrable function such that* $1 < p^- = \inf_{x\in\Omega} p(x)$ *and* $p^+ = \sup_{x\in\Omega} p(x) \leq \infty$*. Let* $f \in W^{-m,p'(x)}(\Omega)$ *where* p' *is such that* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$*. Then there exists at least one weak solution* $u \in W_0^{m,p(x)}(\Omega)$ *for the Dirichlet problem of the form:*

$$A(u) = f$$

where A *is defined by (11) and the Musielak-Orlicz function* φ *is replaced by* $t^{p(x)}$*.*

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