On the hyper-order of solutions of a class of higher order linear differential equations

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Abstract

In this paper, we investigate the growth of solutions of the linear differential equation

$$f^{(k)} + \left(A_{k-1}(z)e^{P_{k-1}(z)} + B_{k-1}(z)\right) f^{(k-1)} + \cdots + \left(A_1(z)e^{P_1(z)} + B_1(z)\right) f' + \left(A_0(z)e^{P_0(z)} + B_0(z)\right) f = 0,$$

where $k \geq 2$ is an integer, $P_j(z)$ $(j = 0, 1, \cdots, k-1)$ are nonconstant polynomials and $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$ are meromorphic functions. Under some conditions, we determine the hyper-order of these solutions.

1 Introduction and statement of the result

Throughout this paper, we use the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [7]). Let $\sigma(f)$ denote the order of growth of a meromorphic function $f(z)$ and $\sigma_2(f)$ the hyper-order of $f(z)$ which is defined by (see [8], [10])

$$\sigma_2(f) = \limsup_{r \to +\infty} \frac{\log \log T(r, f)}{\log r},$$

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where $T(r,f)$ is the characteristic function of Nevanlinna. We define the logarithmic measure of a set $E \subset [1,+\infty)$ by $lm(E) = \int_1^{+\infty} \frac{\chi_E(t)}{t} dt$, where χ_E is the characteristic function of E.

The main purpose of this paper is to study the growth of solutions of the linear differential equations of the form

$$f^{(k)} + \left(A_{k-1}(z)e^{P_{k-1}(z)} + B_{k-1}(z)\right) f^{(k-1)} + \cdots + \left(A_1(z)e^{P_1(z)} + B_1(z)\right) f' + \left(A_0(z)e^{P_0(z)} + B_0(z)\right) f = 0, \quad (1.1)$$

where $k \geq 2$ is an integer, $P_j(z)$ $(j = 0, 1, \cdots, k-1)$ are nonconstant polynomials and $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$ are meromorphic functions.

Many authors have also considered the higher order linear differential equations with entire coefficients. In [1], Belaïdi and Abbas have proved the following result:

Theorem A ([1]) *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \cdots, k-1)$ be nonconstant polynomials, where $a_{0,j}, \cdots, a_{n,j}$ $(j = 0, \cdots, k-1)$ are complex numbers such that $a_{n,j}a_{n,s} \neq 0$ $(j \neq s)$. Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$ be entire functions with $\sigma(A_j) < n$ $(j = 0, 1, \cdots, k-1)$. Suppose that $\arg a_{n,j} \neq \arg a_{n,s}$ $(j \neq s)$ or $a_{n,j} = c_j a_{n,s}$ $(0 < c_j < 1)$ $(j \neq s)$. Then every transcendental solution f of the differential equation*

$$f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \cdots + A_s(z)e^{P_s(z)} f^{(s)} + \cdots + A_0(z)e^{P_0(z)} f = 0 \quad (1.2)$$

is of infinite order and satisfies $\sigma_2(f) = n$.

Furthermore, if $\max\{c_1, \cdots, c_{s-1}\} < c_0$, then every solution $f \not\equiv 0$ of equation (1.2) *is of infinite order and satisfies $\sigma_2(f) = n$.*

Recently, Tu and Yi have obtained the following result for equations of the form (1.2):

Theorem B ([9]) *Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ $(k \geq 2)$ be entire functions with $\sigma(A_j) < n$ $(n \geq 1)$, and let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \cdots, k-1)$ be polynomials with degree n, where $a_{n,j}$ $(j = 0, 1, \cdots, k-1)$ are complex numbers such that $a_{n,0} = |a_{n,0}| e^{i\theta_0}$, $a_{n,s} = |a_{n,s}| e^{i\theta_s}$, $a_{n,0}a_{n,s} \neq 0$ $(1 \leq s \leq k-1)$, θ_0, $\theta_s \in [0, 2\pi)$, $\theta_0 \neq \theta_s$, $A_0 A_s \not\equiv 0$; for $j \neq 0, s$, $a_{n,j}$ satisfies either $a_{n,j} = c_j a_{n,0}$ $(c_j < 1)$ or $\arg a_{n,j} = \theta_s$. Then every solution $f \not\equiv 0$ of equation* (1.2) *is of infinite order and satisfies $\sigma_2(f) = n$.*

In [4] and [8], earlier results can be found on related topics dealing with second order equations, whereas here we deal with $k-$th order equations. In this paper, we extend and improve Theorems A-B from entire solutions to meromorphic solutions by proving the following result:

Theorem 1.1 *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \cdots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \cdots, a_{n,j}$ $(j = 0, 1, \cdots, k-1)$ are complex numbers such that $a_{n,j} \neq 0$ $(j = 0, 1, \cdots, k-1)$. Let $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions with $\sigma(A_j) < n$ and $\sigma(B_j) < n$. Suppose that one of the following statements holds:*
(i) *there exists $d \in \{1, \cdots, k-1\}$ such that $\arg a_{n,j} \neq \arg a_{n,d}$ $(j \neq d)$;*
(ii) *there exists $d \in \{1, \cdots, k-1\}$ such that $a_{n,j} = c_j a_{n,d}$ $(0 < c_j < 1)$ $(j \neq d)$;*
(iii) *there exist d, $s \in \{1, \cdots, k-1\}$ such that $a_{n,d} = |a_{n,d}| e^{i\theta_d}$, $a_{n,s} = |a_{n,s}| e^{i\theta_s}$, θ_d, $\theta_s \in [0, 2\pi)$, $\theta_d \neq \theta_s$ and for $j \in \{0, \cdots, k-1\} \setminus \{d, s\}$, $a_{n,j}$ satisfies either $a_{n,j} = d_j a_{n,d}$ $(d_j < 1)$ or $\arg a_{n,j} = \theta_s$.*

Then every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicity of equation (1.1) *is of infinite order and satisfies $\sigma_2(f) = n$.*

Furthermore, if $\max\{c_1, \cdots, c_{d-1}\} < c_0$ in case (ii), *then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicity of equation* (1.1) *is of infinite order and satisfies $\sigma_2(f) = n$.*

Remark 1.1 Clearly, the method used in linear differential equations with entire coefficients can not deal with the case of meromorphic coefficients. In addition, the proofs of the results in [1] rely heavily on the idea of Lemma 2.3, Lemma 2.4 and Lemma 2.9 in [1]. However, it seems too complicated to deal with our cases. The methods in the proof of Theorem 1.1 are mainly the estimate for the logarithmic derivative of a transcendental meromorphic function of finite order due to Gundersen [6], Lemma 2.2 [2] due to Cao and Yi and Lemma 2.5 [5] due to Chen and Xu.

Remark 1.2 Recently, Chen and Xu [5] have investigated the growth of solutions of differential equations of the above type with meromorphic coefficients. So, it is also interesting to consider the growth and oscillation of meromorphic solutions of non-homogeneous linear differential equations with meromorphic coefficients.

2 Preliminary lemmas

Lemma 2.1 ([6]) *Let $f(z)$ be a transcendental meromorphic function and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure and a constant $B > 0$ that depends only on α and (i, j) (i, j positive integers with $i > j$) such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left| \frac{f^{(i)}(z)}{f^{(j)}(z)} \right| \leq B \left[\frac{T(\alpha r, f)}{r} (\log^{\alpha} r) \log T(\alpha r, f) \right]^{i-j}.$$

Remark 2.1 In [2], Cao and Yi have obtained the following lemma but with no mention of the existence of finite logarithmic set. Here we give the full lemma.

Lemma 2.2 ([2]) *Let $f(z) = g(z)/b(z)$ be a meromorphic function with $\sigma(f) = \sigma \leq +\infty$, where $g(z)$ and $b(z)$ are entire functions satisfying one of the following conditions:*

(i) *g being transcendental and b being polynomial,*
(ii) *g, b all being transcendental and* $\lambda(b) = \sigma(b) < \sigma(g) = \sigma$.
Then there exist a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ *and a set* E_2 *of finite logarithmic measure such that the estimation*

$$\left|\frac{f(z)}{f^{(d)}(z)}\right| \leq r_m^{2d} \ (d \in \mathbb{N})$$

holds for all z satisfying $|z| = r_m \notin E_2, r_m \to +\infty$ *and* $|g(z)| = M(r_m, g)$.

Lemma 2.3 ([3]) *Let* $g(z)$ *be a transcendental meromorphic function of order* $\sigma(g) = \sigma < +\infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_3 \subset (1, +\infty)$ *that has finite logarithmic measure, such that*

$$|g(z)| \leq \exp\left\{r^{\sigma+\varepsilon}\right\}$$

holds for $|z| = r \notin [0,1] \cup E_3, r \to +\infty$.

Remark 2.2 Combining Lemma 2.3 and applying it to $\frac{1}{g(z)}$, it is clear that for any given $\varepsilon > 0$, there exists a set $E_4 \subset (1, +\infty)$ that has finite logarithmic measure, such that

$$\exp\left\{-r^{\sigma+\varepsilon}\right\} \leq |g(z)| \leq \exp\left\{r^{\sigma+\varepsilon}\right\}$$

holds for $|z| = r \notin [0,1] \cup E_4, r \to +\infty$.

Lemma 2.4 *Let* $P(z) = (\alpha + i\beta) z^n + \cdots (\alpha, \beta$ *are real numbers,* $|\alpha| + |\beta| \neq 0)$ *be a polynomial with degree* $n \geq 1$ *and* $A(z)$ *be a meromorphic function with* $\sigma(A) < n$. *Set* $f(z) = A(z) e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_5 \subset (1, +\infty)$ *having finite logarithmic measure such that for any* $\theta \in [0, 2\pi) \setminus H$ $(H = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\})$ *and for* $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, *we have*
(i) *if* $\delta(P, \theta) > 0$, *then*

$$\exp\{(1-\varepsilon)\delta(P,\theta) r^n\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\{(1+\varepsilon)\delta(P,\theta) r^n\}, \tag{2.1}$$

(ii) *if* $\delta(P, \theta) < 0$, *then*

$$\exp\{(1+\varepsilon)\delta(P,\theta) r^n\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\{(1-\varepsilon)\delta(P,\theta) r^n\}. \tag{2.2}$$

Proof. Set $f(z) = h(z) e^{(\alpha+i\beta)z^n}$, where $h(z) = A(z) e^{P_{n-1}(z)}$, $P_{n-1}(z) = P(z) - (\alpha + i\beta) z^n$. Then $\rho(h) = \lambda < n$. By Remark 2.2, for any given ε $(0 < \varepsilon < n - \lambda)$, there exists a set $E_5 \subset (1, +\infty)$ that has finite logarithmic measure, such that for $|z| = r \notin [0,1] \cup E_5, r \to +\infty$

$$\exp\left\{-r^{\lambda+\varepsilon}\right\} \leq |h(z)| \leq \exp\left\{r^{\lambda+\varepsilon}\right\}. \tag{2.3}$$

By $\left|e^{(\alpha+i\beta)\left(re^{i\theta}\right)^n}\right| = e^{\delta(P,\theta)r^n}$ and (2.3), we have

$$\exp\left\{\delta(P,\theta) r^n - r^{\lambda+\varepsilon}\right\} \leq |f(z)| \leq \exp\left\{\delta(P,\theta) r^n + r^{\lambda+\varepsilon}\right\}. \tag{2.4}$$

By $\theta \notin H$, where $H = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$, we see that:
(i) if $\delta(P, \theta) > 0$, then by $0 < \lambda + \varepsilon < n$ and (2.4), we know that (2.1) holds for $r \notin [0, 1] \cup E_5$, $r \to +\infty$;
(ii) if $\delta(P, \theta) < 0$, then by $0 < \lambda + \varepsilon < n$ and (2.4), we know that (2.2) holds for $r \notin [0, 1] \cup E_5$, $r \to +\infty$.

Lemma 2.5 ([5]) *Let $k \geq 2$ be an integer and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions of finite order. If f is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of the equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0,$$

then $\sigma_2(f) \leq \max\{\sigma(A_j) : j = 0, 1, \cdots, k-1\}$.

3 Proof of Theorem 1.1

First of all we prove that equation (1.1) cannot have a transcendental meromorphic solution f with order $\sigma(f) < n$. Assume f is a transcendental meromorphic solution of equation (1.1) with $\sigma(f) = \sigma < n$. Then $\sigma\left(f^{(j)}\right) = \sigma < n$ $(j = 1, \cdots, k)$. Set $\alpha = \max\left\{\sigma,\ \sigma\left(B_j\right)\ \ (j = 0, \cdots, k-1)\right\} < n$.

Suppose that (i) holds. Since $\arg a_{n,j} \neq \arg a_{n,d}$ $(j \neq d)$, there is a ray $\arg z = \theta \in [0, 2\pi) \setminus H$, where $H = \{\theta \in [0, 2\pi) : \delta(P_0, \theta) = 0$ or...or $\delta(P_{k-1}, \theta) = 0\}$ such that $\delta(P_d, \theta) > 0$, $\delta\left(P_j, \theta\right) < 0$ $(j \neq d)$. By Lemma 2.3, for any given ε $(0 < 2\varepsilon < \min\{1, n - \alpha\})$, there exists a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$, we have

$$\left|f^{(k)}(z)\right| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}, \tag{3.1}$$

$$\left|B_d(z) f^{(d)}(z)\right| \leq \exp\left\{r^{\alpha+\varepsilon}\right\} \tag{3.2}$$

and

$$\left|B_j(z) f^{(j)}(z)\right| \leq \exp\left\{r^{\sigma\left(B_j f^{(j)}\right)+\frac{\varepsilon}{2}}\right\}\ \ (j \neq d). \tag{3.3}$$

By Lemma 2.4 and $\sigma\left(A_j f^{(j)}\right) < n$ $(j = 0, 1, \cdots, k-1)$, for the above ε, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_5$, $r \to +\infty$, we have

$$\left|A_d(z) e^{P_d(z)} f^{(d)}(z)\right| \geq \exp\left\{(1-\varepsilon)\,\delta(P_d, \theta)\, r^n\right\} \tag{3.4}$$

and

$$\left|A_j(z) e^{P_j(z)} f^{(j)}(z)\right| \leq \exp\left\{(1-\varepsilon)\,\delta\left(P_j, \theta\right) r^n\right\} < 1\ \ (j \neq d). \tag{3.5}$$

From (3.3) and (3.5), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_3 \cup$

E_5, $r \to +\infty$, we have

$$\begin{aligned}\left|\left(A_j(z)e^{P_j(z)} + B_j(z)\right) f^{(j)}(z)\right| &= \left|A_j(z)e^{P_j(z)} f^{(j)}(z) + B_j(z) f^{(j)}(z)\right| \\ &\le \exp\left\{(1-\varepsilon)\,\delta\left(P_j,\theta\right) r^n\right\} + \exp\left\{r^{\sigma\left(B_j f^{(j)}\right)+\frac{\varepsilon}{2}}\right\} \\ &\le \exp\left\{r^{\sigma\left(B_j f^{(j)}\right)+\varepsilon}\right\} \le \exp\left\{r^{\alpha+\varepsilon}\right\} \quad (j \ne d). \end{aligned} \tag{3.6}$$

By (1.1), we have

$$\begin{aligned}\left|A_d(z)e^{P_d(z)} f^{(d)}(z)\right| \le \left|B_d(z) f^{(d)}(z)\right| + \left|f^{(k)}(z)\right| \\ + \sum_{\substack{j=0\\ j\ne d}}^{k-1} \left|\left(A_j(z)e^{P_j(z)} + B_j(z)\right) f^{(j)}(z)\right|. \end{aligned} \tag{3.7}$$

By (3.1), (3.2), (3.4), (3.6) and (3.7), for all z with $\arg z = \theta \in [0,2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_3 \cup E_5$, $r \to +\infty$, we have

$$\exp\left\{(1-\varepsilon)\,\delta\left(P_d,\theta\right) r^n\right\} \le (k+1)\exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{3.8}$$

This is absurd. Hence $\sigma(f) \ge n$.

Suppose that (ii) holds. Since $a_{n,j} = c_j a_{n,d}$ $(0 < c_j < 1)$ $(j \ne d)$, it follows that $\delta\left(P_j,\theta\right) = c_j\delta\left(P_d,\theta\right)$ $(j \ne d)$. Put $c = \max\left\{c_j\ (j \ne d)\right\}$. Then $0 < c < 1$. We take a ray $\arg z = \theta \in [0,2\pi) \setminus H$, where $H = \{\theta \in [0,2\pi) : \delta\left(P_d,\theta\right) = 0\}$, such that $\delta\left(P_d,\theta\right) > 0$. By Lemma 2.3, for any given ε $(0 < 2\varepsilon < \min\left\{\frac{1-c}{1+c}, n-\alpha\right\})$, there exists a set $E_3 \subset (1,+\infty)$ having finite logarithmic measure such that for all z with $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$, we have (3.1) and

$$\left|B_j(z) f^{(j)}(z)\right| \le \exp\left\{r^{\alpha+\varepsilon}\right\} \quad (j = 0,\cdots,k-1). \tag{3.9}$$

By Lemma 2.4 and $\sigma\left(A_j f^{(j)}\right) < n$ $(j = 0,1,\cdots,k-1)$, for the above ε, there exists a set $E_5 \subset (1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0,2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.4) and

$$\left|A_j(z)e^{P_j(z)} f^{(j)}(z)\right| \le \exp\left\{(1+\varepsilon)\,c\delta\left(P_d,\theta\right) r^n\right\} \quad (j \ne d). \tag{3.10}$$

From (1.1), we have

$$\begin{aligned}\left|A_d(z)e^{P_d(z)} f^{(d)}(z)\right| \le \left|f^{(k)}(z)\right| \\ + \sum_{j=0}^{k-1}\left|B_j(z)f^{(j)}(z)\right| + \sum_{\substack{j=0\\ j\ne d}}^{k-1}\left|A_j(z)e^{P_j(z)} f^{(j)}(z)\right|. \end{aligned} \tag{3.11}$$

By (3.1), (3.4), $(3.9) - (3.11)$ and $0 < 2\varepsilon < n - \alpha$, for all z with $\arg z = \theta \in [0,2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_3 \cup E_5$, $r \to +\infty$, we have

$$\begin{aligned}\exp\left\{(1-\varepsilon)\,\delta\left(P_d,\theta\right) r^n\right\} &\le (k+1)\exp\left\{r^{\alpha+\varepsilon}\right\} \\ &\quad + (k-1)\exp\left\{(1+\varepsilon)\,c\delta\left(P_d,\theta\right) r^n\right\} \\ &= (k-1)\,(1+o(1))\exp\left\{(1+\varepsilon)\,c\delta\left(P_d,\theta\right) r^n\right\}. \end{aligned} \tag{3.12}$$

By $0 < 2\varepsilon < \frac{1-c}{1+c}$ and (3.12), we have

$$\exp\left\{\frac{(1-c)}{2}\delta\left(P_d,\theta\right)r^n\right\} \leq M_1, \tag{3.13}$$

where M_1 (>0) is some constant. This is a contradiction. Hence $\sigma\left(f\right) \geq n$.

Suppose that (iii) holds. Suppose that $a_{n,j_1}, \cdots, a_{n,j_m}$ satisfy $a_{n,j_\gamma} = d_{j_\gamma}a_{n,d}$, $j_\gamma \in \{0,\cdots,k-1\} \setminus \{d,s\}$, $\gamma \in \{1,\cdots,m\}$, $1 \leq m \leq k-2$ and $\arg a_{n,j} = \theta_s$ for $j \in \{0,\cdots,k-1\} \setminus \{d,s,j_1,\cdots,j_m\}$. Choose a constant ρ satisfying $\max\{d_{j_1},\cdots, d_{j_m}\} < \rho < 1$. We divide the proof into two cases:
(**a**) $\rho \leq 0$;
(**b**) $0 < \rho < 1$.

Case (a). $\rho \leq 0$. Since $\theta_d \neq \theta_s$, there is a ray $\arg z = \theta \in [0,2\pi) \setminus H$, where $H = \{\theta \in [0,2\pi) : \delta\left(P_d,\theta\right) = 0 \text{ or } \delta\left(P_s,\theta\right) = 0\}$ such that $\delta\left(P_d,\theta\right) > 0$ and $\delta\left(P_s,\theta\right) < 0$. Hence

$$\delta\left(P_{j_\gamma},\theta\right) = d_{j_\gamma}\delta\left(P_d,\theta\right) < 0 \ \ (\gamma = 1,\cdots,m), \tag{3.14}$$

$$\delta\left(P_j,\theta\right) = \left|a_{n,j}\right|\cos\left(\theta_s + n\theta\right) < 0,\ j \in \{0,\cdots,k-1\} \setminus \{d,s,j_1,\cdots,j_m\}. \tag{3.15}$$

By Lemma 2.3, for any given ε $(0 < 2\varepsilon < \min\{1,n-\alpha\})$ there exists a set $E_3 \subset (1,+\infty)$ having finite logarithmic measure such that for all z with $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$, we have (3.1), (3.2) and (3.3). By Lemma 2.4 and $\sigma\left(A_jf^{(j)}\right) < n$ $(j = 0,1,\cdots,k-1)$, for the above ε, there exists a set $E_5 \subset (1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0,2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.4) and from (3.14) and (3.15), we obtain (3.5). By (3.3) and (3.5), for all z with $\arg z = \theta \in [0,2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_3 \cup E_5$, $r \to +\infty$, we have (3.6). By (3.1), (3.2), (3.4), (3.6) and (3.7), for all z with $\arg z = \theta \in [0,2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_3 \cup E_5$, $r \to +\infty$, we have (3.8). This is absurd. Hence $\sigma\left(f\right) \geq n$.

Case (b). $0 < \rho < 1$. Using the same reasoning as above, there exists a ray $\arg z = \theta \in [0,2\pi) \setminus H$, where H is defined as above, such that $\delta\left(P_d,\theta\right) > 0$ and $\delta\left(P_s,\theta\right) < 0$. Hence

$$\delta\left(-\rho P_d,\theta\right) = -\rho\delta\left(P_d,\theta\right) < 0,\ \delta\left(\left(1-\rho\right)P_d,\theta\right) = \left(1-\rho\right)\delta\left(P_d,\theta\right) > 0, \tag{3.16}$$

$$\delta\left(P_j,\theta\right) = \left|a_{n,j}\right|\cos\left(\theta_s + n\theta\right) < 0,\ j \in \{0,\cdots,k-1\} \setminus \{d,s,j_1,\cdots,j_m\}, \tag{3.17}$$

$$\delta\left(P_j - \rho P_d,\theta\right) < 0,\ j \in \{0,\cdots,k-1\} \setminus \{d,j_1,\cdots,j_m\} \tag{3.18}$$

and

$$\delta\left(P_{j_\gamma} - \rho P_d,\theta\right) = \left(d_{j_\gamma} - \rho\right)\delta\left(P_d,\theta\right) < 0 \ \ (\gamma = 1,\cdots,m). \tag{3.19}$$

By Lemma 2.4 and $\max\{\sigma\left(f^{(k)}\right), \sigma\left(A_jf^{(j)}\right), \sigma\left(B_jf^{(j)}\right) \ (j = 0,1,\cdots,k-1)\} < n$, for any given ε $(0 < 2\varepsilon < 1)$, there exists a set $E_5 \subset (1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0,2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have

$$\left|A_d(z)e^{(1-\rho)P_d(z)}f^{(d)}\left(z\right)\right| \geq \exp\left\{\left(1-\varepsilon\right)\left(1-\rho\right)\delta\left(P_d,\theta\right)r^n\right\}, \tag{3.20}$$

$$\left|e^{-\rho P_d(z)} f^{(k)}(z)\right| \leq \exp\left\{-(1-\varepsilon)\rho\delta(P_d,\theta)r^n\right\} \leq 1, \quad (3.21)$$

$$\left|B_j(z)e^{-\rho P_d(z)} f^{(j)}(z)\right| \leq \exp\left\{-(1-\varepsilon)\rho\delta(P_d,\theta)r^n\right\} \leq 1 \ (j=0,\cdots,k-1) \quad (3.22)$$

and from (3.18) and (3.19) we obtain

$$\left|A_j(z)e^{P_j(z)-\rho P_d(z)} f^{(j)}(z)\right| \leq \exp\left\{(1-\varepsilon)\delta\left(P_j-\rho P_d,\theta\right)r^n\right\} \leq 1 \ (j \neq d). \quad (3.23)$$

By (1.1), we have

$$\left|A_d(z)e^{(1-\rho)P_d(z)} f^{(d)}(z)\right| \leq \left|e^{-\rho P_d(z)} f^{(k)}(z)\right| + \sum_{\substack{j=0\\ j\neq d}}^{k-1}\left|A_j(z)e^{P_j(z)-\rho P_d(z)} f^{(j)}(z)\right| + \sum_{j=0}^{k-1}\left|B_j(z)e^{-\rho P_d(z)} f^{(j)}(z)\right|. \quad (3.24)$$

By $(3.20)-(3.24)$, for all z with $\arg z=\theta\in[0,2\pi)\setminus H$, $|z|=r\notin[0,1]\cup E_5$, $r\to+\infty$, we have

$$\exp\left\{(1-\varepsilon)(1-\rho)\delta(P_d,\theta)r^n\right\} \leq 2k. \quad (3.25)$$

This is absurd. Hence $\sigma(f)\geq n$.

Assume f is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of equation (1.1). By Lemma 2.1, there exist a constant $B>0$ and a set $E_1\subset(1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(d)}(z)}\right| \leq Br\left[T(2r,f)\right]^{j-d+1} \quad (j=d+1,\cdots,k) \quad (3.26)$$

and

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq Br\left[T(2r,f)\right]^{j+1} \quad (j=1,2,\cdots,d-1). \quad (3.27)$$

By (1.1), it follows that the poles of f can only occur at the poles of A_j and B_j $(j=0,\cdots,k-1)$. Note that the poles of f are of uniformly bounded multiplicity. Hence $\lambda(1/f)\leq\max\left\{\sigma(A_j),\sigma(B_j)\ (j=0,\cdots,k-1)\right\}<n$. By Hadamard factorization theorem, we know that f can be written as $f(z)=\frac{g(z)}{b(z)}$, where $g(z)$ and $b(z)$ are entire functions with $\lambda(b)=\sigma(b)=\lambda(1/f)<n\leq\sigma(f)=\sigma(g)$. By Lemma 2.2, there exist a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m\to+\infty$ and a set E_2 of finite logarithmic measure such that the estimation

$$\left|\frac{f(z)}{f^{(d)}(z)}\right| \leq r_m^{2d} \quad (3.28)$$

holds for all z satisfying $|z|=r_m\notin E_2, r_m\to+\infty$ and $|g(z)|=M(r_m,g)$. Set $\beta=\max\left\{\sigma(B_j)\ (j=0,\cdots,k-1)\right\}$.

Suppose that (i) holds. Using the same reasoning as above, there is a ray $\arg z = \theta \in [0, 2\pi) \setminus H$, where H is defined above such that $\delta(P_d, \theta) > 0$, $\delta(P_j, \theta) < 0$ $(j \neq d)$. By Lemma 2.3, for any given ε $(0 < 2\varepsilon < \min\{1, n - \beta\})$, there exists a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$, we have

$$|B_d(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\} \tag{3.29}$$

and

$$\left|B_j(z)\right| \leq \exp\left\{r^{\sigma(B_j)+\frac{\varepsilon}{2}}\right\} \quad (j \neq d). \tag{3.30}$$

By Lemma 2.4, for any given ε $(0 < 2\varepsilon < \min\{1, n - \beta\})$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_5$, $r \to +\infty$, we have

$$\left|A_d(z)e^{P_d(z)}\right| \geq \exp\left\{(1 - \varepsilon)\,\delta(P_d, \theta)\, r^n\right\} \tag{3.31}$$

and

$$\left|A_j(z)e^{P_j(z)}\right| \leq \exp\left\{(1 - \varepsilon)\,\delta(P_j, \theta)\, r^n\right\} < 1 \quad (j \neq d). \tag{3.32}$$

By (3.29) and (3.31), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_3 \cup E_5$, $r \to +\infty$, we have

$$\left|A_d(z)e^{P_d(z)} + B_d(z)\right| \geq (1 - o(1)) \exp\left\{(1 - \varepsilon)\,\delta(P_d, \theta)\, r^n\right\}. \tag{3.33}$$

By (3.30) and (3.32), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_3 \cup E_5$, $r \to +\infty$, we have

$$\begin{aligned} \left|A_j(z)e^{P_j(z)} + B_j(z)\right| &\leq \exp\left\{(1 - \varepsilon)\,\delta(P_j, \theta)\, r^n\right\} + \exp\left\{r^{\sigma(B_j)+\frac{\varepsilon}{2}}\right\} \\ &\leq \exp\left\{r^{\sigma(B_j)+\varepsilon}\right\} \leq \exp\left\{r^{\beta+\varepsilon}\right\} \quad (j \neq d). \end{aligned} \tag{3.34}$$

We can rewrite (1.1) as

$$\begin{aligned} A_d(z)e^{P_d(z)} + B_d(z) = \frac{f^{(k)}}{f^{(d)}} &+ \sum_{j=d+1}^{k-1} \left(A_j(z)e^{P_j(z)} + B_j(z)\right) \frac{f^{(j)}}{f^{(d)}} \\ &+ \sum_{j=0}^{d-1} \left(A_j(z)e^{P_j(z)} + B_j(z)\right) \frac{f^{(j)}}{f} \frac{f}{f^{(d)}}. \end{aligned} \tag{3.35}$$

Hence from (3.26) − (3.28) and (3.33) − (3.35), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r_m \notin [0, 1] \cup E_1 \cup E_2 \cup E_3 \cup E_5$, $r_m \to +\infty$ and $|g(z)| = M(r_m, g)$, we obtain

$$(1 - o(1)) \exp\left\{(1 - \varepsilon)\,\delta(P_d, \theta)\, r_m^n\right\} \leq M_2 r_m^{2d+1} \exp\left\{r_m^{\beta+\varepsilon}\right\} [T(2r_m, f)]^k, \tag{3.36}$$

where M_2 (> 0) is some constant. Thus $0 < 2\varepsilon < \min\{1, n - \beta\}$ implies $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. By Lemma 2.5, we have $\sigma_2(f) = n$.

Suppose that (ii) holds. Using the same reasoning as above, we take a ray $\arg z = \theta \in [0, 2\pi) \setminus H$, where $H = \{\theta \in [0, 2\pi) : \delta(P_d, \theta) = 0\}$, such that $\delta(P_d, \theta) > 0$. By Lemma 2.3, for any given ε $(0 < 2\varepsilon < \min\left\{\frac{1-c}{1+c}, n - \beta\right\})$, there exists a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$, we have

$$\left|B_j(z)\right| \leq \exp\left\{r^{\beta+\varepsilon}\right\} \ \ (j = 0, \cdots, k-1). \tag{3.37}$$

By Lemma 2.4, for any given ε $(0 < 2\varepsilon < \min\left\{\frac{1-c}{1+c}, n - \beta\right\})$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_5$, $r \to +\infty$, we have (3.31) and

$$\left|A_j(z)e^{P_j(z)}\right| \leq \exp\left\{(1+\varepsilon)\, c\delta(P_d, \theta)\, r^n\right\} \ \ (j \neq d). \tag{3.38}$$

By (3.31) and (3.37), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_3 \cup E_5$, $r \to +\infty$, we have (3.33). By (3.37) and (3.38), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_3 \cup E_5$, $r \to +\infty$, we have

$$\left|A_j(z)e^{P_j(z)} + B_j(z)\right| \leq (1 + o(1)) \exp\left\{(1+\varepsilon)\, c\delta(P_d, \theta)\, r^n\right\} \ \ (j \neq d). \tag{3.39}$$

Hence from (3.26) $-$ (3.28) and (3.33), (3.35) and (3.39), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r_m \notin [0, 1] \cup E_1 \cup E_2 \cup E_3 \cup E_5$, $r_m \to +\infty$ and $|g(z)| = M(r_m, g)$, we obtain

$$\begin{aligned}&(1 - o(1)) \exp\left\{(1-\varepsilon)\, \delta(P_d, \theta)\, r_m^n\right\} \\ &\leq M_3 r_m^{2d+1} (1 + o(1)) \exp\left\{(1+\varepsilon)\, c\delta(P_d, \theta)\, r_m^n\right\} \left[T(2r_m, f)\right]^k,\end{aligned} \tag{3.40}$$

where M_3 (> 0) is a constant. By $0 < 2\varepsilon < \frac{1-c}{1+c}$ and (3.40), we have

$$\exp\left\{\frac{(1-c)}{2}\delta(P_d, \theta)\, r_m^n\right\} \leq M_4 r_m^{2d+1} \left[T(2r_m, f)\right]^k, \tag{3.41}$$

where M_4 (> 0) is a constant. Hence (3.41) implies $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. By Lemma 2.5, we have $\sigma_2(f) = n$.

Suppose that (iii) holds.

Case (a). $\rho \leq 0$. Using the same reasoning as above, there exists a ray $\arg z = \theta \in [0, 2\pi) \setminus H$, where H is defined as above, such that $\delta(P_d, \theta) > 0$ and $\delta(P_s, \theta) < 0$. Hence (3.14) and (3.15) hold. By Lemma 2.3, for any given ε $(0 < 2\varepsilon < \min\{1, n - \beta\})$ there exists a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$, we have (3.29) and (3.30). By Lemma 2.4, for the above ε, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_5$, $r \to +\infty$, we have (3.31) and (3.32). By (3.29) and (3.31), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_3 \cup E_5$, $r \to +\infty$, we have (3.33). By (3.30) and (3.32), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0, 1] \cup E_3 \cup E_5$, $r \to +\infty$, we have (3.34). Hence from (3.26) $-$ (3.28) and (3.33) $-$ (3.35), for all

z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r_m \notin [0,1] \cup E_1 \cup E_2 \cup E_3 \cup E_5$, $r_m \to +\infty$ and $|g(z)| = M(r_m, g)$, we obtain (3.36). Thus $0 < 2\varepsilon < \min\{1, n - \beta\}$ implies $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. By Lemma 2.5, we have $\sigma_2(f) = n$.

Case (b). $0 < \rho < 1$. Using the same reasoning as above, there exists a ray $\arg z = \theta \in [0, 2\pi) \setminus H$, where H is defined as above, such that $\delta(P_d, \theta) > 0$ and $\delta(P_s, \theta) < 0$. Hence $(3.16) - (3.19)$ hold. By Lemma 2.4, for any given ε $(0 < 2\varepsilon < 1)$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have

$$\left| A_d(z) e^{(1-\rho)P_d(z)} \right| \geq \exp\left\{ (1-\varepsilon)(1-\rho)\,\delta(P_d, \theta)\, r^n \right\}, \tag{3.42}$$

$$\left| e^{-\rho P_d(z)} \right| \leq \exp\left\{ -(1-\varepsilon)\,\rho\delta(P_d, \theta)\, r^n \right\} \leq 1, \tag{3.43}$$

$$\left| B_j(z)\, e^{-\rho P_d(z)} \right| \leq \exp\left\{ -(1-\varepsilon)\,\rho\delta(P_d, \theta)\, r^n \right\} \leq 1 \ \ (j = 0, \cdots, k-1), \tag{3.44}$$

$$\left| A_j(z) e^{P_j(z) - \rho P_d(z)} \right| \leq \exp\left\{ (1-\varepsilon)\,\delta\left(P_j - \rho P_d, \theta\right) r^n \right\} \leq 1 \ \ (j \neq d). \tag{3.45}$$

We can rewrite (1.1) as

$$\begin{aligned} A_d(z) e^{(1-\rho)P_d(z)} &= -B_d(z)\, e^{-\rho P_d(z)} + e^{-\rho P_d(z)} \frac{f^{(k)}}{f^{(d)}} \\ &\quad + \sum_{j=d+1}^{k-1} \left(A_j(z) e^{P_j(z) - \rho P_d(z)} + B_j(z)\, e^{-\rho P_d(z)} \right) \frac{f^{(j)}}{f^{(d)}} \\ &\qquad + \sum_{j=0}^{d-1} \left(A_j(z) e^{P_j(z) - \rho P_d(z)} + B_j(z)\, e^{-\rho P_d(z)} \right) \frac{f^{(j)}}{f} \frac{f}{f^{(d)}}. \end{aligned} \tag{3.46}$$

By $(3.26) - (3.28)$ and $(3.42) - (3.46)$, for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r_m \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r_m \to +\infty$ and $|g(z)| = M(r_m, g)$, we obtain

$$\exp\left\{ (1-\varepsilon)(1-\rho)\,\delta(P_d, \theta)\, r_m^n \right\} \leq M_5 r_m^{2d+1} \left[T(2r_m, f) \right]^k, \tag{3.47}$$

where $M_5\ (> 0)$ is some constant. Thus $0 < 2\varepsilon < 1$ implies $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. By Lemma 2.5, we have $\sigma_2(f) = n$.

If $\arg a_{n,j} = \theta_s$ $(j \neq d, s)$, then $\arg a_{n,j} \neq \arg a_{n,d}$ $(j \neq d)$ and by case (i), it follows that every transcendental solution f of equation (1.1) is of infinite order and satisfies $\sigma_2(f) = n$.

Suppose now that $\max\{c_1, \cdots, c_{d-1}\} < c_0$ in case (ii). If f is a rational solution of (1.1), then by $\max\{c_1, \cdots, c_{d-1}\} < c_0$, the hypotheses of case *(ii)* and

$$\begin{aligned} f = -\Bigg(&\frac{1}{A_0(z) e^{P_0(z)} + B_0(z)} f^{(k)} + \frac{A_{k-1}(z) e^{P_{k-1}(z)} + B_{k-1}(z)}{A_0(z) e^{P_0(z)} + B_0(z)} f^{(k-1)} \\ &+ \cdots + \frac{A_1(z) e^{P_1(z)} + B_1(z)}{A_0(z) e^{P_0(z)} + B_0(z)} f' \Bigg), \end{aligned} \tag{3.48}$$

we obtain a contradiction since the left side of equation (3.48) is a rational function but the right side is a transcendental meromorphic function.

Now we prove that equation (1.1) cannot have a nonzero polynomial solution. Suppose that $c' = \max\{c_1, \cdots, c_{d-1}\} < c_0$ and let $f(z)$ be a nonzero polynomial solution of equation (1.1) with $\deg f(z) = q$. We take a ray $\arg z = \theta \in [0, 2\pi) \setminus H$, where H is defined as above, such that $\delta(P_d, \theta) > 0$. By Lemma 2.3, for any given ε $\left(0 < 2\varepsilon < \min\left\{\frac{1-c}{1+c}, \frac{c_0-c'}{c_0+c'}, n-\beta\right\}\right)$, there exists a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$, we have (3.37). By Lemma 2.4, for the above ε, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.31) and (3.38). By (3.31) and (3.37), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_3 \cup E_5$, $r \to +\infty$, we have (3.33) and by (3.37) and (3.38), for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_3 \cup E_5$, $r \to +\infty$, we have (3.39). If $q \geq d$, by (1.1), (3.33) and (3.39), we obtain for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_3 \cup E_5$, $r \to +\infty$

$$\begin{aligned} M_6 r^{q-d}(1-o(1)) \exp\{(1-\varepsilon)\delta(P_d,\theta) r^n\} &\leq \left|A_d(z)e^{P_d(z)} + B_d(z)\right| \left|f^{(d)}(z)\right| \\ &\leq \sum_{j \neq d} \left|A_j(z)e^{P_j(z)} + B_j(z)\right| \left|f^{(j)}(z)\right| \\ &\leq M_7 r^q (1+o(1)) \exp\{(1+\varepsilon) c\delta(P_d,\theta) r^n\}, \end{aligned} \tag{3.49}$$

where $M_6\ (>0)$, $M_7\ (>0)$ are constants. By (3.49), we get

$$\exp\left\{\frac{(1-c)}{2}\delta(P_d,\theta) r^n\right\} \leq M_8 r^d, \tag{3.50}$$

where $M_8\ (>0)$ is some constant. Hence (3.50) is a contradiction. If $q < d$, by (1.1), (3.33) and (3.39), we obtain for all z with $\arg z = \theta \in [0, 2\pi) \setminus H$, $|z| = r \notin [0,1] \cup E_3 \cup E_5$, $r \to +\infty$

$$\begin{aligned} M_9 r^{d-1}(1-o(1)) \exp\{(1-\varepsilon) c_0\delta(P_d,\theta) r^n\} &\leq \left|A_0(z)e^{P_0(z)} + B_0(z)\right| |f(z)| \\ &\leq \sum_{j=1}^{d-1} \left|A_j(z)e^{P_j(z)} + B_j(z)\right| \left|f^{(j)}(z)\right| \\ &\leq M_{10} r^{d-2}(1+o(1)) \exp\{(1+\varepsilon) c'\delta(P_d,\theta) r^n\}, \end{aligned} \tag{3.51}$$

where $M_9\ (>0)$, $M_{10}\ (>0)$ are constants. By (3.51), we get

$$\exp\left\{\frac{(c_0-c')}{2}\delta(P_d,\theta) r^n\right\} \leq \frac{M_{11}}{r}, \tag{3.52}$$

where $M_{11}\ (>0)$ is some constant. This is a contradiction. Therefore, if $\max\{c_1, \cdots, c_{d-1}\} < c_0$, then every meromorphic solution of equation (1.1) is of infinite order and satisfies $\sigma_2(f) = n$.

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