Quadric Veronesean Caps

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Abstract

In 2008, Ferrara Dentice and Marino provided a characterization theorem for Veronesean caps in $\mathsf{PG}(N, \mathbb{K})$, with $\mathbb{K}$ a skewfield. This result extends the theorem for the finite case proved by J.A. Thas and Van Maldeghem in 2004. However, although the statement of this theorem is correct, the proof given by Ferrara Dentice and Marino is incomplete, as they borrow some lemmas from the paper of J.A. Thas and Van Maldeghem, which are proved using counting arguments and hence require a different approach in the infinite case. In this paper we use the Veblen-Young theorem to fill these gaps. Moreover, we then use this classification of Veronesean caps to provide a further general geometric characterization.

1 Introduction

Veronesean varieties are fundamental objects in geometry, be it in classical algebraic geometry or modern finite geometry. In the past decades, several characterization results were proved for both quadric Veroneseans and Hermitian Veroneseans in the finite case, many of them purely combinatorial, but some of them rather geometric in nature. Two examples of the latter are (1) the characterization as unions of ovals or ovoids with an additional assumption on the tangent lines or planes, see [1], [4] and [5]; these characterizations also hold for certain projections of the varieties, (2) the characterization as representation of a projective space in another projective space where lines of the former are ovals or ovoids in the latter, see [6] and [7].

Since the formulation of the assumptions of the above characterizations are independent of the finiteness, one can wonder whether these also hold in the general

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(infinite) case. A first attempt towards this was made by Ferrara Dentice and Marino (in "Classification of Veronesean caps", *Discrete Math.* **308** (2008), 299–302), who considered the characterization of type (1) for quadric Veroneseans. However, their proof contains two serious gaps, as firstly they neglected to prove that the tangent lines at a fixed point x to the ovals containing x and meeting a fixed oval not through x fill up a plane, which is crucial in showing that the cap endowed with the ovals is a projective space; in the finite case, this just follows by the numbers. Secondly, in the case of an infinite field, one needs to show that this projective space is necessarily finite-dimensional (this is trivial in the finite case). Once this proved, it is a routine exercise to reformulate the proof in [5] count-free. In the present paper, we fill these gaps by directly showing that the cap endowed with the ovals is a finite-dimensional projective space using Veblen's axiom. Then we go on proving a type (2) characterization for quadric Veronesean varieties (valid in the general infinite case, but at the same time providing an alternative proof for the finite case).
The paper is organized as follows. In Section 2, we introduce the necessary notions: we review the Veblen-Young theorem, which is crucial in our arguments, define quadric Veroneseans, and state our main results. In Section 3 we prove Theorem 2.2. In Section 4, we prove Theorem 2.3.

2 Notation and main results

2.1 Axiomatization of projective spaces

A good exposition on the foundations of projective and polar spaces can be found on Peter Cameron's website, and the paragraph below is based on these lecture notes. At the end of the 19th century a lot of work was done on the axiomatization of projective spaces, starting with Pasch. This work culminated in 1910 when Veblen and Young provided a beautiful characterization of projective spaces [8] based on the following axiom.
Veblen's axiom
If a line intersects two sides of a triangle but does not contain their intersection then it also intersects the third side.
Before stating the Veblen-Young theorem, we recall that a *thick linear space* is an incidence structure such that any line contains at least three points, and such that any two distinct points are contained in a unique line.

Theorem 2.1 (Veblen-Young theorem)**.** *Let $(X, \mathcal{L})$ be a thick linear space satisfying Veblen's axiom. Then one of the following holds:*

(1) $X = \mathcal{L} = \emptyset$.

(2) $|X| = 1$, $\mathcal{L} = \emptyset$.

(3) $\mathcal{L} = \{X\}, |X| \geq 3$.

(4) $(X, \mathcal{L})$ *is a projective plane.*

(5) $(X, \mathcal{L})$ *is a projective space over a skew field, not necessarily of finite dimension.*

2.2 Veronesean caps

An *oval* C in a projective plane π is a set of points of π such that no line of π intersects C in at least 3 points, and for every point $x \in C$, there is a unique line L through x intersecting C in only x. The line L is called the *tangent line* at x to C and denoted $T_x(C)$.
Let X be a spanning point set of $\mathrm{PG}(N, \mathbb{K})$, $N \in \mathbb{N}$, $N \geq 3$, with $\mathbb{K}$ any skew field, and let Σ be a collection of planes of $\mathrm{PG}(N, \mathbb{K})$ such that, for any $\pi \in \Sigma$, the intersection $\pi \cap X$ is an oval $X(\pi)$ in π (and then, for $x \in X(\pi)$, we sometimes denote $T_x(X(\pi))$ simply by $T_x(\pi)$). Then $\mathcal{V} = (X, \Sigma)$ is called a *Veronesean cap* if the following properties hold :

(V1) Any two points x and y lie in a unique element of Σ, denoted by $[x, y]$.

(V2) If $\pi_1, \pi_2 \in \Sigma$, with $\pi_1 \neq \pi_2$, then $\pi_1 \cap \pi_2 \subset X$.

(V3) If $x \in X$ and $\pi \in \Sigma$, with $x \notin \pi$, then all lines $T_x([x, y])$, $y \in \pi \cap X$, are contained in a common plane of $\mathrm{PG}(N, \mathbb{K})$, denoted by $T(x, \pi)$.

In [6], it is proved that for $n \geq 2$ the following are examples of Veronesean caps.
Quadric Veroneseans
Let $\mathbb{K}$ be a (commutative) field and n a natural number greater than or equal to 1. The *quadric Veronesean $\mathcal{V}_n$ of index n* is the set of points of the projective space $\mathsf{PG}(n(n+3)/2, \mathbb{K})$ with generic element

$$(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, x_0x_2, \ldots, x_0x_n, x_1x_2, \ldots, x_1x_n, \ldots, x_{n-1}x_n),$$

where $(x_0, x_1, \ldots, x_n)$ is a point of $\mathsf{PG}(n, \mathbb{K})$. Equivalently, if we consider a point of $\mathsf{PG}(n(n+3)/2, \mathbb{K})$ with projective coordinates

$$(y_{00}, y_{11}, \ldots, y_{nn}, y_{01}, y_{02}, \ldots, y_{0n}, y_{12}, \ldots, y_{1n}, \ldots, y_{n-1,n}),$$

then it belongs to $\mathcal{V}_n$ if and only if $\mathrm{rank}(y_{ij}) = 1$, with $y_{ij} = y_{ji}$ if $i > j$.
The following theorem is our first main result and is the generalization of the finite case, proved in [6].

Theorem 2.2. *Let X be a Veronesean cap in $\Pi = \mathrm{PG}(N, \mathbb{K})$. Then $\mathbb{K}$ is a field and there exists a natural number $n \geq 2$ (called the index of X), a projective space $\Pi' := \mathsf{PG}(n(n+3)/2, \mathbb{K})$ containing Π, a subspace R of Π' skew to Π, and a quadric Veronesean $\mathcal{V}_n$ of index n in Π', with $R \cap \mathcal{V}_n = \emptyset$, such that X is the (bijective) projection of $\mathcal{V}_n$ from R onto Π. The subspace R can be empty, in which case X is projectively equivalent to $\mathcal{V}_n$.*

The above was also stated by Ferrara Dentice and Marino, but their argument contains a gap. To be more precise, let $\mathcal{V} = (X, \Sigma)$ be a Veronesean cap, where X is a set of points in $\mathsf{PG}(N, \mathbb{K})$, for some skew field $\mathbb{K}$, and Σ its collection of planes. Associated with $\mathcal{V}$ we can consider the geometry $\mathcal{P}$ having point set X and as line set $\mathcal{L}$ the set Σ, endowed with the natural incidence. Then Ferrara Dentice and Marino proved the above theorem under the extra assumption that $(X, \mathcal{L})$ is a finite-dimensional projective space (in fact, they derived that $(X, \mathcal{L})$ is a projective space from the unproved and unreferenced fact that, in (V3), the

tangent lines at any point $x \in X$ are not only contained in a common plane, but cover the whole plane). In Section 3 we will prove that $(X, \mathcal{L})$ is a projective space using the Veblen-Young theorem. Moreover, we show that $(X, \mathcal{L})$ is finite-dimensional. The proof of Theorem 2.2 is then finished by quoting the proof for the finite case contained in [5].
As an application, we will show the following characterization, which basically replaces Condition (V2) with a dimension restriction, and (V3) with the condition that the geometry of points and ovals is a projective space.

Theorem 2.3. *Let X be a spanning set of points in the projective space $\mathsf{PG}(d, \mathbb{K})$, with $\mathbb{K}$ any skew field of order at least 3. Suppose that*

(V1*) *X contains a set of plane ovals and that any two points x, y of X are contained in exactly one of these ovals; we denote this oval by $X[x, y]$;*

(V2*) *the set X endowed with all subsets $X[x, y]$, has the structure of the point-line geometry of a projective space $\mathsf{PG}(n, \mathbb{F})$, for some skew field $\mathbb{F}$, $n \geq 3$, or of some projective plane Π (and we put $n = 2$ in this case);*

(V3*) $d \geq \frac{1}{2}n(n+3)$.

Then $d = \frac{1}{2}n(n+3)$ and X is the point set of a quadric Veronesean of index n in $\mathsf{PG}(d, \mathbb{K})$. In particular, $\mathbb{K}$ and $\mathbb{F}$ are commutative, $\mathbb{F} \equiv \mathbb{K}$ if $n \geq 3$, and Π is isomorphic to $\mathsf{PG}(2, \mathbb{K})$ if $n = 2$.

3 Proof of the Main Result

Let $\mathcal{V} = (X, \Sigma)$ be a Veronesean cap, where X is a set of points in $\mathsf{PG}(N, \mathbb{K})$, for some skew field $\mathbb{K}$, and Σ its collection of planes.
Associated with $\mathcal{V}$ we can consider the geometry $\mathcal{P}$ having point set X and line set the set Σ, endowed with the natural incidence.

Theorem 3.1. *$\mathcal{P}$ is a projective space.*

Proof. Recall that we denote by $[x, y]$ the unique element of Σ through $x, y \in X$; so $[x, y]$ is a plane. Then we write $X[x, y]$ for $X \cap [x, y]$. The set of ovals $X[x, y]$ will be denoted by Ω.
Let x_{12}, x_{23} and x_{13} be three points of X not contained in a common oval of Ω and denote $C_1 = X[x_{12}, x_{13}]$, $C_2 = X[x_{12}, x_{23}]$ and $C_3 = X[x_{13}, x_{23}]$. Let C_4 be an oval of Ω intersecting C_1 in a point x_{14} and C_2 in a point x_{24}, both different from x_{12}. Our purpose is to show that Veblen's axiom holds, which means that we have to show that C_4 intersects C_3. Of course, we may assume that $C_3 \neq C_4$ and that C_4 does not contain x_{13} nor x_{23}. First we claim that $V := \langle C_1, C_2, C_3 \rangle$ contains C_4 and may be assumed to be of dimension 5.
Indeed, let us first show that V contains C_4. Since both $T_{x_{13}}(C_3)$ and $T_{x_{13}}(C_1)$ belong to $\langle C_1, C_3 \rangle \subseteq V$, it follows by Condition (V3) applied to the point x_{13} and the oval C_2 that also $T_{x_{13}}([x_{13}, x_{24}])$ does, and hence $[x_{13}, x_{24}] = \langle T_{x_{13}}([x_{13}, x_{24}]), x_{24} \rangle$ is contained in V. Likewise, applying (V3) to x_{24} and C_1 and reasoning as above it follows that C_4 is contained in V.

Clearly V cannot be 3-dimensional by (V2). Now, if V were 4-dimensional, then, again by (V2), C_4 and C_3 would meet, and Veblen's axiom would follow automatically. So we may assume that V is 5-dimensional.
Now we project $V \setminus \langle C_2 \rangle$ from $\langle C_2 \rangle$ onto a plane π of V disjoint from $\langle C_2 \rangle$. The ovals C_3 and C_4 together with their tangents at their intersection point with C_2 are mapped onto two full lines of π, say L_3 and L_4, respectively. Let x be the intersection of L_3 and L_4. There are basically four different possibilities.

(1) *There is a point x_i of $C_i \setminus C_2$ projected onto x from $\langle C_2 \rangle$, for $i = 3, 4$, and $x_3 \neq x_4$.*

In this case, since the space $\langle x_3, x_4, C_2 \rangle = \langle x, C_2 \rangle$ is 3-dimensional, the line $\langle x_3, x_4 \rangle$ meets the plane $\langle C_2 \rangle$ in a point y. This implies that the plane $[x_3, x_4]$ intersects $\langle C_2 \rangle$ in y, implying $y \in X$ by (V2), contradicting $X[x_3, x_4]$ being an oval.

(2) *There is a point x_3 of $C_3 \setminus C_2$ projected onto x from $\langle C_2 \rangle$, and the tangent line $T_{x_{24}}(C_4) := L_4$ to C_4 at x_{24} projects onto x from $\langle C_2 \rangle$.*

In this case, clearly L_4 is contained in $\langle C_2, x_3 \rangle$, which also contains $T_{x_{24}}(C_2)$. Hence, by our axioms, the 3-space $\langle C_2, x_3 \rangle$ also contains $T_{x_{24}}([x_{13}, x_{24}])$ (since the ovals C_2, C_4 and $X[x_{13}, x_{24}]$ all intersect C_1). Similarly, since the ovals $X[x_{13}, x_{24}], C_2$ and $X[x_3, x_{24}]$ all meet the conic C_3, the line $T_{x_{24}}([x_3, x_{24}])$ belongs to $\langle C_2, x_3 \rangle$, which implies that $X[x_3, x_{24}]$ belongs to the 3-space $\langle C_2, x_3 \rangle$ and so $[x_3, x_{24}]$ meets $\langle C_2 \rangle$ in a line, contradicting our axioms.

(3) *The tangent lines $T_{x_{2i}}(C_i) =: L_i$ to C_i at x_{2i} project onto x from $\langle C_2 \rangle$, for all $i \in \{3, 4\}$.*

In this case, as above, the 3-space $\langle C_2, x \rangle$ contains $T_{x_{24}}([x_{13}, x_{24}])$. It follows that the 4-space $U := \langle C_2, x, x_{13} \rangle$ contains $X[x_{13}, x_{24}], C_2$ and C_3. But, as above, one easily deduces that U also contains C_1, and so U coincides with V, a contradiction.

(4) The only remaining possibility is that *there is a point z of $(C_3 \cap C_4) \setminus C_2$ projected onto x from $\langle C_2 \rangle$*. But then $C_3 \cap C_4$ is nonempty, and that is exactly what we had to prove.

Hence we have shown that Veblen's axiom holds. ■

Remark At this point it is not yet clear why $\mathcal{P}$ is finite-dimensional. If $\mathcal{P}$ is finite-dimensional we call the dimension of $\mathcal{P}$ the *index* of the Veronesean cap.
To finish the proof of Theorem 2.2 for index $n = 2$, we first claim that in this case the point set of $\mathcal{P}$ generates a 5-dimensional space. Indeed, the argument in the third paragraph of the proof of Theorem 3.1 shows that $\mathcal{P}$ generates a space of dimension at most 5, and (V2) implies that this dimension is at least 4. Suppose that the dimension is exactly 4. We consider a point $x \in X$ and three ovals C_1, C_2, C_3. Then any 3-space containing C_1 and not containing the tangent lines at x to C_2 and C_3, respectively, intersects $C_2 \cup C_3$ in two points x_2, x_3 distinct from x, with $x_i \in C_i$, $i = 1, 2$. Then the intersection of $[x_2, x_3]$ with C_1 lies on the line of $\mathsf{PG}(4, \mathbb{K})$ spanned by x_2, x_3, contradicting the fact that $[x_2, x_3]$ is an oval. Hence the claim follows and $N = 5$.

We project from a projective line intersecting X in two points x and y onto a 3-dimensional space Γ skew to this line. Considering the projections of all ovals through x or y, except for $X[x, y]$ itself, we obtain two sets of affine lines spanning Γ and such that each affine line of one set meets every affine line of the other set. It follows easily that the corresponding (projective) lines form the two generator sets of a hyperbolic quadric Q, from which two generators are removed, one of each class. Hence $\mathbb{K}$ is a field.

Remark The missing generators contain the projections of the tangents $T_x([x, z])$ and $T_y([y, z])$, for z ranging through the points of $X \setminus \{x, y\}$. Hence the subspace Σ_y generated by y and the tangents at x of all ovals containing x is 3-dimensional (which also follows from (V3)). So the images of the planes of the ovals through these points yield two opposite reguli.

Now the general case for finite index to prove that $\mathbb{K}$ is a field follows as in [6]. Finally, to exclude the possibility of $\mathcal{P}$ being infinite-dimensional, the above argument with the two opposite reguli shows

Lemma 3.2. *If $x \in X$ and $\pi \in \Sigma$ with $x \notin \pi$, then $T(x, \pi)\setminus\{x\}$ is the disjoint union of $T_x([x, y])\setminus\{x\}$, with y ranging over $X \cap \pi$.*

This is Lemma 2.1 from [5]. Similarly as in that article it now follows that the tangent space $T(x)$ of a Veronesean cap has the same dimension as the projective space $\mathcal{P}$ (here, $T(x)$ is the space generated by all the tangents at x to conics $X[x, y]$, $y \in X \setminus \{x\}$). Since the tangent space is contained in $\mathsf{PG}(N, \mathbb{K})$, and N is finite, it follows immediately that $\mathcal{P}$ is finite-dimensional.

4 An application of quadric Veronesean caps

Using the classification of Veronesean caps, we can now show Theorem 2.3. In order to do so, we show (V2) and (V3). But, as in the finite case (see Section 3 of [6]), one shows that, if $n \geq 3$, the space spanned by the points of X corresponding to a plane of $\mathsf{PG}(n, \mathbb{F})$ has dimension 5. Hence it suffices to consider the case $n = 2$.

For ease of notation, we will call *oval* any oval of the form $X[x, x']$, with $x, x' \in X$.

Proof of Theorem 2.3

Take two distinct points $x, y \in X$. Let C_1, C_2 be two distinct ovals through x not containing y. Denote $H := \langle C_1, C_2, y\rangle$. Let C be an arbitrary oval through y, but not through x. Then C meets $C_1 \cup C_2$ in two distinct points and hence contains three noncollinear points of H and is thus contained in H. It follows easily that $X \subseteq H$ and so H coincides with $\mathsf{PG}(5, \mathbb{K})$. This firstly shows (V2) and secondly implies that the projections of $C_1 \setminus \{x\}$ and $C_2 \setminus \{x\}$ from the line $\langle x, y\rangle$ onto a solid Γ skew to $\langle x, y\rangle$ are two non-coplanar affine lines A_1 and A_2, respectively (an *affine line* is just the point set of a line with one point removed). As in the remark above the subspace Γ_y generated by y and the tangents at x of all ovals containing x is 3-dimensional. Replacing y by any other point y' of X distinct from x and such that $y' \notin X[x, y]$, we see that all mentioned tangents together with y' are also contained in a solid $\Gamma_{y'}$. If $\Gamma_y = \Gamma_{y'}$, then it would contain two ovals. Renaming them as C_1, C_2 and picking a point not on these, we obtain a contradiction to the above result that H is 5-dimensional. Hence all tangents at x are contained in the plane $\Gamma_y \cap \Gamma_{y'}$ and the theorem is proved. ■

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