Homoclinic solutions for second order Hamiltonian systems with small forcing terms*

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Abstract

The existence of homoclinic solutions is obtained for a class of nonautonomous second order Hamiltonian systems $\ddot{u}(t) + \nabla V(t, u(t)) = f(t)$ as the limit of the $2kT$-periodic solutions which are obtained by the Mountain Pass theorem, where $V(t, x) = -K(t, x) + W(t, x)$ is T-periodic with respect to t, $T > 0$, and $W(t, x)$ satisfies the superquadratic condition: $W(t, x)/|x|^2 \to +\infty$ as $|x| \to \infty$ uniformly in t, which needs not to satisfy the global Ambrosetti-Rabinowitz condition.

1 Introduction and main results

In this paper, we put our attention to the existence of homoclinic orbits for the second order Hamiltonian system

$$\ddot{u}(t) + \nabla V(t, u(t)) = f(t), \quad \forall t \in R, \tag{1}$$

where $f : R \to R^N$ is a continuous, bounded function. As usual, we say that a solution $u(t)$ of problem (1) is nontrivial homoclinic(to 0) if $u \neq 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$. Here and subsequently, $\nabla V(t, x)$ denotes the gradient with respect to the x variable, and $(\cdot, \cdot) : R^N \times R^N \to R$ denotes the standard inner product in R^N and $|\cdot|$ is the induced norm.

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The existence of homoclinic orbits is a very important problem in the theory of Hamiltonian systems. It has been studied by many authors (see[1-13]). In 1990, Rabinowitz in [10] showed the existence of homoclinic orbits for problem (1) as the limit of the $2kT$-periodic solutions of problem (1) when $f = 0$ and the function V considered by the author is of the form

$$V(t,x) = -\frac{1}{2}(L(t)x,x) + W(t,x), \tag{2}$$

where L is a continuous T-periodic positive definite symmetric matrix valued function for all $t \in [0,T]$, W is T-periodic and satisfies the so-called global Ambrosetti-Rabinowitz condition, that is,

(W_1) there exists a constant $\lambda > 2$ such that

$$0 < \lambda W(t,x) \leq (x, \nabla W(t,x))$$

for every $t \in R$ and $x \in R^N \setminus \{0\}$. As we know, condition (W_1) implies that

(W_1') $W(t,x)/|x|^2 \to +\infty$ as $|x| \to \infty$ uniformly in t,

which is weaker than (W_1). Then, by replacing (W_1) with (W_1'), the authors in [8] obtained the existence of homoclinic orbits for problem (1) while $f = 0$ and V is of the form (2). Via the same method of Rabinowitz in [10], Izydorek and Janczewska in [5] proved problem (1) possesses a nontrivial homoclinic solution when $V(t,x) = -K(t,x) + W(t,x)$ rather than the form (2), and K is assumed to be periodic in t, satisfying the pinching condition $b_1|x|^2 \geq K(t,x) \geq b_2|x|^2$. After then, by weakening the pinching condition, Tang and Xiao in [12] generalized the results of [5], which are the following theorems.

Theorem A([12]). *Suppose that V and f satisfy (W_1) and the following conditions*

(V) $V(t,x) = -K(t,x) + W(t,x)$, where $K, W : R \times R^N \to R$ are C^1-maps, T-periodic with respect to $t, T > 0$,

(K_1) there are constants $b > 0$ and $\gamma \in (1,2]$ such that

$$K(t,0) = 0, \quad K(t,x) \geq b|x|^\gamma$$

for all $(t,x) \in R \times R^N$,

(K_2) there is a constant $\theta \in [2,\lambda)$ such that

$$(x, \nabla K(t,x)) \leq \theta K(t,x)$$

for all $(t,x) \in R \times R^N$,

(W_2) $\nabla W(t,x) = o(|x|)$ as $x \to 0$ uniformly with respect to t,

(f)

$$0 < \int_R |f(t)|^2 dt < 2\left(\min\left\{\frac{\nu}{2}, b\nu^{\gamma-1} - m\nu^{\lambda-1}\right\}\right)^2,$$

where $m = \sup\{W(t,x) | t \in [0,T], x \in R^N, |x| = 1\}$, and $\nu \in (0,1]$ such that

$$b\nu^{\gamma-1} - m\nu^{\lambda-1} = \max_{x\in[0,1]}\left(bx^{\gamma-1} - mx^{\lambda-1}\right).$$

Then problem (1) possesses a nontrivial homoclinic solution.

When $f = 0$, under one stronger condition on K, they also proved system (1) possesses a nontrivial homoclinic solution, which is the following theorem

Theorem B([12]). *Suppose that $f = 0$ and V satisfies (V), (K_1), (W_1), (W_2) and the following condition*

(K_2') there is a constant $\theta \in [2, \lambda)$ such that

$$K(t,x) \leq (x, \nabla K(t,x)) \leq \theta K(t,x)$$

for all $(t,x) \in R \times R^N$.

Then problem (1) possesses a nontrivial homoclinic solution.

Motivated by the papers above, in this paper, we will obtain the homoclinic solution of problem (1) by using the more general condition (W_1') rather than (W_1). The main results are the following theorems.

Theorem 1.1. *Suppose that $f \neq 0$ and V satisfies (V), (K_1), (W_1') and the following conditions*

(K_2'') $(x, \nabla K(t,x)) \leq 2K(t,x)$ for all $(t,x) \in R \times R^N$,

(W_2') $\nabla W(t,x) = o(|x|^{\gamma-1})$ as $x \to 0$ uniformly with respect to t,

(W_3) there are constants $\beta \geq 0$ and $d_1 > 0$ such that

$$|W(t,x)| \leq d_1 |x|^{\beta}$$

for all $(t,x) \in R \times R^N$,

(W_4) there exist constants $\mu > \max\{\beta - \gamma, 1\}$, $d_2 > 0$ and function $g \in L^1(R, R^+)$ such that

$$(x, \nabla W(t,x)) - 2W(t,x) \geq d_2 |x|^{\mu} - g(t)$$

for all $(t,x) \in R \times R^N$.

Then there is a constant $\delta > 0$ such that, for any f satisfying

$$\max\left\{ \int_R |f(t)|^2 dt, \int_R |f(t)|^{\mu/(\mu-1)} dt \right\} < \delta, \tag{3}$$

system (1) possesses at least one nontrivial homoclinic solution.

Theorem 1.2. *Suppose that $f = 0$ and V satisfies (V), (K_1), (W_1'), (W_2'), (W_3) and the following conditions*

(K_2''') there is a constant $2 \geq \rho > 0$ such that

$$\rho K(t,x) \leq (x, \nabla K(t,x)) \leq 2K(t,x)$$

for all $(t,x) \in R \times R^N$,

(W_4') there exist constants $\mu > \beta - \gamma$, $d_2 > 0$ and function $g \in L^1(R, R^+)$ such that

$$(x, \nabla W(t,x)) - 2W(t,x) \geq d_2|x|^\mu - g(t)$$

for all $(t,x) \in R \times R^N$.

Then problem (1) possesses a nontrivial homoclinic solution.

Remark 1.1. Condition (K_2''') implies $K(t,0) = 0$ and (K_2'').

Remark 1.2. There are functions K and W which satisfy our Theorem 1.1 and Theorem 1.2 without satisfying the corresponding assumptions in [5, 12]. For example, let

$$K(t,x) = |x|^{\frac{11}{6}} + |x|^{\frac{9}{5}}, \quad W(t,x) = \begin{cases} |x|^2 \ln|x|^2 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}$$

where $t \in R$, $x \in R^N$, then $V(t,x) = -K(t,x) + W(t,x)$ cannot be represented as the form $V(t,x) = -K_0(t,x) + W_0(t,x)$ with $K_0(t,x)$ and $W_0(t,x)$ satisfying Theorem A or Theorem B because W satisfies (W_1') and does not satisfy (W_1) while V satisfies our conditions with $b = \frac{1}{2}$, $\gamma = \rho = \frac{9}{5}$, $\beta = \frac{11}{4}$, $d_1 = \mu = 2$, $d_2 = 1$, $g(t) = 0$.

2 Proof of Theorems

For each $k \in N$, let $L^2_{2kT}(R, R^N)$ denote the Hilbert space of $2kT$-periodic functions on R with values in R^N under the norm

$$\|u\|_{L^2_{2kT}(R,R^N)} := \left(\int_{-kT}^{kT} |u(t)|^2 dt \right)^{1/2},$$

and $L^\infty_{2kT}(R, R^N)$ be a space of $2kT$-periodic essentially bounded measurable functions from R into R^N under the norm

$$\|u\|_{L^\infty_{2kT}(R,R^N)} := esssup\{|u(t)| : t \in [-kT, kT]\}.$$

In order to obtain a homoclinic solution of problem (1), we consider a sequence of systems of differential equations:

$$\ddot{u}(t) + \nabla V(t, u(t)) = f_k(t), \tag{4}$$

where, for each $k \in N$, $f_k : R \to R^N$ is a $2kT$-periodic extension of restriction of f to the interval $[-kT, kT]$.

For each $k \in N$, let $E_k := W^{1,2}_{2kT}(R, R^N)$ denote the Hilbert space of $2kT$-periodic function from R to R^N under the norm

$$\|u\|_{E_k} := \left(\int_{-kT}^{kT} (|\dot{u}(t)|^2 + |u(t)|^2) dt \right)^{1/2}.$$

Moreover, let $\eta_k : E_k \to [0, +\infty)$ be given by

$$\eta_k(u) := \left(\int_{-kT}^{kT} (|\dot{u}(t)|^2 + 2K(t, u(t))dt \right)^{1/2}, \tag{5}$$

and $I_k : R \to R^N$ be the corresponding functional of (4) defined by

$$I_k(u) = \int_{-kT}^{kT} \left(\frac{1}{2}|\dot{u}(t)|^2 + K(t, u(t)) - W(t, u(t)) + (f_k(t), u(t)) \right) dt, \tag{6}$$

then one can easily check that $I_k \in C^1(E_k, R)$ and

$$\langle I_k'(u), v \rangle = \int_{-kT}^{kT} ((\dot{u}(t), \dot{v}(t)) - (\nabla V(t, u(t)), v(t)) + (f_k(t), v(t)))\, dt. \tag{7}$$

It follows from (5) and (6) that

$$I_k(u) = \frac{1}{2}\eta_k^2(u) + \int_{-kT}^{kT} (-W(t, u(t)) + (f_k(t), u(t)))dt. \tag{8}$$

Now, we prove the existence of a homoclinic solution of problem (1) as the limit of the $2kT$-periodic solutions of system (4) which are obtained via the Mountain Pass theorem. We have divided the proof of Theorem 1.1 into a sequence of lemmas. We can obtain a conclusion directly from the estimation made in [12], which is our first lemma.

Lemma 2.1. *There is a positive constant C which is independent of k such that for each $k \in N$ and $u \in E_k$ the following inequality holds*

$$\|u\|_{L^\infty_{2kT}(R,R^N)} \leq C\|u\|_{E_k}. \tag{9}$$

Lemma 2.2. *Suppose that (K_2'') holds. Then we have*

$$K(t, x) \leq K\left(t, \frac{x}{|x|}\right)|x|^2 \tag{10}$$

for all $t \in [0, T]$ and $|x| \geq 1$.

Proof. Set $f(s) = s^{-2}K(t, s\xi)$. By (K_2''), we have

$$\begin{aligned} f'(s) &= -2s^{-3}K(t, s\xi) + s^{-2}(\nabla K(t, s\xi), \xi) \\ &= s^{-3}(-2K(t, s\xi) + (\nabla K(t, s\xi), s\xi)) \\ &\leq 0, \end{aligned}$$

then if $s \geq 1$ we have $f(s) \leq f(1)$, that is,

$$s^{-2}K(t, s\xi) \leq K(t, \xi),$$

set $s = |x|$ and $\xi = x/|x|$, we obtain our inequality. ∎

By (V) we can set

$$M := \sup\{K(t,x)|\ t \in [0,T], x \in R^N, |x| \leq 1\},$$

then from Lemma 2.2 we have

$$K(t,x) \leq M(|x|^2 + 1) \tag{11}$$

for all $(t,x) \in R \times R^N$.

Lemma 2.3. *Suppose that $f \neq 0$ and V satisfies (V), (K_1), (K_2''), (W_1'), (W_2'), (W_3) and (W_4), then there is a constant $\delta > 0$ such that, for any f satisfying (3), system (4) possesses a $2kT$-periodic solution $u_k \in E_k$ for every $k \in N$.*

Proof. It is known that the Mountain Pass theorem holds when the usual (PS) condition is replaced by condition (C). Then we apply the Mountain Pass theorem to obtain the critical point of I_k under condition (C).

First of all, we prove a property of W. It follows from (W_2') that, for any $\varepsilon > 0$, there exists $\sigma > 0$ such that

$$|\nabla W(t,x)| \leq \gamma\varepsilon|x|^{\gamma-1}, \quad |x| \leq \sigma,\ \forall t \in [0,T],$$

which implies that

$$\begin{aligned} |W(t,x)| &= \left|\int_0^1 (\nabla W(t,sx),x)ds\right| \\ &\leq \int_0^1 |\nabla W(t,sx)||x|ds \\ &\leq \int_0^1 \gamma\varepsilon|sx|^{\gamma-1}|x|ds \\ &= \varepsilon|x|^\gamma. \end{aligned} \tag{12}$$

We can choose $\varepsilon = \frac{1}{2}b$, then there is a $1 \geq \sigma_0 > 0$ such that (12) holds when $|x| \leq \sigma_0$ for all $t \in [0,T]$.

Our proof involves three steps.

Step 1: I_k satisfies condition (C). We can choose $\delta > 0$ such that $\delta < \frac{\sigma_0}{2C}\min\{1,b\}$. Assumption (W_3) yields $W(t,0) = 0$ which means $I_k(0) = 0$. Then we show that I_k satisfies the (C) condition. Assume that $\{u_j\}_{j\in N} \subset E_k$ is a sequence such that $\{I_k(u_j)\}_{j\in N}$ is bounded and $\|I_k'(u_j)\| \to 0$ as $j \to \infty$. Then there exists a constant $C_k > 0$ such that

$$I_k(u_j) \leq C_k, \quad \|I_k'(u_j)\|(1 + \|u_j\|_{E_k}) \leq C_k. \tag{13}$$

Then $\{u_j\}$ is bounded. If not, passing to a subsequence if necessary, we can suppose that $\|u_j\|_{E_k} \to \infty$ as $j \to \infty$. By (13), (K_2''), (W_4) and (3) we have

$$\begin{aligned}
3C_k &\geq 2I_k(u_j) + \|I_k'(u_j)\|(1 + \|u_j\|_{E_k}) \\
&\geq 2I_k(u_j) - \langle I_k'(u_j), u_j \rangle \\
&\geq \int_{-kT}^{kT} ((\nabla W(t, u_j(t)), u_j(t)) - 2W(t, u_j(t))) + \int_{-kT}^{kT} (f_k(t), u_j(t))dt \\
&\geq d_2 \int_{-kT}^{kT} |u_j(t)|^\mu dt - \int_{-kT}^{kT} g(t)dt - \delta \left(\int_{-kT}^{kT} |u_j(t)|^\mu dt \right)^{1/\mu} \\
&\geq d_2 \int_{-kT}^{kT} |u_j(t)|^\mu dt - \delta \left(\int_{-kT}^{kT} |u_j(t)|^\mu dt \right)^{1/\mu} - G \qquad (14)
\end{aligned}$$

for some $G > 0$. Since $\mu > 1$, it follows from (14) that, there is $D_k > 0$ such that

$$\int_{-kT}^{kT} |u_j(t)|^\mu dt \leq D_k. \qquad (15)$$

Moreover, from (W_3) and (W_4) we can conclude $\beta \geq \mu$, then by (6), (3), (W_3), (15) and Lemma 2.1 we obtain

$$\begin{aligned}
\frac{1}{2}\eta_k^2(u_j) &\leq I_k(u_j) + \int_{-kT}^{kT} W(t, u_j(t))dt - \int_{-kT}^{kT} (f_k(t), u_j(t))dt \\
&\leq C_k + d_1 \int_{-kT}^{kT} |u_j(t)|^\beta dt + \delta \left(\int_{-kT}^{kT} |u_j(t)|^\mu dt \right)^{1/\mu} \\
&\leq C_k + \delta D_k^{1/\mu} + d_1 \int_{-kT}^{kT} |u_j(t)|^\beta dt \\
&\leq C_k + \delta D_k^{1/\mu} + d_1 C^{\beta-\mu} \|u_j\|_{E_k}^{\beta-\mu} \int_{-kT}^{kT} |u_j(t)|^\mu dt \\
&\leq C_k + \delta D_k^{1/\mu} + d_1 C^{\beta-\mu} D_k \|u_j\|_{E_k}^{\beta-\mu}. \qquad (16)
\end{aligned}$$

Since $\mu > \beta - \gamma$, it follows from (16) that there is a constant $\gamma_0 \in (\beta - \mu, \gamma)$ such that

$$\frac{\eta_k^2(u_j)}{\|u_j\|_{E_k}^{\gamma_0}} \to 0 \qquad (17)$$

as $j \to \infty$. When j is big enough, we have $\|u_j\|_{E_k} \geq 1$, by (K_1) and Lemma 2.1, we get

$$\begin{aligned}
\eta_k^2(u_j) &\geq \int_{-kT}^{kT} |\dot{u}_j(t)|^2 dt + 2b \int_{-kT}^{kT} |u_j(t)|^\gamma dt \\
&\geq \int_{-kT}^{kT} |\dot{u}_j(t)|^2 dt + 2bC^{\gamma-2} \|u_j\|_{E_k}^{\gamma-2} \int_{-kT}^{kT} |u_j(t)|^2 dt \\
&\geq \min\{1, 2bC^{\gamma-2}\} \left(\int_{-kT}^{kT} |\dot{u}_j(t)|^2 dt + \|u_j\|_{E_k}^{\gamma-2} \int_{-kT}^{kT} |u_j(t)|^2 dt \right) \\
&\geq \min\{1, 2bC^{\gamma-2}\} \|u_j\|_{E_k}^\gamma,
\end{aligned}$$

which implies that

$$\frac{\eta_k^2(u_j)}{\|u_j\|_{E_k}^{\gamma_0}} \to \infty,$$

as $j \to \infty$. This is a contradiction. Then $\{u_j\}_{j\in N}$ is bounded in E_k. By a standard argument, we see that $\{u_j\}_{j\in N}$ has a convergent subsequence in E_k. Hence I_k satisfies the (C) condition.

Step 2: Now, we show that there exist constants $\varrho,\ \alpha > 0$ independent of k such that $I_k \geq \alpha$ on $\partial B_\varrho(0) = \{u \in E_k|\ \|u\|_{E_k} = \varrho\}$. Set

$$\varrho = \frac{\sigma_0}{C}, \qquad \alpha = \frac{\frac{1}{2}\min\{1,b\}\sigma_0^2 - C\delta\sigma_0}{C^2} > 0, \tag{18}$$

which implies $0 < \|u\|_{L^\infty_{2kT}} \leq \sigma_0 \leq 1$. It follows from (8), (K_1), (12) and (3) that

$$\begin{aligned} I_k(u) &= \frac{1}{2}\eta_k^2(u) + \int_{-kT}^{kT} (-W(t,u(t)) + (f_k(t),u(t)))\,dt \\ &\geq \frac{1}{2}\int_{-kT}^{kT} |\dot{u}(t)|^2 dt + b\int_{-kT}^{kT} |u(t)|^\gamma dt - \frac{1}{2}b\int_{-kT}^{kT} |u(t)|^\gamma dt \\ &\qquad + \int_{-kT}^{kT} (f_k(t),u(t))dt \\ &\geq \frac{1}{2}\int_{-kT}^{kT} |\dot{u}(t)|^2 dt + \frac{1}{2}b\int_{-kT}^{kT} |u(t)|^\gamma dt - \delta\|u\|_{E_k} \\ &\geq \frac{1}{2}\min\{1,b\}\left(\int_{-kT}^{kT} |\dot{u}(t)|^2 dt + \int_{-kT}^{kT} |u(t)|^\gamma dt\right) - \delta\|u\|_{E_k} \\ &\geq \frac{1}{2}\min\{1,b\}\|u\|_{E_k}^2 - \delta\|u\|_{E_k}. \end{aligned} \tag{19}$$

By the definition of ϱ and α, if $\|u\|_{E_k} = \varrho$, (19) implies $I_k(u) \geq \alpha$.

Step 3: We only need to prove that for each $k \in N$ there is $e_k \in E_k$ such that $\|e_k\|_{E_k} > \varrho$ and $I_k(e_k) \leq 0$. By (8) and (11), for every $r \in R \setminus \{0\}$ and $u \in E_k \setminus \{0\}$, the following inequality holds

$$\begin{aligned} I_k(ru) \leq\ & \left(\frac{1}{2}\int_{-kT}^{kT} |\dot{u}(t)|^2 dt + M\int_{-kT}^{kT} |u(t)|^2 dt\right)|r|^2 - \int_{-kT}^{kT} W(t,ru)dt \\ & + |r|\delta\|u\|_{E_k} + 2kTM. \end{aligned} \tag{20}$$

Fix $Q \in C_0^\infty(-T,T) \setminus \{0\} \subset E_1$, then there exists $t_0 \in (-T,T)$ such that $Q(t_0) \neq 0$, which implies that there are $\delta_0 > 0$, $L_1 > 0$ such that

$$|Q(t)| \geq L_1 \tag{21}$$

for all $|t - t_0| < \delta_0$. By (W_1') and (W_3), we can conclude, there exists $L_2 > 0$ such that

$$W(t,x) \geq -L_2 \tag{22}$$

for all $(t,x) \in R \times R^N$. Moreover, (W_1') also implies that for every $\zeta > 0$, there exists $L_3 > 0$ such that

$$\frac{W(t,x)}{|x|^2} \geq \zeta \tag{23}$$

for all $|x| \geq L_3$ uniformly in $t \in R$. When $r \geq L_3/L_1$, combining (21), (22), (23) we have

$$\begin{aligned}\int_{-T}^{T} \frac{W(t,rQ)}{|r|^2}dt &= \int_{-T}^{t_0-\delta_0} \frac{W(t,rQ)}{|r|^2}dt + \int_{t_0-\delta_0}^{t_0+\delta_0} \frac{W(t,rQ)}{|r|^2}dt + \int_{t_0+\delta_0}^{T} \frac{W(t,rQ)}{|r|^2}dt \\ &\geq -\frac{2L_2(T-\delta_0)}{|r|^2} + \int_{t_0-\delta_0}^{t_0+\delta_0} \frac{W(t,rQ)}{|rQ|^2}|Q|^2dt \\ &\geq -\frac{2L_2L_1^2(T-\delta_0)}{L_3^2} + 2\delta_0 L_1^2\zeta,\end{aligned}$$

then by the arbitrariness of $\zeta > 0$ we obtain

$$\int_{-T}^{T} \frac{W(t,rQ)}{|r|^2}dt \to +\infty \quad as \quad |r| \to +\infty. \tag{24}$$

Hence (20) implies that there exists $r_0 \in R \setminus \{0\}$ such that $\|r_0Q\|_{E_1} > \varrho$ and $I_1(r_0Q) < 0$. Set $e_1(t) = r_0Q(t)$ and $e_k(t) = e_1(t)$. Then $e_k \in E_k$, $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \varrho$ and $I_k(e_k) = I_1(e_1) < 0$ for each $k \in N$. By the Mountain Pass theorem, I_k possesses a critical value $c_k \geq \alpha$ given by

$$c_k = \inf_{g\in\Gamma_k} \max_{s\in[0,1]} I_k(g(s)), \tag{25}$$

where

$$\Gamma_k = \{g \in C([0,1], E_k) |\ g(0) = 0,\ \ g(1) = e_k\}.$$

Hence, for each $k \in N$, there exists $u_k \in E_k$ such that

$$I_k(u_k) = c_k, \qquad I_k'(u_k) = 0. \tag{26}$$

Then the function u_k is a desired classical $2kT$-periodic solution of system (4). ■

Lemma 2.4. *Let $u_k \in E_k$ be the solution of system (4) which satisfies (26) for all $k \in N$. Then there is a constant $M_1 > 0$ independent of k such that*

$$\|u_k\|_{E_k} \leq M_1 \tag{27}$$

for all $k \in N$.

Proof. For each $k \in N$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$ where e_k is defined in Lemma 2.3. Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in N$ and $s \in [0,1]$. Therefore, by (25) we have

$$c_k \leq \max_{s\in[0,1]} I_1(g_1(s)) \equiv M_0, \tag{28}$$

where M_0 is independent of $k \in N$, then from (26) we obtain

$$I_k(u_k) \leq M_0, \quad \|I_k^{'}(u_k)\|(1 + \|u_k\|_{E_k}) = 0. \tag{29}$$

In a way similar to proof of *Step* 1 in Lemma 2.3, there exists $M_1 > 0$ independent of k such that

$$\|u_k\|_{E_k} \leq M_1$$

for all $k \in N$, which completes the proof. ∎

Lemma 2.5. *Let $u_k \in E_k$ be the solution of system (4) which satisfies (27) for $k \in N$. Then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in N}$ convergent to u_0 in $C^1_{loc}(R, R^N)$.*

Proof. In order to finish the proof via the Arzelà-Ascoli theorem, we divide our proof into two steps.

First, we show that $\{\dot{u}_k\}_{k\in N}$ and $\{\ddot{u}_k\}_{k\in N}$ are uniformly bounded sequence. By (27), we know that $\{u_k\}_{k\in N}$ is a uniformly bounded sequence, and combining Lemma 2.1 we get

$$\|u_k\|_{L^\infty_{2kT}} \leq C\|u_k\|_{E_k} \leq CM_1. \tag{30}$$

Since u_k is a $2kT$-periodic solution of system (4), it follows that

$$\ddot{u}_k(t) = -\nabla V(t, u_k(t)) + f_k(t) \tag{31}$$

for every $t \in [-kT, kT)$, then we have

$$\begin{aligned} |\ddot{u}_k(t)| &\leq |\nabla V(t, u_k(t))| + |f_k(t)| = |\nabla V(t, u_k(t))| + |f(t)| \\ &\leq |\nabla V(t, u_k(t))| + \sup_{t\in R} |f(t)| \end{aligned}$$

for $k \in N$. By (30) and (V) we conclude that there is a constant $M_2 > 0$ independent of k such that

$$\|\ddot{u}_k\|_{L^\infty_{2kT}} \leq M_2. \tag{32}$$

Finally, from the Mean Value Theorem, for each $k \in N$ and $t \in R$, there is $t_k \in [t-1, t]$ such that

$$\dot{u}_k(t_k) = \int_{t-1}^{t} \dot{u}_k(s)ds = u_k(t) - u_k(t-1),$$

and

$$\dot{u}_k(t) = \int_{t_k}^{t} \ddot{u}_k(s)ds + \dot{u}_k(t_k),$$

hence

$$\begin{aligned} |\dot{u}_k(t)| &= \left| \int_{t_k}^{t} \ddot{u}_k(s)ds + u_k(t) - u_k(t-1) \right| \\ &\leq \int_{t-1}^{t} |\ddot{u}_k(s)|ds + |u_k(t) - u_k(t-1)|. \end{aligned}$$

By (30) and (32), we obtain

$$\begin{aligned}\|\dot{u}_k\|_{L^\infty_{2kT}} &\leq \int_{t-1}^{t} |\ddot{u}_k(s)|ds + |u_k(t) - u_k(t-1)| \\ &\leq M_2 + 2CM_1\end{aligned}$$

for each $k \in N$.

Second, we need to prove that $\{u_k\}_{k\in N}$ and $\{\dot{u}_k\}_{k\in N}$ are equicontinuous. Actually, by (32) we get

$$|\dot{u}_k(t_1) - \dot{u}_k(t_2)| \leq \left|\int_{t_2}^{t_1} \ddot{u}_k(s)ds\right| \leq \int_{t_2}^{t_1} |\ddot{u}_k(s)|ds \leq M_2|t_1 - t_2|$$

for each $k \in N$ and $t_1, t_2 \in R$, which shows $\{\dot{u}_k\}_{k\in N}$ is equicontinuous, and $\{u_k\}_{k\in N}$ remains in the same way. Then there is a subsequence $\{u_{k_j}\}_{k\in N}$ convergent to u_0 in $C^1_{loc}(R, R^N)$ by the Arzelà-Ascoli theorem. ■

Lemma 2.6. *Let $u_0 : R \to R^N$ be a function determined by Lemma 2.5. Then u_0 is a nontrivial homoclinic solution of problem (1).*

Proof The proof will be divided into three steps.

Step 1: we will show that u_0 satisfies (1). By Lemma 2.3 and Lemma 2.5, we have $u_{k_j} \to u_0$ in $C^1_{loc}(R, R^N)$ as $j \to \infty$, and

$$\ddot{u}_{k_j}(t) = -\nabla V(t, u_{k_j}(t)) + f_{k_j}(t)$$

for each $j \in N$ and $t \in [-k_jT, k_jT)$. Take $a, b \in R$ such that $a < b$. There exists $j_0 \in N$ such that for all $j > j_0$ and for every $t \in [a, b]$ we have

$$\ddot{u}_{k_j}(t) = -\nabla V(t, u_{k_j}(t)) + f(t).$$

In consequence, for $j > j_0$, $\ddot{u}_{k_j}(t)$ is continuous in $[a, b]$ and $\ddot{u}_{k_j}(t) \to -\nabla V(t, u_0(t)) + f(t)$ uniformly on $[a, b]$. So it follows that $\ddot{u}_{k_j}$ is a classical derivative of $\dot{u}_{k_j}$ in (a, b) for each $j > j_0$. Moreover, since $\dot{u}_{k_j} \to \dot{u}_0$ uniformly on $[a, b]$, we get

$$-\nabla V(t, u_0(t)) + f(t) = \ddot{u}_0(t)$$

for every $t \in (a, b)$. Since a and b are arbitrary, we conclude that u_0 satisfies (1).

Step 2: We prove that $u_0(t) \to 0$ as $t \to \pm\infty$. For every $l \in N$, there is $j_0 \in N$ such that

$$\int_{-lT}^{lT} \left(|u_{k_j}(t)|^2 + |\dot{u}_{k_j}(t)|^2\right) dt \leq \|u_{k_j}\|^2_{E_{k_j}} \leq M_1^2$$

for all $j > j_0$. From this and Lemma 2.5 it follows that

$$\int_{-lT}^{lT} \left(|u_0(t)|^2 + |\dot{u}_0(t)|^2\right) dt \leq M_1^2$$

for each $l \in N$. Letting $l \to +\infty$, we obtain

$$\int_{-\infty}^{+\infty} \left(|u_0(t)|^2 + |\dot{u}_0(t)|^2\right) dt \leq M_1^2,$$

then

$$\int_{|t|\geq r} \left(|u_0(t)|^2 + |\dot{u}_0(t)|^2\right) dt \to 0 \tag{33}$$

as $r \to +\infty$. Fix $t \in R$, then we have

$$|u_0(t)| \leq |u_0(\omega)| + \left|\int_\omega^t \dot{u}_0(s)ds\right| \tag{34}$$

for each $\omega \in R$. From (34) and Hölder inequality we obtain

$$\begin{aligned}
|u_0(t)| &\leq \int_{t-1}^t \left(|u_0(\omega)| + \left|\int_\omega^t \dot{u}_0(s)ds\right|\right) d\omega \\
&\leq \left(\int_{t-1}^t \left(|u_0(\omega)| + \left|\int_\omega^t \dot{u}_0(s)ds\right|\right)^2 d\omega\right)^{1/2} \\
&\leq \left(2\int_{t-1}^t \left(|u_0(\omega)|^2 + \left|\int_\omega^t \dot{u}_0(s)ds\right|^2\right) d\omega\right)^{1/2} \\
&\leq \sqrt{2}\left(\int_{t-1}^t \left(|u_0(\omega)|^2 + \int_\omega^t |\dot{u}_0(s)|^2 ds\right) d\omega\right)^{1/2} \\
&\leq \sqrt{2}\left(\int_{t-1}^t |u_0(\omega)|^2 d\omega + \int_{t-1}^t\int_{t-1}^t |\dot{u}_0(s)|^2 ds d\omega\right)^{1/2} \\
&\leq \sqrt{2}\left(\int_{t-1}^t \left(|u_0(s)|^2 + |\dot{u}_0(s)|^2\right) ds\right)^{1/2},
\end{aligned} \tag{35}$$

then by (33), we obtain $u_0(t) \to 0$ as $t \to \pm\infty$.

Step 3: We now show that $\dot{u}_0(t) \to 0$ as $t \to \pm\infty$. Similar to (35) we obtain

$$|\dot{u}_0(t)|^2 \leq 2\int_{t-1}^t \left(|\dot{u}_0(s)|^2 + |\ddot{u}_0(s)|^2\right) ds \tag{36}$$

for each $t \in R$. From (33), one has

$$\int_{t-1}^t |\dot{u}_0(s)|^2 ds \to 0 \tag{37}$$

as $t \to \pm\infty$. And since u_0 is a solution of problem (1), we have

$$\int_{t-1}^t |\ddot{u}_0(s)|^2 ds = \int_{t-1}^t \left(|\nabla V(s, u_0(s))|^2 + |f(s)|^2\right) ds - 2\int_{t-1}^t (\nabla V(s, u_0(s)), f(s))ds.$$

From (K_1) and (W_2'), we can conclude that $\nabla K(s,0) = 0$ and $\nabla W(s,0) = 0$, which yield $\nabla V(s,0) = 0$ for all $s \in R$. Since $V(s,x)$ is T-periodic with respect to s, $\nabla V(s,x)$ has the same property. Then for every $s \in [0,T]$ and $\varepsilon > 0$, there is $\rho_s > 0$ such that

$$|\nabla V(w,x)| < \varepsilon$$

for all $w \in B(s;\rho_s) \cap [0,T]$ and $|x| < \rho_s$, which implies $B(s;\rho_s)(s \in [0,T])$ is an open coverage of $[0,T]$. By the compactness of $[0,T]$, we can see that there exist $B(s_1;\rho_{s_1}), B(s_2;\rho_{s_2}), \cdots, B(s_m;\rho_{s_m})$ such that $[0,T] \subset \cup_{i=1}^m B(s_i;\rho_{s_i})$. Let $\rho_0 = \min\{\rho_{s_1}, \rho_{s_2}, \cdots, \rho_{s_m}\}$, then we have

$$|\nabla V(s,x)| < \varepsilon$$

for all $|x| < \rho_0$ and uniformly in $s \in [0,T]$. Since $u_0(s) \to 0$ as $s \to \pm\infty$, there is $p > 0$ such that $|u_0(s)| < \rho_0$ for $|t| \geq p$. Hence, when $|t| \geq p+1$,

$$\int_{t-1}^{t} |\nabla V(s, u_0(s))|^2 ds < \varepsilon^2.$$

Noting that $\int_{t-1}^{t} |f(s)|^2 ds \to 0$ as $t \to \pm\infty$, we have

$$\int_{t-1}^{t} |\ddot{u}_0(s)|^2 ds \to 0, \tag{38}$$

then we obtain our conclusion.

Since $\nabla V(t,0) = 0$, then $u = 0$ is not a solution of problem (1) for $f \neq 0$, which shows $u_0 \neq 0$. ∎

From Lemma 2.3 - Lemma 2.6, we complete the proof of Theorem 1.1. Finally, we will prove Theorem 1.2.

Proof of Theorem 1.2. Under conditions of Theorem 1.2, the conclusions of Lemma 2.1 - Lemma 2.4 for the system (1) are still true, which means there is a $2kT$-periodic solution $u_k \in E_k$ satisfies

$$\ddot{u}(t) + \nabla V(t, u(t)) = 0 \tag{39}$$

for $k \in N$. Since V is T-periodic with respect to t, we can see $u_k(t+nT)$ is still a $2kT$-periodic solution of (39) for every $n \in Z$. By replacing earlier, if necessary, u_k by $u_k(t+nT)$ for some $n \in Z$, we can assume that the maximum of u_k occurs in $[-T,T]$.

Similar to the proofs of Lemma 2.5 and Lemma 2.6, we choose a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ convergent to a u_0 in $C^1_{loc}(R,R^N)$, u_0 is a homoclinic solution of problem (1). Finally, we have to show that $u_0 \neq 0$. As Rabinowitz in [10], we set

$$\psi(s) = \max_{t\in[0,T],|u|\leq s} \frac{(\nabla W(t,u),u)}{|u|^2}$$

for $s > 0$ and $\psi(0) = 0$. Then it is easy to verify that ψ is continuous, nondecreasing and $\psi(s) \to +\infty$ as $s \to +\infty$. By the definition of ψ, we have

$$\int_{-k_jT}^{k_jT} (\nabla W(t, u_{k_j}(t)), u_{k_j}(t))dt \le \psi(\|u_{k_j}\|_{L^\infty_{2k_jT}})\|u_{k_j}\|^2_{E_{k_j}} \tag{40}$$

for all $j \in N$. Since $I'_{k_j}(u_{k_j})u_{k_j} = 0$, it follows from (7) that

$$\int_{-k_jT}^{k_jT} (\nabla W(t, u_{k_j}(t)), u_{k_j}(t))dt = \int_{-k_jT}^{k_jT} |\dot{u}_{k_j}(t)|^2 dt + \int_{-k_jT}^{k_jT} (\nabla K(t, u_{k_j}(t)), u_{k_j}(t))dt. \tag{41}$$

From (40), (41), (K_1), (K_2'''), Lemma 2.1 and (27), we obtain

$$\begin{aligned}
\psi(\|u_{k_j}\|_{L^\infty_{2k_jT}})\|u_{k_j}\|^2_{E_{k_j}} &\ge \int_{-k_jT}^{k_jT} |\dot{u}_{k_j}(t)|^2 dt + \int_{-k_jT}^{k_jT} (u_{k_j}(t), \nabla K(t, u_{k_j}(t)))dt \\
&\ge \int_{-k_jT}^{k_jT} |\dot{u}_{k_j}(t)|^2 dt + b\rho \int_{-k_jT}^{k_jT} |u_{k_j}(t)|^\gamma dt \\
&\ge \int_{-k_jT}^{k_jT} |\dot{u}_{k_j}(t)|^2 dt + b\rho (C\|u_{k_j}\|_{E_k})^{\gamma-2} \int_{-k_jT}^{k_jT} |u_{k_j}(t)|^2 dt \\
&\ge \int_{-k_jT}^{k_jT} |\dot{u}_{k_j}(t)|^2 dt + b\rho (CM_1)^{\gamma-2} \int_{-k_jT}^{k_jT} |u_{k_j}(t)|^2 dt \\
&\ge C_1 \|u_{k_j}\|^2_{E_{k_j}},
\end{aligned}$$

where $C_1 = \min\{1, b\rho(CM_1)^{\gamma-2}\}$, and hence

$$\psi(\|u_{k_j}\|_{L^\infty_{2k_jT}}) \ge C_1 > 0. \tag{42}$$

By the property of ψ, there is a constant $C_2 > 0$ such that

$$\|u_{k_j}\|_{L^\infty_{2k_jT}} \ge C_2 \tag{43}$$

for each $j \in N$. Consequently we get

$$\max_{t\in[-T,T]} |u_{k_j}(t)| = \|u_{k_j}\|_{L^\infty_{2k_jT}} \ge C_2, \quad j \in N,$$

which implies that

$$\max_{t\in[-T,T]} |u_0(t)| \ge C_2.$$

Hence $u_0 \ne 0$. The proof is completed. ■

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