Rigidity theorem for complete spacelike submanifold in $S_q^{n+p}(1)$ with constant scalar curvature *

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Abstract

In this paper, the complete spacelike submanifold with parallel normalized mean curvature vector and constant normalized scalar curvature is discussed in $(n+p)$-dimensional connected semi-Riemannian manifold $S_q^{n+p}(1)$ $(1 \leq q \leq p)$ and a rigidity theorem is obtained.

1 Introduction

Let L_s^m be an m-dimensional connected semi-Riemannian manifold of index s $(s \geq 0)$; this is called a semi-definite space of index s. In particular, $L_q^{n+p}(c)$ is an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c, of index $q(1 \leq q \leq p)$. It is called an indefinite space form of index q. according to whether $c > 0$, $c = 0$ or $c < 0$, it is denoted by $S_q^{n+p}(c)$, $\mathbb{R}_q^{n+p}(c)$ or $\mathbb{H}_q^{n+p}(c)$. A submanifold immersed in $L_q^{n+p}(c)$ is said to be spacelike if the induced metric in M from that of the ambient space $L_q^{n+p}(c)$ is positive definite. Spacelike submanifolds usually appear in the study of question related to causality in general relativity.

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More precisely, level sets of a function of global time are spacelike submanifolds. Also, spacelike hypersurfaces with constant mean curvature are convenient as initial hypersurfaces for the Cauchy problem in arbitrary space time and for studying the propagation of gravitational radiation. Aiyama [1] had proved that a compact spacelike submanifold with parallel mean curvature vector and flat normal bundle in de Sitter space $S_p^{n+p}(c)$ is totally umbilical. Alias et al. [2] obtained also some rigidity results for spacelike submanifolds with parallel mean curvature vector in pseudo-Riemannian space forms $N_p^{n+p+1}(c)$. Cheng [3] generalized the results obtained in [4] to complete spacelike submanifolds in de Sitter space $S_p^{n+p}(c)$. Li [5] extended Montiel's result in [6] for complete spacelike submanifolds with parallel mean curvature vector with two topological ends. Liu [7], characterized the complete spacelike submanifolds M^n, with parallel mean curvature vector satisfying $H^2 = 4(n-1)c/n^2 (c > 0)$ in de Sitter space $S_p^{n+p}(c)$. He shows that M^n is totally umbilical, or M^n is the hyperbolic cylinder in $S_p^{n+p}(c)$ or M^n has unbounded volume and positive Ricci curvature. Recently, in [8], Baek et al., obtained an optimal estimate of the squared norm of the second fundamental form for complete spacelike hypersurfaces with constant mean curvature in a locally symmetric Lorentz space satisfying some curvature conditions and characterized the totally umbilical hypersurfaces. In particular, semi-Riemannian space forms $N_p^{n+p}(c)$ are examples of locally symmetric semi-Riemannian spaces. In this paper we extend the last result to higher codimensional spacelike submanifolds with parallel mean curvature vector in a semi-Riemannian space form $N_p^{n+p}(c)$. Moreover we extend also to spacelike submanifolds a gap theorem obtained by Brasil et al. in [9] and Chaves and Sousa in [10] for hypersurfaces. In the context of submanifolds, there is a well known result of Ishihara [11] that, for an n-dimensional complete maximal spacelike submanifold M^n immersed in $N_p^{n+p}(c)$, if $c \geq 0$, then M^n is totally geodesic and if $c < 0$, then $0 \leq S \leq -npc$.

In [12], Alias and Romero studied the complete maximal spacelike submanifolds M^n in $S_q^{n+p}(c)$. They prove that if M^n is compact maximal spacelike in $S_q^{n+p}(c)$ with $Ric(M) \geq (n-1)c$, then M^n is totally geodesic. M. Mariano [13] studied the complete spacelike submanifolds M^n with parallel mean curvature and second fundamental form locally timelike in a semi-Riemannian space form $N_q^{n+p}(c)$.

In this paper, we apply Cheng-Yau's technique to complete submanifolds in $S_q^{n+p}(1)$ in order to prove the following results

Theorem 1.1. *Let* $x : M^n \to S_q^{n+p}(1)$ $(n \geq 3, 1 \leq q \leq p)$ *be a substantial isometric immersion of a complete Riemannian manifold. Assume that the normalized mean curvature vector of* M^n *in* $S_q^{n+p}(1)$ *is parallel and constant normalized scalar curvature* R *satisfying* $R \leq 1$. *if the squared norm of the second fundamental form* S *satisfies* $\sup S \leq 2\sqrt{n-1}$, *then either*

(1) $S = nH^2$, M^n *is totally umbilical submanifold and* $S = n(1-R)$; *or*

(2) $\sup S = 2\sqrt{n-1}$ *and* M^n $(n = 2)$ *is totally umbilical submanifold, or* M^n $(n \geq 3)$ *lies in a totally geodesic submanifold* $S_1^{n+1}(1)$ *of* $S_q^{n+p}(1)$ *and* M^n *isometric to a hyperbolic cylinder* $S^{n-1}(1-\tanh^2 r) \times H^1(1-\coth^2 r)$.

Theorem 1.2. *Let M^n be a complete spacelike submanifold in $S_q^{n+p}(1)$ $(1 \le q \le p)$ with parallel normalized mean curvature vector. If $\sup K$ denote the function that assigns to each point of M^n the supremum of the sectional curvature at that point, there exists a constant $\beta(n,q,H)$ such that if $\sup K \le \beta(n,q,H)$, then either*

(1) $n = 2$ *and M^2 is totally umbilical or*

(2) $n \ge 3$ *and M^n is totally geodesic.*

2 Preliminaries

Let $S_q^{n+p}(1)$ be an $(n+p)$-dimensional semi-Riemannian space with index q $(1 \le q \le p)$. Let M^n be an n-dimensional connected spacelike submanifold immersed in $S_q^{n+p}(1)$. We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_{n+p}$ in $S_q^{n+p}(1)$ such that at each point of M^n, $e_1, \cdots, e_n$ span the tangent of M^n and from an orthonormal frame there. We use the following convention on the range of indices:

$$1 \le A, B, C, \cdots, \le n+p; \ \ 1 \le i, j, k, \cdots, \le n; \ \ n+1 \le \alpha, \beta, \gamma, \cdots, \le n+p.$$

Take the correspondent dual coframe $\{\omega_1, \cdots, \omega_{n+p}\}$ such that the semi-Riemannian metric of $S_q^{n+p}(1)$ is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \varepsilon_A \omega_A^2$, we define

$$\varepsilon_A = 1(1 \le A \le n+p-q); \ \ \varepsilon_A = -1(n+p-q+1 \le A \le n+p).$$

then the structure equations for $S_q^{n+p}(1)$ are given by

$$d\omega_A = -\sum_B \varepsilon_B \omega_{BA} \wedge \omega_A, \quad \varepsilon_B \omega_{BA} + \varepsilon_A \omega_{AB} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_C \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D. \tag{2.2}$$

$$K_{ABCD} = \varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Next, we restrict those forms to M^n. First of all we get $\omega_\alpha = 0$, so the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since $0 = d\omega_\alpha = -\sum_j \omega_{\alpha j} \wedge \omega_j$, from Cartan's lemma, we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \ \ h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.4}$$

Set

$$B = \sum_{\alpha,i,j} \varepsilon_\alpha h_{ij}^\alpha \omega_i \omega_j e_\alpha, \ \ h = \frac{1}{n} \sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha \ \ and \ \ H = |h| = \frac{1}{n} \sqrt{\sum_\alpha (\sum_i h_{ii}^\alpha)^2}$$

the second fundamental form, the mean curvature vector and the mean curvature of M^n, respectively.

Using the structure equations we obtain the Gauss equation

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_{\alpha} \varepsilon_\alpha (h^\alpha_{il}h^\alpha_{jk} - h^\alpha_{ik}h^\alpha_{jl}). \tag{2.5}$$

The normalized scalar curvature R is given by

$$n(n-1)R = n(n-1) - n^2H^2 + S, \tag{2.6}$$

where $S = \sum_{\alpha,i,j}(h^\alpha_{ij})^2$ is the squared norm of the second fundamental form of M^n.

We also have the structure equations of the normal bundle of M^n

$$d\omega_\alpha = \sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0, \tag{2.7}$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2}\sum_{i,j} R_{\alpha\beta ij}\omega_i \wedge \omega_j. \tag{2.8}$$

where

$$R_{\alpha\beta ij} = \sum_l (h^\alpha_{il}h^\beta_{lj} - h^\alpha_{jl}h^\beta_{li}). \tag{2.9}$$

Denote the first and the second covariant derivatives of h_{ij} as h_{ijk} and h_{ijkl}; we have

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_k h^\alpha_{ik}\omega_{kj} - \sum_\beta \varepsilon_\alpha\varepsilon_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{2.10}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_m h^\alpha_{mjk}\omega_{ml} + \sum_m h^\alpha_{imk}\omega_{mj} + \sum_m h^\alpha_{ijm}\omega_{mk} - \sum_\beta \varepsilon_\alpha\varepsilon_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{2.11}$$

Then we have the Codazzi equation and Ricci's identity

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik}, \tag{2.12}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta \varepsilon_\alpha\varepsilon_\beta h^\beta_{ij}R_{\alpha\beta kl}. \tag{2.13}$$

Next, we compute the Laplacian $\Delta h^\alpha_{ij} = \sum_k h_{ijkk}$. From (2.11) and (2.13), it follows that

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} + \sum_{m,k} h^\alpha_{km}R_{mijk} + \sum_{m,i} h^\alpha_{km}R_{mkjk} + \sum_{k,\beta} \varepsilon_\alpha\varepsilon_\beta h^\beta_{ik}R_{\alpha\beta jk}. \tag{2.14}$$

Now, suppose that the second fundamental form is locally timelike. Then, we can assume that

$$B' = -\sum_{\alpha=n+p-q+1}^{n+p} \sum_{i,j} h^\alpha_{ij}\omega_i\omega_j e_\alpha, \tag{2.15}$$

and $\sum_{i,j} h_{ij}^{\alpha} = 0$ for all $\alpha \leq n+p-q$. hence,

$$h = \frac{1}{n} \sum_{\alpha=n+p-q+1}^{n+p} (\sum_i h_{ii}^{\alpha}) e_{\alpha}. \tag{2.16}$$

Therefore h is timelike in $S_q^{n+p}(1)$ and

$$S = \sum_{\alpha=n+p-q+1}^{n+p} \sum_{i,j} (h_{ij}^{\alpha})^2. \tag{2.17}$$

Recall that M^n is a submanifold with parallel mean curvature vector h if $\nabla^{\perp} \frac{h}{H} \equiv 0$, where $\nabla^{\perp}$ is the normal connection of M^n in $S_q^{n+p}(1)$. If $H \neq 0$, we choose $e_{n+p-q+1} = h/H$. Thus

$$\sum_k h_{kki}^{\alpha} = 0, \quad H^{\alpha} H^{n+p-q+1} = H^{n+p-q+1} H^{\alpha}. \tag{2.18}$$

$$H^{n+p-q+1} = \frac{1}{n} \mathrm{tr} h^{n+p-q+1} = H \ \ and \ \ H^{\alpha} = \frac{1}{n} \mathrm{tr} h^{\alpha} = 0, \ \ \alpha \neq n+p-q+1, \tag{2.19}$$

where h^{α} denotes the matrix $[h_{ij}^{\alpha}]$. Let us define

$$\Phi_{ij}^{n+p-q+1} = h_{ij}^{n+p-q+1} - H\delta_{ij}, \ \ \Phi_{ij}^{\alpha} = h_{ij}^{\alpha}, \ \ \alpha \neq n+p-q+1. \tag{2.20}$$

Therefore

$$\Phi^{n+p-q+1} = H^{n+p-q+1} - HI, \ \ \Phi^{\alpha} = H^{\alpha}, \ \ \alpha \neq n+p-q+1, \tag{2.21}$$

where Φ^{α} denotes the matrix (Φ_{ij}^{α}). Then

$$|\mu|^2 = |\Phi^{n+p-q+1}|^2 = \mathrm{tr}(H^{n+p-q+1})^2 - nH^2, \tag{2.22}$$

$$|\tau|^2 = \sum_{\alpha \neq n+p-q+1} |\Phi^{\alpha}|^2 = \sum_{\beta \neq n+p-q+1} (h_{ij}^{\beta})^2, \tag{2.23}$$

and

$$\mathrm{tr}(\Phi^{\alpha}) = 0, \ \ \forall \alpha. \tag{2.24}$$

Thus,

$$S = \sum_{\alpha=n+p-q+1}^{n+p} |\Phi^{\alpha}|^2. \tag{2.25}$$

By (2.22), (2.23), and (2.25), we get

$$S = |\Phi|^2 + nH^2 = |\mu|^2 + |\tau|^2 + nH^2, \tag{2.26}$$

and so

$$\Delta S = \Delta(\mathrm{tr}(H^{n+p-q+1})^2) + \Delta(|\tau|^2). \tag{2.27}$$

We will need the following lemma.

Lemma 2.1. [14] *Let $\mu_1, \cdots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = B$, with B is constant, then*

$$|\sum_i \mu_i^3| \le \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}},$$

and equality holds if and only if

$$\mu_1 = \cdots = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}B},\ \mu_n = \sqrt{\frac{n-1}{n}B}.$$

Lemma 2.2. *Let M^n be a spacelike submanifold in $S_q^{n+p}(1)$ $(1 \le q \le p)$. Suppose that the normalized scalar curvature R is constant and $R \le 1$. Then*

$$\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \ge n^2 |\nabla H|^2$$

and the symmetric tensor T defined by (2.35) *is positive semi-definite. Moreover,*

i) when $R < 1$, if the equality holds on M, then H is constant and T is positive definite;

ii) when $R = 1$, if the equality occurs on M, then either H is constant or M lies in a totally geodesic subspace S_1^{n+1} of S_p^{n+p} and, in the former case, the matrix h^{n+1} has rank 1.

Lemma 2.3. [15] *Let $a_1, \cdots, a_n; b_1, \cdots, b_n (n \ge 2)$ be real numbers satisfying $\sum_i b_i = 0$. Then*

$$\sum_{i,j} a_i a_j (b_i - b_j)^2 \ge -\frac{n}{\sqrt{n-1}} (\sum_i a_i^2)(\sum_i b_i^2).$$

By substituting (2.5) and (2.9) in (2.14), we obtain

$$\begin{aligned}
\frac{1}{2}\Delta(\mathrm{tr}(H^{n+p-q+1})^2) &= \sum_{i,j,k} (h_{ijk}^{n+p-q+1})^2 + \sum_{i,j} h_{ij}^{n+p-q+1} \Delta h_{ij}^{n+p-q+1} \\
&= \sum_{i,j,k} (h_{ijk}^{n+p-q+1})^2 + \sum_{i,j} h_{ij}^{n+p-q+1} (nH)_{ij} + n \mathrm{tr} H_{n+p-q+1}^2 \\
&\quad - n^2 H^2 - nH \mathrm{tr} H_{n+p-q+1}^3 + [\mathrm{tr} H_{n+p-q+1}^2]^2 \\
&\quad + \sum_{\beta > n+p-q+1} [\mathrm{tr} H_{n+p-q+1} H_\beta]^2, \qquad (2.28)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\Delta|\tau|^2 &= \sum_{i,j,k,\alpha > n+p-q+1} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha > n+p-q+1} h_{ij}^\alpha \Delta h_{ij}^\alpha \\
&= \sum_{i,j,k,\alpha > n+p-q+1} (h_{ijk}^\alpha)^2 + n|\tau|^2 - nH \sum_{\alpha > n+p-q+1} \mathrm{tr}(H_\alpha^2 H_{n+p-q+1}) \\
&\quad + \sum_{\alpha > n+p-q+1} [\mathrm{tr}(H_{n+p-q+1} H_\alpha)]^2 + \sum_{\alpha,\beta > n+p-q+1} [\mathrm{tr}(H_\alpha H_\beta)]^2. \quad (2.29)
\end{aligned}$$

It is easy to check that $\mathrm{tr}(\mu) = \mathrm{tr}(\Phi^{n+p-q+1}) = 0$. By using Lemma 2.1, we obtain

$$\begin{aligned} -nH\mathrm{tr}H_{n+p-q+1}^3 &\geq -\frac{n(n-2)}{\sqrt{n(n-1)}}H|\mu|^3 - 3nH^2|\mu|^2 - n^2H^4 \\ &\geq -\frac{n-2}{2\sqrt{n-1}}|\mu|^2(anH^2 + \frac{1}{a}|\mu|^2) - 3nH^2|\mu|^2 - n^2H^4, \end{aligned} \tag{2.30}$$

where a is any positive number.

Letting $a = (n-2\sqrt{n-1})/(n-2)$, and substituting (2.30) into (2.28), we obtain

$$\begin{aligned} \frac{1}{2}\Delta(\mathrm{tr}(H^{n+p-q+1})^2) &\geq \sum_{i,j,k}(h_{ijk}^{n+p-q+1})^2 + \sum_{i,j} h_{ij}^{n+p-q+1}(nH)_{ij} \\ &\quad +|\mu|^2\{n - \frac{n}{2\sqrt{n-1}}S\}. \end{aligned} \tag{2.31}$$

For a given $\alpha > n+p-q+1$, we may choose $\{e_1, e_2 \cdots, e_n\}$, such that $h_{ij}^{\alpha} = h_{ii}^{\alpha}\delta_{ij}$, $h_{ij}^{n+p-q+1} = h_{ii}^{n+p-q+1}\delta_{ij}$. Then

$$-nH\mathrm{tr}(H_\alpha^2 H_{n+p-q+1}) + [\mathrm{tr}(H_{n+p-q+1}H_\alpha)]^2 = \frac{1}{2}\sum_{i,j} h_{ii}^{n+p-q+1}h_{jj}^{n+p-q+1}(h_{ii}^{\alpha} - h_{jj}^{\alpha})^2,$$

Since $\sum_i h_{ii}^{\alpha} = n\mathrm{tr}H^{\alpha} = 0$, we can use Lemma 2.3 and obtain

$$\begin{aligned} &-nH\sum_{\alpha>n+p-q+1}\mathrm{tr}(H_\alpha^2 H_{n+p-q+1}) + \sum_{\alpha>n+p-q+1}[\mathrm{tr}(H_{n+p-q+1}H_\alpha)]^2 \\ &\qquad \geq -\frac{n}{2\sqrt{n-1}}|\tau|^2 S. \end{aligned} \tag{2.32}$$

Substituting (2.32) into (2.29), we obtain

$$\frac{1}{2}\Delta|\tau|^2 \geq \sum_{i,j,k,\alpha>n+p-q+1}(h_{ijk}^{\alpha})^2 + |\tau|^2\{n - \frac{n}{2\sqrt{n-1}}S\}. \tag{2.33}$$

From (2.27), (2.31) and (2.33), we obtain

$$\frac{1}{2}\Delta S \geq \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{i,j} h_{ij}^{n+p-q+1}(nH)_{ij} + (S - nH^2)\{n - \frac{n}{2\sqrt{n-1}}S\}. \tag{2.34}$$

Let $T = \Sigma_{i,j}T_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = nH\delta_{ij} - h_{ij}^{n+p-q+1}. \tag{2.35}$$

According to Cheng-Yau [16], we introduce the operator $\Box$ associated to T acting on any C^2-function f by

$$\Box f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+p-q+1})f_{ij}.$$

Choosing $f = H$ in above expression, we have

$$\begin{aligned}\Box(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+p-q+1})(nH)_{ij} \\ &= \frac{1}{2}\Delta(n^2H^2) - |\mathrm{grad}(nH)|^2 - \sum_{i,j} h_{ij}^{n+p-q+1}(nH)_{ij}. \end{aligned} \tag{2.36}$$

From (2.34) to (2.36) and Lemma 2.2, we obtain the next Lemma has an essential role in the proofs of our results.

Lemma 2.4. *Let M^n be a complete spacelike submanifold in $S_q^{n+p}(1)$, $(1 \leq q \leq p)$ with parallel normalized mean curvature vector and constant normalized scalar curvature R, $R \leq 1$. Then the following inequality holds*

$$\Box(nH) \geq |\Phi|^2\left(n - \frac{n}{2\sqrt{n-1}}S\right). \tag{2.37}$$

The following Lemma appeared in [17], for $p = 1$. Like in the proof of Proposition 2.2 in [18], we have

Lemma 2.5. *Let M be a complete spacelike submanifold in $S_q^{n+p}(1)$ $(1 \leq q \leq p)$ with constant normalized scalar curvature R, $R \leq 1$. If the mean curvature H of M^n is bounded, then there is a sequence of points $\{p_k\} \in M^n$ such that $\lim_{k\to\infty} nH(p_k) = n\sup H$, $\lim_{k\to\infty} |\nabla nH(p_k)| = 0$ and $\limsup_{k\to\infty}(\Box(nH)(p_k)) \leq 0$.*

We recall the following indefinite version of a lemma due to Erbacher [19].

Lemma 2.6. *Let ψ: $M_s^n \to Q_t^{n+p}(c)$ be an isometric immersion of a connected indefinite Riemannian manifold into a space form. If there exists a k-dimensional parallel normal subbundle $L(p)$ which contains the first normal space $N_1(p)$ for all $p \in M_s^n$, then there exists a $(n+p+k)$-dimensional totally geodesic submanifold Q^{n+p-k} of $Q_t^{n+p}(c)$ such that $\psi(M_s^n) \subset Q^{n+p-k}$, i.e., ψ admits a reduction of codimension to k.*

Lemma 2.7. [20][21] *Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bound from below. Let F be a C^2-function bounded from below on M^n. Then there is a sequence of points $\{p_k\}$ in M^n, such that*

$$\lim_{k\to\infty} |\nabla F(p_k)| = 0, \quad \limsup_{k\to\infty} \Delta F(p_k) \geq 0, \quad \lim_{k\to\infty} F(p_k) = \inf F.$$

3 Proof of the theorem

Proof of Theorem 1.1. The following relations may be readily from the (2.6) and (2.25)

$$H^2 = \frac{S - n(n-1)(R-1)}{n^2}, \tag{3.1}$$

$$|\Phi|^2 = \frac{(n-1)S + n(n-1)(R-1)}{n}. \tag{3.2}$$

$$|\Phi|^2 = n(n-1)(R-1+H^2). \tag{3.3}$$

By our assumption $\sup S \le 2\sqrt{n-1}$ and (3.1), we can know H is bounded.

By our assumptions, from Lemma 2.5, there is a sequence of points p_k in M^n, such that

$$\lim_{k\to\infty}(nH(p_k)) = n\sup H, \quad \limsup_{k\to\infty}(\Box(nH)(p_k)) \le 0. \tag{3.4}$$

As R is constant, it is clear from (3.1) and (3.3) that $\lim_{k\to\infty}(S(p_k)) = \sup S$ and $\lim_{k\to\infty}(\Phi(p_k)) = \sup\Phi$.

From Lemma 2.4 and (3.4), we obtain

$$\sup|\Phi|^2(n - \frac{n}{2\sqrt{n-1}}\sup S) = \limsup_{k\to\infty}(\Box(nH)(p_k)) = 0. \tag{3.5}$$

Then $\sup|\Phi|^2 = 0$ or $\sup S = 2\sqrt{n-1}$.

(1) If $\sup|\Phi|^2 = 0$, then $S = nH^2$, M^n is totally umbilical submanifold and $S = n(1-R)$.

(2) If $\sup S = 2\sqrt{n-1}$, we get H is constant.

(i) If $n = 2$, according to Cheng [3] had proved the theorem, we can know M^n is totally umbilical submanifold.

(ii) If $n \ge 3$ and $\sup S = 2\sqrt{n-1}$, then all the estimates employed to derive this inequality are, actually, equalities and keeping in mind Theorem 1 [22] and Lemma 2.6, we can obtain M^n lies in a totally geodesic submanifold $S_1^{n+1}(1)$ of $S_q^{n+p}(1)$. From the equality in lemma 2.1, we know M^n isometric to a hyperbolic cylinder $S^{n-1}(1-\tanh^2 r)\times H^1(1-\coth^2 r)$ in $S_1^{n+1}(1)$. ∎

Proof of Theorem 1.2. By (2.9), (2.14), (2.18) and (2.19), we obtain

$$\sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha}h_{ki}^{\beta}R_{\alpha\beta jk} = \frac{1}{2}\sum_{\alpha,\beta} N(H^\alpha H^\beta - H^\beta H^\alpha)$$

and

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + n\sum_{\alpha,i,j}h_{ij}^{\alpha}H_{ij}^{\alpha} + \frac{1}{2}\sum_{\alpha,\beta}N(H^\alpha H^\beta - H^\beta H^\alpha)\\ &\quad + \sum_{\alpha,i,j,m,k}h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + \sum_{\alpha,i,j,m,k}h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk},\end{aligned} \tag{3.6}$$

where $N(A) = \mathrm{tr}(AA^t)$, for all matrix $A = (a_{ij})$.

Next, we will obtain a pointiest estimate for the last two terms. For each fixed α, let λ_i^α be an eigenvalue of h^α and denotes by $\sup K$ the supremum of the sectional curvature at a point p of M^n. Then

$$\begin{aligned}2\Bigg(\sum_{i,j,m,k}h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + \sum_{i,j,k,m}h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}\Bigg) &= \sum_{i,k}(\lambda_i^\alpha - \lambda_k^\alpha)^2 R_{ikik}\\ &\le (\sup K)\sum_{i,k}(\lambda_i^\alpha - \lambda_k^\alpha)^2\\ &= 2n(\sup K)N(\Phi^\alpha).\end{aligned} \tag{3.7}$$

Therefore,

$$\sum_{\alpha,i,j,m,k} h^{\alpha}_{ij}h^{\alpha}_{km}R_{mijk} + \sum_{\alpha,i,j,m,k} h^{\alpha}_{ij}h^{\alpha}_{mi}R_{mkjk} \le n(\sup K)\sum_{\alpha} N(\Phi^{\alpha}) = n(\sup K)|\Phi|^2. \quad (3.8)$$

On the other hand, since

$$\sum_{\alpha\ge n+p-q+1} \mathrm{tr}(H^{n+p-q+1}(H^{\alpha})^2) = \sum_{\alpha\ge n+p-q+1} \mathrm{tr}(\Phi^{n+p-q+1}(\Phi^{\alpha})^2) + H|\Phi|^2 + 2H\mathrm{tr}(\Phi^{n+p-q+1})^2 + nH^2, \quad (3.9)$$

and

$$\sum_{\alpha,\beta=n+p-q+1} [\mathrm{tr}(H^{\alpha}H^{\beta})]^2 = \sum_{\alpha,\beta=n+p-q+1} [\mathrm{tr}(\Phi^{\alpha}\Phi^{\beta})]^2 + 2nH^2\mathrm{tr}(\Phi^{n+p-q+1})^2 + n^2H^4.$$

By applying Lemma 2.3 to Φ^{α} and $\Phi^{n+p-q+1}$, we obtain

$$\begin{aligned} |\mathrm{tr}(\Phi^{n+p-q+1}(\Phi^{\alpha})^2)| &\le \frac{n-2}{\sqrt{n(n-1)}}|\Phi^{n+p-q+1}||\Phi^{\alpha}|^2 \\ &\le \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3. \end{aligned} \quad (3.10)$$

Using the Cauchy-Schwartz inequality, it is easy to prove that

$$|\Phi|^4 \le q\sum_{\alpha=n+p-q+1}^{n+p} (N(\Phi^{\alpha}))^2 \le q\sum_{\alpha=n+p-q+1}^{n+p} [\mathrm{tr}(\Phi^{\alpha}\Phi^{\beta})]^2. \quad (3.11)$$

So, we obtain

$$\begin{aligned} &\sum_{\alpha,i,j,m,k} h^{\alpha}_{ij}h^{\alpha}_{km}R_{mijk} + \sum_{\alpha,i,j,m,k} h^{\alpha}_{ij}h^{\alpha}_{mi}R_{mkjk} \\ &\quad = n|\Phi|^2 - nH\sum_{\alpha}\mathrm{tr}(H^{n+p-q+1}(H^{\alpha})^2) + \sum_{\alpha,\beta}[\mathrm{tr}(H^{\alpha}H^{\beta})]^2 \\ &\qquad + \frac{1}{2}\sum_{\alpha,\beta} N(H^{\alpha}H^{\beta} - H^{\beta}H^{\alpha}) \\ &\quad \ge |\Phi|^2\left(\frac{|\Phi|^2}{q} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi| + n - nH^2\right). \end{aligned} \quad (3.12)$$

For technical reason, we will write the expression (3.6) for the Laplacian of S as

$$\begin{aligned} \frac{1}{2}\Delta S \ \ge\ & (1-a)\left(\sum_{\alpha,i,j,m,k} h^{\alpha}_{ij}h^{\alpha}_{km}R_{mijk} + \sum_{\alpha,i,j,m,k} h^{\alpha}_{ij}h^{\alpha}_{mi}R_{mkjk}\right) \\ & + a\left(\sum_{\alpha,i,j,m,k} h^{\alpha}_{ij}h^{\alpha}_{km}R_{mijk} + \sum_{\alpha,i,j,m,k} h^{\alpha}_{ij}h^{\alpha}_{mi}R_{mkjk}\right) \end{aligned} \quad (3.13)$$

From (3.8), (3.11), (3.12) and (2.25), if $a \geq 1$, we obtain

$$\frac{1}{2}\Delta|\Phi|^2 = \frac{1}{2}\Delta S \geq a|\Phi|^2\Bigg(\frac{|\Phi|^2}{q} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi| + n\Big[1 - H^2 + (\frac{1-a}{a})\sup K\Big]\Bigg). \quad (3.14)$$

Let $\lambda_i^{n+p-q+1}$ be an eigenvalue of $h^{n+p-q+1}$. we know

$$\begin{aligned} Ric(e_i) &= (n-1) - nHh_{ii}^{n+p-q+1} + \sum_k (h_{ik}^{n+p-q+1})^2 \\ &= \left(\lambda_i^{n+p-q+1} - \frac{nH}{2}\right)^2 + (n-1) - \frac{n^2H^2}{4} \\ &\geq (n-1) - \frac{n^2H^2}{4}. \end{aligned}$$

So, we know *Ricci* curvature of M^n is bounded from blow. By Lemma 2.7, Thus we may apply *Omori* and *Yau*'s result [21] to the function $F = \frac{1}{\sqrt{1+|\Phi|^2}}$, which is a positive smooth function on M^n. Like in the proof the theorem [23], then

$$0 \geq \sup|\Phi|^2\Bigg(\frac{\sup|\Phi|^2}{q} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sup|\Phi| + n\Big[1 - H^2 + (\frac{1-a}{a})\sup K\Big]\Bigg). \quad (3.15)$$

Let

$$\beta(n,q,H) = \frac{a}{4(a-1)(n-1)}(4(n-1) - [q(n-2)^2 + 4(n-1)]H^2).$$

If $\sup K \leq \beta(n,q,H)$, it can be easily checked that

$$\sup|\Phi|^2\Bigg(\frac{\sup|\Phi|^2}{q} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sup|\Phi| + n\Big[1 - H^2 + (\frac{1-a}{a})\sup K\Big]\Bigg) \geq 0. \quad (3.16)$$

Moreover, the equality holds if and only if $\sup K = \beta(n,q,H)$ and $\sup|\Phi| = nq(n-2)/(2\sqrt{n(n-1)})$. Thus, if $\sup K < \beta(n,q,H)$, from (3.14) and (3.15), we conclude that $\sup|\Phi| = 0$ and M^n is totally umbilical.

If $\sup K = \beta(n,q,H)$, we will suppose that M^n is not totally umbilical and derive a contradiction. First, let us prove that $q = 1$. all the estimates employed to derive this inequality are, actually, equalities. From (3.10) and (3.11), we deduce that

$$\limsup_{k\to\infty}(N(\Phi^{n+p-q+1}(p_k))) = \limsup_{k\to\infty}(|\Phi|^2(p_k)) = \sup|\Phi|^2. \quad (3.17)$$

$$\sup|\Phi|^4 = q\sum_{\alpha=n+p-q+1}^{n+p} \limsup_{k\to\infty}(N(\Phi^\alpha))^2(p_k) = \\ q\sum_{\alpha=n+p-q+1}^{n+p} (\limsup_{k\to\infty} N(\Phi^\alpha)(p_k))^2. \quad (3.18)$$

From (2.23), (2.25), (3.16) and (3.17), we have $\sup|\Phi|^4 = q\sup|\Phi|^4$ and which implies $q = 1$.

Next, let us prove that $\sup K = 0$. Since h is parallel and the equality holds in (3.6), (3.9) and (3.12), we can get

$$0 = \limsup_{k\to\infty}\frac{1}{2}\Delta|\Phi|^2(p_k) = n(\sup K)\sup|\Phi|^2 = n(\sup K)(\sup|\Phi|)^2.$$

Therefore, $\sup K = 0$.

Now, we are in position to prove that M^n is totally umbilical. Observe that $\sup K = 0$ and $q = 1$ yield

$$0 = \sup K = \beta(n,q,H) = \frac{a}{4(a-1)(n-1)}(4(n-1) - n^2H^2).$$

Hence $n^2H^2 = 4(n-1)$, according to Montiel [6], either M^n is a totally umbilical hypersurface or $n > 2$ and the supremum of the scalar curvature of M^n is equal to $(n-2)^2$.

Because M^n is not totally umbilical, we conclude that the supremum of the scalar curvature of M^n is equal to $(n-2)^2$, which contradicts the fact that $\sup K = 0$. Therefore, M^n is totally umbilical.

AS a is arbitrary, taking the limit for $a \to \infty$ in

$$\sup K \le \beta(n,q,H) = \frac{a}{4(a-1)(n-1)}(4(n-1) - [q(n-2)^2 + 4(n-1)]H^2),$$

we obtain

$$\sup K \le \beta(n,q,H) = \frac{1}{4(n-1)}(4(n-1) - [q(n-2)^2 + 4(n-1)]H^2).$$

Moreover, since M^n is totally umbilical, if $n \ge 3$, we have

$$1 - H^2 = \sup K \le \frac{1}{4(n-1)}(4(n-1) - [q(n-2)^2 + 4(n-1)]H^2),$$

thus $q(n-2)H^2 \le 0$, which implies $H = 0$ and shows that M^n is totally geodesic.

So the proof is concluded. ■

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References

[1] A.J. Goddard, Some remarks on the existence of spacelike hypersurfaces of constant mean curvature, Math. Proc. Cambridge Phil. Soc. 82 (1977) 489-495.

[2] L.J. Alias, F.J.M. Estudillo, A. Romero, Spacelike submanifolds with parallel mean curvature in pseudo- Riemannian space forms, Tsukuba J. Math. 21 (1997) 169-179.

[3] Q.M. Cheng, Complete spacelike submanifolds in a De Sitter space with parallel mean curvature vector, Math. Z. 206 (1991) 333-339.

[4] K. Akutagawa, On spacelike hypersurfaces with constant mean curvature in the de Sitter space, Math. Z. 196 (1987) 13-19.

[5] H. Li, Complete spacelike submanifolds in De Sitter space with parallel mean curvature vector satisfying $H^2 = 4(n-1)/n^2$, Ann. Global Anal. Geom. 15 (1997) 335-345.

[6] S. Montiel, A characterization of hyperbolic cylinders in the De Sitter space, Tohoku Math. J. 48 (1996) 23-31.

[7] X. Liu, Complete spacelike submanifolds in De Sitter space, Acta Math. Univ. Comenianae. (2001) 241-249.

[8] J.O. Baek, Q.M. Cheng, Y.J. Suh, Complete spacelike hypersurfaces in locally symmetric Lorentz space, J. Geom. Phys. 49 (2004) 231-247.

[9] A. Brasil, A.G. Colares, O. Palmas, Complete spacelike hypersurfaces with constant mean curvature in the De Sitter space: a gap theorem, Illinois J. Math. 47 (2003) 847-866.

[10] R.M.B. Chaves and L.A.M. Sousa, Jr., Some application of a Simon's type formula for complete spacelike submanifolds in a semi-Riemannian space form, Diff. Geom. Appl. 25 (2007) 419-432.

[11] T. Ishihara, Maximal space-like submanifolds of a pseudo-Riemannian space of constant curvature, Michigan Math. J. 35 (1988) 345-352.

[12] L.J. Alias, A. Romero, Integral formulas for compact spacelike n-submanifolds in De Sitter spaces. Applications to the parallel mean curvature vector case. Manuscripta math. 87 (1995) 405-416.

[13] M. Mariano, On complete spacelike submanifolds with second fundamental form local timelike in a semi-Riemannian space form, J. Geom. Phys. 60 (2010) 720-728.

[14] M. Okumura. Hypersurfaces and a pinching problem on the second fundamental tensor. Am.J.math. 96(1974)207-213.

[15] J. Zhang, Spacelike submanifold in the de Sitter space S_p^{n+p} with constant scalar curvature. Appl. Math. J. Chinese Univ. Ser. B, 20 (2005) 183-196.

[16] S.Y. Cheng, S.T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces, Ann. of Math. 104 (1976) 407-419.

[17] F.E.C. Camargo, et al. Rigidity theorems for complete spacelike hypersurfaces with constant scalar curvature in De Sitter space, Diff. Geom. Appl. 26 (2008) 592-599.

[18] F.E.C. Camargo, et al. New characterizations of complete spacelike submanifolds in semi-Riemannian space forms, Kodai Math. J. 32 (2009) 209-230.

[19] J. Erbacher, Reduction of the codimension of an isometric immersion, J. Diff. Geom. 5 (1971) 333-340.

[20] O.Omori, Isometric immersions of Riemannian manifolds, J.Math.Sci.Japan,19 (1967) 205-214.

[21] S.T. Yau. Harmonic functions on complete Riemannian manifolds. Commun.Pure Appl.Math. 28 (1975) 201-228.

[22] S. T. Yau, submanifolds with parallel mean curvature I, Amer. J. math. 2 (1974) 346-366.

[23] S. Zhang and B. Wu, Rigidity theorems for complete spacelike hypersurfaces with constant scalar curvature in locally symmetric Lorentz spaces, J. Geom. Phys. 60 (2010) 333-340.

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