Double density by moduli and statistical convergence

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Abstract

By using unbounded modulus functions we introduce a new concept of density for sets of pairs of natural numbers. Consequently, we obtain a generalization of the notion of statistical convergence of double sequences which is studied and characterized. As an application, we prove that 'Pringsheim convergence' is equivalent to 'module statistical convergence for every unbounded modulus function'.

1 Introduction

The concept of statistical convergence was first defined by Steinhaus ([16]) and also independently by Fast ([4]). In [8], Kolk begins to study its applications to Banach spaces. In [3] the authors find a remarkable connection of statistical convergence with some classical properties; concretely, Banach spaces with separable duals are characterized, in a way which cannot be reproduced with usual convergence. In [2], the weakly unconditionally Cauchy series are also characterized by means of statistical convergence. In [6] and [10], certain summability matrices are used to characterize the statistical convergence of simple and double sequences, respectively. Other works studying this convergence are [5], [7] and [11].

Let $A \subset \mathbb{N} = \{1, 2, \ldots\}$. We denote by $|A|$ the cardinality of A and if $n \in \mathbb{N}$ we denote $A(n) = \{i \in A : i \leq n\}$. The density of A is defined by

$$d(A) = \lim_n \frac{|A(n)|}{n},$$

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in case this limit exists.

A sequence $(x_n)_n$ in a normed space X is said to be statistically convergent to some $x \in X$, and we write $\operatorname{stlim}_k x_k = x$, if for each $\varepsilon > 0$ we have $d(\{i \in \mathbb{N} : \|x_i - x\| > \varepsilon\}) = 0$. Analogously, $(x_n)_n$ is said to be statistically Cauchy if for each $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists an integer $m \geq n$ such that $d(\{i \in \mathbb{N} : \|x_i - x_m\| < \varepsilon\}) = 1$.

Fast ([4]) proved that $\operatorname{stlim}_n x_n = x$ if and only if there exists $A \subseteq \mathbb{N}$ with $d(A) = 1$ and $\lim_{n \in A} x_n = x$. Fridy ([5]) proved that in a Banach space, a sequence is statistically convergent if and only if it is statistically Cauchy.

Moricz ([11]), working with double sequences, proved the analogous versions to Fast's and Fridy's results. Namely, if $A \subseteq \mathbb{N}^2$ we say that the density of A, in case it exists, is the double limit $d_2(A) = \lim_{p,q} \frac{|A(p,q)|}{pq}$, where $A(p,q) = \{(i,j) \in A : i \leq p, j \leq q\}$. The double sequence $(x_{ij})_{ij}$ is said to be statistically convergent to x_0 if for each $\varepsilon > 0$ we have $d_2(\{(i,j) : \|x_{ij} - x_0\| < \varepsilon\}) = 1$, or equivalently $d_2(\{(i,j) : \|x_{ij} - x_0\| > \varepsilon\}) = 0$. Statistically Cauchy double sequences are defined similarly. Consequently, Moricz proved that $(x_{ij})_{ij}$ is statistically convergent if and only if there exists $A \subset \mathbb{N}^2$ with $d_2(A) = 1$ and such that $(x_{ij})_{(i,j) \in A}$ is convergent to x_0 in Pringsheim's sense. He also proved that the concepts 'statistically Cauchy' and 'statistically convergent' are the same in Banach spaces.

We recall that $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called a modulus function if it satisfies:

1. $f(x) = 0$ if and only if $x = 0$.
2. $f(x + y) \leq f(x) + f(y)$ for every $x, y \in \mathbb{R}^+$
3. f is increasing.
4. f is continuous from the right at 0.

From these properties it is clear that a modulus function must be continuous on $\mathbb{R}^+$. Examples of moduli are $f(x) = \frac{x}{1+x}$ and $f(x) = x^p$ with $0 < p \leq 1$.

The notion of a modulus function was introduced by Nakano ([12]). This notion was used by Ruckle ([15]) and Maddox ([9]) to introduce and discuss some properties of sequence spaces and by Pehlivan ([13]) to generalize the strong almost convergence.

In [1], a new concept of density of a subset A of $\mathbb{N}$ is defined by means of an unbounded modulus function f, as $d_f(A) = \lim_n \frac{f(|A(n)|)}{f(n)}$, if this limit exists, and we will say that a sequence $(x_i)_i$ is f-statistically convergent to x if for every $\varepsilon > 0$ we have $d_f(\{i \in A : \|x_i - x\| > \varepsilon\}) = 0$. The concept of f-statistically Cauchy sequence is also defined and it is proved that if X is a complete space and $(x_i)_i$ is an f-statistically Cauchy sequence, then $(x_i)_i$ is f-statistically convergent. Furthermore, in [1] it is also proved that:

1. $(x_i)_i$ is f-statistically convergent to x if and only if there exists $A \subseteq \mathbb{N}$ with $d_f(A) = 0$ and $\lim_{i \in \mathbb{N} \setminus A} x_i = x$.
2. If $f - \operatorname{stlim}_i x_i = x$ for every unbounded modulus f then $\lim_i x_i = x$.

In this paper we define and study the f-statistical convergence of double sequences for unbounded moduli f and we obtain results similar to those of Moricz in [11] with the statistical convergence of double sequences. Let us remark that this generalization is completely nontrivial, since the apparently natural generalization of double density lacks even a basic property which will be stated. We also obtain a characterization of Pringsheim's convergence inspired by the results of [1]. Thus, every double limit we use will be considered in Pringsheim's sense unless otherwise stated.

2 Basic results and examples

Let f be an unbounded modulus function. The f**-density** of a set $A \subseteq \mathbb{N}^2$ is defined by

$$d_{2,f}(A) = \lim_{i,j} \lim_{p,q} \frac{f(|A(p,q,i,j)|)}{f(pq)}$$

in case the outer limit exists, where $A(p,q,i,j) = \{(a,b) \in A : i \leq a \leq p, j \leq b \leq q\}$. Note that if the modulus function is $f(x) = x$ then clearly $d_{2,f}(A) = d_2(A)$. Although this definition might seem unnatural at first, in example 2.1 we will show that the simpler expression $\lim_{p,q} \frac{f(|A(p,q)|)}{f(pq)}$ is not an appropriate density, since it is natural to expect that f-statistical convergence is implied by usual convergence.

Let X be a normed space and $(x_{ij})_{ij}$ a double sequence in X. We will say that $(x_{ij})_{ij}$ is f**-statistically convergent** to x, and we will write $f - \mathrm{stlim}_{ij}\, x_{ij} = x$ if for every $\varepsilon > 0$ we have that $d_{2,f}(\{(i,j) \in \mathbb{N}^2 : \|x_{ij} - x\| > \varepsilon\}) = 0$. Similarly, we will say that $(x_{ij})_{ij}$ is f**-statistically Cauchy** if for every $\varepsilon > 0$ and for every $l \in \mathbb{N}$ there exist $M, N \geq l$ such that $d_{2,f}(\{(i,j) \in \mathbb{N}^2 : \|x_{ij} - x_{MN}\| > \varepsilon\}) = 0$.

Given $A \subseteq \mathbb{N}^2$, if there exist $m, n \in \mathbb{N}$ such that $(i,j) \notin A$ if $i \geq m$ and $j \geq n$ then $d_{2,f}(A) = 0$. Therefore, if $\lim_{i,j} x_{ij} = x$ then $f - \mathrm{stlim}_{i,j}\, x_{ij} = x$.

The next examples are intended to clarify some aspects of this convergence, as well as its relation to usual convergence.

EXAMPLES 2.1 Let $f(x) = \log(x+1)$. Take $A \subseteq \mathbb{N}^2$ such that $\lim_{p,q} \frac{|A(p,q,i,j)|}{\sqrt{pq}} = 1$ for every $i, j \in \mathbb{N}$ and let $B = \mathbb{N}^2 \setminus A$, then $d_{2,f}(B) = d_2(B) = 1$ and $d_2(A) = 0$. However, $d_{2,f}(A) = \frac{1}{2}$. From this we deduce that $\mathrm{stlim}_{i,j}\, \chi_B(i,j) = 1$ but it is false that $f - \mathrm{stlim}_{i,j}\, \chi_B(i,j) = 1$, where χ_B is the characteristic function of B.

Besides, if we take $C = \{(i,j) \in \mathbb{N}^2 : i > 1 \text{ and } j > 1\}$, it is clear that $\lim_{i,j} \chi_C(i,j) = 1$ whereas if we had used the 'wrong' notion of f-density we would have

$$\limsup_{p,q} \frac{\log(1 + |(\mathbb{N}^2 \setminus C)(p,q)|)}{\log(1+pq)} = \limsup_{p,q} \frac{\log(p+q)}{\log(pq)} = 1$$

which means that the 'wrong' f-statistical limit of $(\chi_C(i,j))_{i,j}$ is not 1 (indeed, it does not even exist). •

From the next proposition it will follow at once that f-statistical convergence implies statistical convergence to the same limit.

Proposition 2.2. *If $A \subset \mathbb{N}^2$ is such that $d_{2,f}(A) = 0$ then $d_2(A) = 0$.*

Proof. Since $\lim_{i,j} \lim_{p,q} \frac{f(|A(p,q,i,j)|)}{f(pq)} = 0$, given $k \in \mathbb{N}$, there exist $i_0, j_0 \in \mathbb{N}$ such that if $i \geq i_0$ and $j \geq j_0$ then $\lim_{p,q} \frac{f(|A(p,q,i,j)|)}{f(pq)} < \frac{1}{k}$. Fix $t_0 = \max\{i_0, j_0\}$ and let $r_0 > k(t_0 - 1)$ be such that if $p, q \geq r_0$ then

$$\frac{f(|A(p,q,t_0,t_0)|)}{f(pq)} \leq \frac{1}{k}.$$

By the subadditivity of f, this implies

$$f(|A(p,q,t_0,t_0)|) < \frac{1}{k} f(pq) \leq \frac{1}{k} k \left(f \left(\frac{1}{k} pq \right) \right) = f \left(\frac{1}{k} pq \right)$$

and we deduce $|A(p,q,t_0,t_0)| < \frac{1}{k} pq$, as f is increasing. Therefore, if $p, q \geq r_0$ we will have $\frac{|A(p,q)|}{pq} = \frac{|A(p,q,t_0,t_0)|}{pq} + \frac{|A(t_0-1,q,1,t_0)|}{pq} + \frac{|A(p,t_0-1,t_0,1)|}{pq} + \frac{|A(t_0-1,t_0-1,1,1)|}{pq} < \frac{1}{k} + (\frac{1}{p} + \frac{1}{q} + \frac{t_0-1}{pq})(t_0 - 1) < \frac{4}{k}$. ■

In the first example in 2.1 we have seen that given $B \subseteq \mathbb{N}^2$, the fact $d_{2,f}(B) = 1$ does not imply $d_{2,f}(\mathbb{N}^2 \setminus B) = 0$. However, the converse is true under a natural assumption on the modulus.

Theorem 2.3. *Let f be a modulus function such that $d_{2,f}(\mathbb{N}^2) = 1$. If $A \subseteq \mathbb{N}^2$ satisfies $d_{2,f}(A) = 0$ then $d_{2,f}(\mathbb{N}^2 \setminus A) = 1$.*

Proof. Given $\varepsilon > 0$ there exist $i_0, j_0, p_0, q_0 \in \mathbb{N}$ such that if $i \geq i_0$, $j \geq j_0$, $p \geq p_0$ and $q \geq q_0$ then

$$\frac{f(|\mathbb{N}^2(p,q,i,j)|)}{f(pq)} \geq 1 - \frac{\varepsilon}{2} \quad \text{and} \quad \frac{f(|A(p,q,i,j)|)}{f(pq)} < \frac{\varepsilon}{2}$$

which immediately implies

$$\frac{f(|(\mathbb{N}^2 \setminus A)(p,q,i,j)|)}{f(pq)} \geq 1 - \varepsilon.$$ ■

Note that in the previous theorem we have required the hypothesis that the f-density of the whole space is 1. In [1], the analogous theorem in the one-dimensional case was proved without that hypothesis. In the propositions and example that follow we will see that this hypothesis can be weakened but not removed.

Proposition 2.4. *Let f be a modulus function. If $d_{2,f}(\mathbb{N}^2)$ exists it must be* 1.

Proof. It is enough to prove that for every $k \in \mathbb{N}$ there exists $(p_n)_n$ strictly increasing, divergent and such that

$$\frac{f(p_n^2 - kp_n)}{f(p_n^2)} \xrightarrow{n} 1$$

Suppose not, then there exist $k \in \mathbb{N}$, $\alpha > 1$ and $p_0 \in \mathbb{N}$ such that if $x \in \mathbb{R}$, $x \geq p_0$ then

$$\frac{f(x^2)}{f(x^2 - kx)} > \alpha$$

Let $M \in \mathbb{N}$ be such that $M > \max\{p_0/k, \log_\alpha 4\}$. Define $x_1 = Mk$ and inductively construct $(x_n)_n$ as the only strictly increasing sequence such that $x_{n+1}^2 - kx_{n+1} = x_n^2$. It is easy to prove that $x_n \leq Mk + kn - k$ for every $n \in \mathbb{N}$. In particular, $x_{M+1} \leq 2Mk$ and

$$4 \geq \frac{f(4M^2k^2)}{f(M^2k^2)} \geq \frac{f(x_{M+1}^2)}{f(x_1^2)} = \frac{f(x_2^2)}{f(x_1^2)}\frac{f(x_3^2)}{f(x_2^2)} \cdots \frac{f(x_{M+1}^2)}{f(x_M^2)} \geq \alpha^M$$

which is in contradiction with the choice of M. ∎

As a consequence, we have the announced weakening of the hypothesis.

Corollary 2.5. *Let f be a modulus function such that $d_{2,f}(\mathbb{N}^2)$ exists. If $A \subseteq \mathbb{N}^2$ satisfies $d_{2,f}(A) = 0$ then $d_{2,f}(\mathbb{N}^2 \setminus A) = 1$.*

There still remains the natural question: Can it happen that $d_{2,f}(\mathbb{N}^2)$ does not exist? The next example answers this affirmatively.

EXAMPLE 2.6

Let us construct a modulus g such that $\liminf_{i,j} \liminf_{p,q} \frac{g(|\mathbb{N}^2(p,q,i,j)|)}{g(pq)} \leq \frac{1}{2}$ and thus $d_{2,g}(\mathbb{N}^2)$ do not exist.

Consider $a_n = 2^{2^{n-1}}, b_n = \left(\frac{1}{10}\right)^{2^{n-1}}, y_n = a_n^2 - a_n$ and $z_n = a_n^2$. It is clear that $y_1 < z_1 < y_2 < z_2 < y_3 < \dots$. It is also straightforward to prove that $z_k \leq \min\{2y_k, \frac{1}{2}y_{k+1}\}$ for every $k \in \mathbb{N}$. Now define g by $g(0) = 0$, $g(1) = 1$, $g(2) = 1 + b_1$ and recursively, on the positive integers, as follows:

$$g(x) = \begin{cases} g(y_k) + g(x - y_k) & \text{if } x \in (y_k, z_k],\ k \in \mathbb{N} \\ g(z_k) + b_{k+1}(x - z_k) & \text{if } x \in (z_k, y_{k+1}],\ k \in \mathbb{N} \end{cases}$$

Next, extend g piecewise linearly to obtain a function $g : [0, +\infty) \to [0, +\infty)$ which is, by construction, strictly increasing. To see that g is a modulus we only need to prove the subadditivity. To begin with, it is straightforward to prove inductively that for each $k \in \mathbb{N}$, if $a \leq b \leq z_k$ then $g(b) - g(a) \geq (b - a)b_{k+1}$.

Take $x, u, v \in \mathbb{N}$ satisfying $u + v = x$. First, suppose $x \in (y_{k+1}, z_{k+1}]$. We will assume that $x - y_{k+1} < u \leq z_k$ since the remaining cases are simpler. Then $z_k < v < y_{k+1}$ and thus $g(u) - g(x - y_{k+1}) \geq b_{k+1}(y_{k+1} - v)$, which is equivalent to $g(x) = g(y_{k+1}) + g(x - y_{k+1}) \leq g(u) + g(v)$. Now suppose $x \in (z_k, y_{k+1}]$, we have by definition the useful equalities $g(x) = g(z_k) + b_{k+1}(x - z_k) = g(y_k) + g(z_{k-1}) + b_{k+1}(x - z_k) = 2g(z_{k-1}) + b_k(y_k - z_{k-1}) + b_{k+1}(x - z_k)$. We will assume $u, v \in (z_{k-1}, z_k]$ since the remaining cases are simpler. If $u \in (z_{k-1}, y_k]$ and $v \in (y_k, z_k]$ we get $g(u) + g(v) = g(z_{k-1}) + b_k(u - z_{k-1}) + g(y_k) + g(v - y_k) \geq g(y_k) + g(z_{k-1}) + b_k(u - z_{k-1}) + b_k(v - y_k) = g(y_k) + g(z_{k-1}) + b_k(x - z_k) > g(x)$. If $u, v \in (z_{k-1}, y_k]$ we get $g(u) + g(v) = 2g(z_{k-1}) + b_k(u + v - z_{k-1}) = g(x) + b_k(x - y_k) - b_{k+1}(x - z_k)) > g(x)$. If $u, v \in (y_k, z_k]$ we get $g(u) + g(v) = 2g(y_k) + g(u - y_k) + g(v - y_k) \geq g(y_k) + g(z_{k-1}) + b_k(u + v - y_k - z_{k-1}) = g(y_k) + g(z_{k-1}) + b_k(x - z_k) > g(x)$.

We conclude that

$$\liminf_n \frac{g(n^2 - n)}{g(n^2)} \leq \lim_n \frac{g(a_n^2 - a_n)}{g(a_n^2)} = \lim_n \frac{g(a_{n-1}^2) + b_n(a_n^2 - 2a_n)}{g(a_n^2 - a_n) + g(a_n)} =$$

$$= \lim_n \frac{g(a_{n-1}^2) + b_n(a_n^2 - 2a_n)}{2g(a_{n-1}^2) + b_n(a_n^2 - 2a_n)} = \frac{1}{2}.$$

•

The previous modulus can be considered a pathological example, since many of the common moduli satisfy the aforementioned hypothesis, as the two theorems following the next lemma will show.

Lemma 2.7. *If a modulus function f satisfies $\sup_m \liminf_r \frac{f(r(1-\frac{1}{m}))}{f(r)} = 1$ then $d_{2,f}(\mathbb{N}^2) = 1$.*

Proof. It suffices to prove that $\liminf_{p,q} \frac{f((p-i+1)(q-j+1))}{f(pq)} = 1$ for every $i, j \in \mathbb{N}$. Take $k = \max\{i, j\}$, then $f((p - i + 1)(q - j + 1)) \geq f(pq - k(p + q))$ and for every $m \in \mathbb{N}$,

$$\liminf_{p,q} \frac{f(pq - k(p+q))}{f(pq)} = \liminf_{\substack{p \geq 2km \\ q \geq 2km}} \frac{f((\frac{p}{2} - k)q + (\frac{q}{2} - k)p)}{f(pq)} \geq$$

$$\geq \liminf_{\substack{p \geq 2km \\ q \geq 2km}} \frac{f(pq(1 - \frac{1}{m}))}{f(pq)} = \liminf_r \frac{f(r(1 - \frac{1}{m}))}{f(r)}$$

from which the conclusion follows. ∎

For the next theorem, a function $f : [0, \infty) \to [0, \infty)$ will be called eventually concave if there exists $a \geq 0$ such that f restricted to $[a, \infty)$ is concave.

Theorem 2.8. *If a modulus function f is eventually concave then $d_{2,f}(\mathbb{N}^2) = 1$.*

Proof. Let $n_0 \in \mathbb{N}$ be such that if $p, q \geq n_0$ and $t \in [0, 1]$ then $f(tp + (1 - t)q) \geq tf(p) + (1 - t)f(q)$. For every $m \in \mathbb{N}$ we have

$$\liminf_r \frac{f(r(1 - \frac{1}{m}))}{f(r)} \geq \liminf_{r \geq 2n_0} \frac{\frac{2}{m}f(r/2) + (1 - \frac{2}{m})f(r)}{f(r)} \geq 1 - \frac{2}{m}.$$ ∎

Theorem 2.9. *If a modulus function f satisfies* $\inf_{n\in\mathbb{N}} \frac{f(n)}{n} > 0$ *then* $d_{2,f}(\mathbb{N}^2) = 1$.

Proof. Call $\alpha = \inf_{n\in\mathbb{N}} \frac{f(n)}{n}$. Since f is subadditive, it is known that $\lim_n \frac{f(n)}{n} = \alpha$ (see [14], pp. 23 and 198). In addition,

$$\alpha = \lim_{x\to\infty} \frac{f(\lfloor x\rfloor)}{\lfloor x\rfloor}\frac{\lfloor x\rfloor}{x} \leq \lim_{x\to\infty} \frac{f(x)}{x} \leq \lim_{x\to\infty} \frac{f(\lceil x\rceil)}{\lceil x\rceil}\frac{\lceil x\rceil}{x} = \alpha$$

which proves that $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \alpha > 0$. In consequence,

$$\lim_r \frac{f(r(1-\frac{1}{m}))}{f(r)} = \frac{\alpha(1-\frac{1}{m})}{\alpha} = 1 - \frac{1}{m}. \qquad \blacksquare$$

Our last examples in this section show that the two previous theorems cannot be immediately deduced from each other.

EXAMPLES 2.10 On one hand, it is well known that $f(x) = \log(x+1)$ is an eventually concave function with $\inf_{n\in\mathbb{N}} \frac{f(n)}{n} = 0$. On the other hand, if g is the function constructed in example 2.6, by virtue of the two previous theorems we know that it must satisfy $\inf_{n\in\mathbb{N}} \frac{g(n)}{n} = 0$ and cannot be eventually concave. If we consider $f(x) = g(x) + x$, it is clear that $\inf_{n\in\mathbb{N}} \frac{f(n)}{n} > 0$ and f is still not eventually concave, since we are adding just a linear function to g. •

3 Main results

As mentioned in the introduction, in this section we will generalize the results in [11], which can be retrieved by considering in our results the modulus function $f(x) = x$. Theorems 3.1 and 3.2 are the analogous to theorem 1 and proposition 1 (reformulated as theorem 2) in [11], respectively. Whereas the proof of the former is similar, we have used a different technique for the latter.

Afterwards, a characterization of Pringheim's convergence will be proved, in a similar fashion to that in [1].

Theorem 3.1. *Let X be a Banach space, f an unbounded modulus and $(x_{ij})_{ij}$ a double sequence of X, then $(x_{ij})_{ij}$ is an f-statistically Cauchy sequence if and only if $(x_{ij})_{ij}$ is f-statistically convergent.*

Proof. We suppose that $f - \text{stlim}_{i,j}\, x_{ij} = x$, then given $\varepsilon > 0$ and $l \in \mathbb{N}$ there exist $M, N \geq l$ such that $\|x_{MN} - x\| < \varepsilon$ then $A = \{(i,j) \in \mathbb{N} : \|x_{ij} - x_{MN}\| > \varepsilon\} \subseteq \{(i,j) \in \mathbb{N} : \|x_{ij} - x\| > \varepsilon/2\}$ and we can deduce that $d_{2,f}(A) = 0$.

Conversely, assume that $(x_{ij})_{ij}$ is an f-statistically Cauchy sequence. For $\varepsilon = 1$ there exist $M_1, N_1 \in \mathbb{N}$ such that $d_{2,f}(\{(i,j) : \|x_{ij} - x_{M_1N_1}\| \geq 1\}) = 0$. For $\varepsilon = 1/2$ there exist $M_2, N_2 > \max\{M_1, N_1\}$ such that $d_{2,f}(\{(i,j) : \|x_{ij} - x_{M_2N_2}\| \geq 1/2\}) = 0$. Inductively we obtain two sequences $(N_k)_k$ and $(M_k)_k$ of natural numbers such that $M_{n+1}, N_{n+1} > \max\{M_n, N_n\}$ and $d_{2,f}(\{(i,j) : \|x_{ij} - x_{M_kN_k}\| \geq 1/k\}) = 0$.

Given $r, s \in \mathbb{N}$ there exists $(a, b) \in \mathbb{N}^2$ such that $\|x_{ab} - x_{M_r N_r}\| < 1/r$ and $\|x_{ab} - x_{M_s N_s}\| < 1/s$, thus $\|x_{M_r N_r} - x_{M_s N_s}\| < 1/r + 1/s$ and we deduce that $(x_{M_k N_k})_k$ is a Cauchy sequence, so it converges to some $x \in X$. Let $\varepsilon > 0$ and take $t \in \mathbb{N}$ with $t > 2/\varepsilon$ and $\|x_{M_t N_t} - x\| < \varepsilon/2$. Consider $A = \{(i, j) \in \mathbb{N}^2 : \|x_{ij} - x\| \geq \varepsilon\} \subseteq \{(i, j) \in \mathbb{N}^2 : \|x_{ij} - x_{M_t N_t}\| \geq \varepsilon/2\} \subseteq \{(i, j) \in \mathbb{N}^2 : \|x_{ij} - x_{M_t N_t}\| \geq 1/t\}$, it is thus clear that $d_{2,f}(A) = 0$. ■

Theorem 3.2. *Let $(x_{ij})_{ij}$ be a sequence in a normed space X and f an unbounded modulus. Then $f - \mathrm{stlim}_{ij}\, x_{ij} = x$ if and only if there exists $A \subseteq \mathbb{N}^2$ such that $d_{2,f}(A) = 0$ and $\lim_{(i,j)\in\mathbb{N}^2\setminus A} x_{ij} = x$.*

Proof. Suppose that $f - \mathrm{stlim}_{i,j}\, x_{ij} = x$. For every $n \in \mathbb{N}$ define $B_n = \{(i, j) \in \mathbb{N}^2 : \|x_{ij} - x\| > \frac{1}{n}\}$, we have $d_{2,f}(B_n) = 0$ and $B_n \subset B_{n+1}$. Just from the definition of density, given $n \in \mathbb{N}$ there exists $r_n \in \mathbb{N}$ such that for every $i, j, p, q \geq r_n$ we have $\frac{f(|B_n(p,q,i,j)|)}{f(pq)} < \frac{1}{2^n}$, and we may also assume that $(r_n)_n$ is a strictly increasing sequence.

Define

$$A_1 = \bigcup_{n\in\mathbb{N}} (B_n \cap [r_n, r_{n+1}) \times [r_n, +\infty))$$

$$A_2 = \bigcup_{n\in\mathbb{N}} (B_n \cap [r_n, +\infty) \times [r_n, r_{n+1}))$$

and consider $A = A_1 \cup A_2$. Note that we can also write, by the construction,

$$A \subseteq \bigcup_{n\in\mathbb{N}} \{(a, b) \in B_n : r_n \leq \min\{a, b\} < r_{n+1}\}.$$

Therefore, if $i, j \geq r_n$ for some $n \in \mathbb{N}$ then

$$\frac{f(|A(p, q, i, j)|)}{f(pq)} \leq \sum_{m\geq n} \frac{f(|B_m(p, q, r_m, r_m)|)}{f(pq)} < \sum_{m\geq n} \frac{1}{2^m} = \frac{1}{2^{n-1}}$$

which clearly implies $d_{2,f}(A) = 0$.

Now let us see $\lim_{(i,j)\in\mathbb{N}^2\setminus A} x_{ij} = x$. Let $\varepsilon > 0$ and take $n \in \mathbb{N}$ satisfying $n > \frac{1}{\varepsilon}$. Let $(i, j) \in \mathbb{N}^2 \setminus A$ be such that $i, j \geq r_n$. If we assume that $i \leq j$ then there exists $s \geq n$ such that $i \in [r_s, r_{s+1})$ and we know $(i, j) \notin A_1$; this implies $(i, j) \notin B_s$ and $\|x_{ij} - x\| \leq \frac{1}{s} \leq \frac{1}{n} < \varepsilon$. Analogously if $j \leq i$, this time by using A_2.

For the converse, given $\varepsilon > 0$ there exist $(i_0, j_0) \in \mathbb{N}^2 \setminus A$ such that if $(i, j) \notin A$ and $i \geq i_0, j \geq j_0$ then $\|x_{ij} - x\| \leq \varepsilon$. Thus, $\{(i, j) \in \mathbb{N}^2 : \|x_{ij} - x\| > \varepsilon\} \subset A \cup \{(i, j) \in \mathbb{N}^2 : i < i_0 \text{ or } j < j_0\}$, which implies $d_{2,f}(\{(i, j) \in \mathbb{N}^2 : \|x_{ij} - x\| > \varepsilon\}) = 0$. ■

The next three lemmas are oriented towards a characterization of Pringsheim's convergence in terms of f-statistical convergence. This will be accomplished in Theorem 3.6.

Lemma 3.3. *If $A \subseteq \mathbb{N}$ is infinite and $(b_k)_k$ is an increasing divergent sequence, then there exists an unbounded module f such that $\lim_k \frac{f(|A(k)|)}{f(b_k)} = 1$.*

Proof. We shall write A as a strictly increasing sequence, $A = (a_k)_k$. Let us define $g : \mathbb{N} \to \mathbb{N}$ by $g(1) = 1, g(2) = a_2$ and if $k \geq 2$ then $g(k+1) = \max\{a_{1+g(k)}, b_{g(k)}\}$.

We have by construction that g is strictly increasing and

$$|A(g(k+1))| \geq 1 + g(k).$$

Now define $f : [0, \infty) \to [0, \infty)$ by $f(0) = 0$, for $n \in \mathbb{N}$ let $f(g(n)) = n$ and finally extend f to be piecewise linear in the remaining intervals.

We observe that $\frac{f(g(2))-f(g(1))}{g(2)-g(1)} \leq 1$ and if $k \geq 2$ then $g(k+1) - g(k) \geq g(k) - g(k-1)$ and that is why $\frac{(k+1)-k}{g(k+1)-g(k)} \leq \frac{k-(k-1)}{g(k)-g(k-1)}$, which means that $\frac{f(g(k+1))-f(g(k))}{g(k+1)-g(k)} \leq \frac{f(g(k))-f(g(k-1))}{g(k)-g(k-1)}$, therefore the corresponding slopes of the segments that form the graph of f are decreasing and thus if $x, y \in [0, +\infty)$ then $f(x+y) \leq f(x) + f(y)$. Consequently, f is an unbounded modulus. Let us show that $\lim_k \frac{f(|A(k)|)}{f(b_k)} = 1$.

For every $k > g(2)$ there exists $n \in \mathbb{N}$ such that $g(n+1) \leq k \leq g(n+2)$. Then

$$\frac{f(|A(k)|)}{f(b_k)} \geq \frac{f(|A(g(n+1))|)}{f(b_{g(n+2)})} \geq \frac{f(1+g(n))}{f(g(n+3))} \geq \frac{f(g(n))}{f(g(n+3))} = \frac{n}{n+3} \to 1. \quad \blacksquare$$

Lemma 3.4. *If $B \subseteq \mathbb{N}$ is infinite and we consider $D_B = \{(p,q) \in \mathbb{N}^2 : p = q \in B\}$, then there exists an unbounded module g such that*

$$\limsup_{i,j} \limsup_{p,q} \frac{g(|D_B(p,q,i,j)|)}{g(p\,q)} = 1.$$

Proof. We consider $B(p,i) = \{n \in B : i \leq n \leq p\}$ and we have that for every modulus f and $i \in \mathbb{N}$,

$$1 \geq \frac{f(|B(p,i)|)}{f(|B(p)|)} \geq \frac{f(|B(p)|) - f(|B(i-1)|)}{f(|B(p)|)} \xrightarrow{p} 1$$

On the other hand, by the above lemma, taking $b_k = k^2$ there exists an unbounded modulus g such that

$$\lim_p \frac{g(|B(p)|)}{g(p^2)} = 1$$

so, for every $i \in \mathbb{N}$,

$$\lim_i \lim_p \frac{g(|D_B(p,p,i,i)|)}{g(p^2)} = \lim_i \lim_p \frac{g(|B(p,i)|)}{g(p^2)} = 1. \quad \blacksquare$$

Lemma 3.5. *Let $(p_k)_k$ and $(q_k)_k$ be strictly increasing sequences and consider $A = \{(p_k, q_k) : k \in \mathbb{N}\}$, $B = \{\max\{p_k, q_k\} : k \in \mathbb{N}\}$ and D_B as in lemma 3.4. Given $i, j \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every $p \geq i$ and $q \geq j$,*

$$|D_B(p,q,i,j)| < |A(p,q,i,j)| + n.$$

Proof. Given $i, j \in \mathbb{N}$ let $n \in \mathbb{N}$ be such that $p_n > i$ and $q_n > j$. Taking into account that $(p_k)_k$ and $(q_k)_k$ are increasing, this implies

$$|A(p,q,1,1) \setminus A(p,q,i,j)| < n$$

for every $p \geq i$, $q \geq j$. The desired result is obtained from the following chain of simple facts:

$$|D_B(p,q,i,j)| \leq |D_B(p,q,1,1)| = |D_B(\min\{p,q\}, \min\{p,q\}, 1, 1)| =$$

$$= |A(\min\{p,q\}, \min\{p,q\}, 1, 1)| \leq |A(p,q,1,1)| < |A(p,q,i,j)| + n. \qquad \blacksquare$$

Theorem 3.6. *Let $(x_{ij})_{ij}$ be a sequence in X. If for every unbounded modulus f there exists $f - \operatorname{stlim} x_{ij}$ then all these limits are the same $x \in X$ and $(x_{ij})_{ij}$ also converges to x in Pringsheim's sense.*

Proof. Suppose that $(x_{ij})_{ij}$ is not a Cauchy sequence. Then there exist two strictly increasing sequences of natural numbers $(p_k)_k$ and $(q_k)_k$ and $\varepsilon_0 > 0$ such that $\|x_{p_r q_r} - x_{p_s q_s}\| > \varepsilon_0$ whenever $r \neq s$. Let $A = \{(p_k, q_k) \in \mathbb{N}^2 : k \in \mathbb{N}\}$. For every unbounded modulus f, if $x_f = f - \operatorname{stlim} x_{ij}$ then we have that

$$A_f = \left\{(i,j) \in \mathbb{N} : \|x_{ij} - x_f\| > \frac{\varepsilon_0}{2}\right\}$$

satisfies $d_{2,f}(A_f) = 0$ and it is clear from the definitions that $A \setminus A_f$ has at most one element. We deduce that $d_{2,f}(A) = 0$ for every unbounded modulus f.

Now define $B = \{\max\{p_k, q_k\} : k \in \mathbb{N}\}$, we have by lemma 3.5 that $d_{2,f}(D_B) \leq d_{2,f}(A) = 0$ for every unbounded modulus f. However, taking g_B as in lemma 3.4, the fact that the density $d_{2,g_B}(D_B)$ exists implies $d_{2,g_B}(D_B) = 1$, which is a contradiction. Therefore $(x_{ij})_{ij}$ is a Cauchy sequence and it it easy to check that it must converge to x_f for every unbounded modulus f. In particular, all these limits are the same. $\blacksquare$

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