Uniform stabilization of the Riemannian wave equation with linear lower order term

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Abstract

The Riemannian wave equation with linear lower order term and unspecified behavior of the nonlinear feedback f is considered. Using the method in $[LT]$ we prove that the energy of the solution decays faster than the solution of some associated differential equation. The decay rate of a general second order hyperbolic equation with polynomial growth at the origin of f is also discussed.

1 Introduction; statement of main result

Consider the following Riemannian wave equation with linear lower order term

$$\begin{cases} y_{tt} - \Delta_g y - \langle D\varphi, Dy\rangle_g = 0 \text{ in } Q = \Omega \times]0,T[\,, \\ y = 0 \text{ on } \Sigma_0 = \Gamma_0 \times]0,T[\,, \\ \frac{\partial y}{\partial n} + by_t + f(y_t) = 0 \text{ on } \Sigma_1 = \Gamma_1 \times]0,T[\,, \\ y(0) = y_0, y_t(0) = y_1 \text{ in } \Omega. \end{cases} \tag{P}$$

Where $T > 0$, Ω is an open bounded set of M with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \neq \varnothing$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \varnothing$, where M is a finite dimensional Riemann manifold with metric $g(.,.) = \langle .,.\rangle_g$ and norm $\|.\|_g : g(X,Y) = \langle X, Y\rangle_g = \sum_{i,j=1}^{n} g_{ij}\alpha_i\beta_j$ and $\|X\|_g = (g(X,X))^{\frac{1}{2}}$ for all $X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in M_x$. Here, for each $x \in M$, M_x denote the tangent space of M at x. Δ_g is the Laplace Beltrami operator

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on the manifold M defined by $\Delta_g y := \frac{1}{\sqrt{\det(g_{ij}(x))}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g_{ij}(x))} a_{ij}(x) \frac{\partial y}{\partial x_j} \right)$, with $(a_{ij}(x)) = (g_{ij}(x))^{-1}$. n is the unit outward normal field along the boundary Γ, D is the Levi Civita connection on M and $\dfrac{\partial y}{\partial n} = \langle Dy, n \rangle_g$ is the normal derivative.

The following assumptions are made on φ, b, f and Ω :

$(H1)$ $\varphi \in W^{2,\infty}(\Omega)$ such that for some positive constants φ_* and φ^* we have $\varphi_* \leq \varphi(\varkappa) \leq \varphi^*$ for all $\varkappa \in \overline{\Omega}$.

$(H2)$ There exists two positive constants b_* and b^* such that $0 < b_* \leq b(\varkappa) \leq b^*$ for all $\varkappa \in \Gamma_1$.

$(H3)$ $f : \mathbb{R} \to \mathbb{R}$ is an increasing function of class C^1 such that for some positive constants c_1 and c_2 we have

$$c_1 |s|^2 \leq f(s)\, s \leq c_2 |s|^2 \text{ for } |s| \geq 1, \tag{1}$$

and

$$f(s)\, s > 0 \text{ for all } s \neq 0.$$

$(H4)$ There exists a function $d : \overline{\Omega} \to \mathbb{R}^+$ of class C^3 verifying

$$\inf_{\Omega} \|Dd\|_g > 0, \tag{2}$$

and for some constant $m_0 > 0$.

$$D^2 d(X, X) \geq m_0 \|X\|_g^2, \text{ for all } X \in M_x. \tag{3}$$

where D^2 is the Hessian with respect to the metric g.
Moreover, we take $\Gamma_0 = \left\{ x \in \Gamma : \langle Dd, n \rangle_g \leq 0 \right\}$ and $\Gamma_1 = \left\{ x \in \Gamma : \langle Dd, n \rangle_g \geq h_0 \right\}$ for some constant $h_0 > 0$.

Remarks

1/ We remark here that no growth conditions at the origin are imposed on the nonlinear feedback f. But, by virtue of Assumption $(H3)$, we can always (See $[LT]$) construct a concave, strictly increasing function $l : \mathbb{R}_+ \to \mathbb{R}$ such that $l(0) = 0$ and

$$l(f(s)\, s) \geq |s|^2 + |f(s)|^2 \text{ for } |s| \leq 1.$$

2/ The well-posedness of problem (P) can be established by the Faedo-Galerkin method:

For all $(y_0, y_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, the system (P) has a unique solution $y \in C(0, T; H^1_{\Gamma_0}(\Omega)) \cap C^1(0, T; L^2(\Omega))$.

If $(y_0, y_1) \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$ such that $\frac{\partial y_0}{\partial n} + b y_1 + f(y_1) = 0$ on Γ_1 then the system (P) has a unique solution y verifying $y \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega))$, $y_t \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega))$ and $y_{tt} \in L^\infty(0, T; L^2(\Omega))$.

Over the last decade, the problem of the uniform decay rates of the wave equation which include first order terms and defined on an Euclidean domain

($\Omega \subset \mathbb{R}^n$) have been considered in $[CS, Gue]$. In all these works, uniform decay rate estimates are obtained under strong hypothesis on the first order terms. The inclusion of these terms produce serious additional difficulties since we do not have any information about their influence on the energy of the solution, specially, about the signal of the derivative of the energy.

Concerning the variable problems, the Riemann geometric methods have been introduced to obtain certain a priori inequalities (observability, uniform stabilization and Carleman estimates) of certain classes of PDEs defined on an open bounded set Ω of a finite dimensional Riemannian manifold M (see for example $[LTY, TY]$).

The main goal of this paper is to show that, without any condition on the linear first order term, the energy of the solution of the system (P) decays faster than the solution of some associated differential equation. For this end, we use the energy (multiplier) method, where we introduce a new geometric multiplier $Dd(\varphi)\,y$ to handle the linear first order term. In order to absorb the lower order term with respect to the energy, we combine the idea in $[FF]$ with the one of I. Lasiecka and D. Tataru in $[LT]$. Finally, we conclude by employing the Lasiecka's and Tataru's abstract stabilization inequalities.

The result of this paper generalizes the corresponding case of a second order hyperbolic equation and a linear growth of f at the origin which was proved in $[Ham]$.

As it is well know, the presence of the first order term provides the non dissipation for the usual energy. For this reason, we have to consider an equivalent energy E of the system (P) defined, for all $t \geq 0$, by $E(t) = \frac{1}{2}\int\limits_{\Omega} e^{\varphi}\left(|y_t|^2 + \|Dy\|_g^2\right) d\Omega$, where $d\Omega$ is the volume element in the Riemannian metric g and we shall see in lemma 3 below that it is a decreasing function.

We state, now, the main result of this paper.

Theorem 1. *For some* $T_0 > 0$

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right) \text{ for all } t > T_0,$$

where $S(t)$ *is the solution of the following differential equation*

$$\begin{cases} S_t(t) + q(S(t)) = 0 \\ S(0) = E(0). \end{cases}$$

Here, for all $s > 0$, $q(s) = s - (I+p)^{-1}(s)$, *with* $p(s) = e^{\varphi_*}\left(cI + \widetilde{l}\right)^{-1}(Ks)$, $\widetilde{l}(s) = l\left(\frac{s}{mes\Sigma_1}\right)$, $K = \frac{1}{Ce^{\varphi^*} mes\Sigma_1}$ *and* $c = \frac{c_1^{-1} + c_2}{e^{\varphi^*} mes\Sigma_1}$.

This paper is organized as follows. In section 2, we present some preliminary identities with which we are working. Then, in section 3, we will give two energy inequalities that we will use in the proof of the main result. Section 4 is devoted to the absorption of the lower order term. In section 5, we complete the proof of the main result. Finally, we study the case of second order hyperbolic equation with polynomial behavior at the origin of the nonlinear feedback.

2 Preliminary identities

We collect here an identity and two formulas to be invoked in the sequel.

An identity $[TY]$**.** For any function q and vector field h on M, we have

$$\langle Dq, D(h(q))\rangle_g = Dh(Dq, Dq) + \frac{1}{2}h\left(\|Dq\|_g^2\right), \tag{4}$$

where $h(u) := \langle h, Du\rangle_g$ for all function u.

Green's theorem $[TY]$**.** If $q_1, q_2 \in H^2(\Omega)$ then

$$\int_\Omega (\Delta_g q_1) q_2 d\Omega = \int_\Gamma \frac{\partial q_1}{\partial n} q_2 d\Gamma - \int_\Omega \langle Dq_1, Dq_2\rangle_g d\Omega.$$

Here $d\Gamma$ is the surface element in the Riemannian metric g.

Divergence theorem $[TY]$**.**

$$\int_\Omega divX d\Omega = \int_\Gamma \langle X, n\rangle_g d\Gamma,$$

where $divX$ is the Riemannian divergence of the vector field X.

We need also to

Lemma 2. *For any function $q \in C^1(\overline{\Omega})$ and vector field h on M we have*

$$\int_\Omega q_1 h(q_2) d\Omega = \int_\Gamma \langle h, n\rangle_g q_1 q_2 d\Gamma - \int_\Omega q_2 div(q_1 h) d\Omega.$$

Proof. It's sufficient to see that

$$q_1 h(q_2) = div(q_1 q_2 h) - q_2 div(q_1 h)$$

integrate over Ω, and use the divergence theorem. ∎

3 Energy inequalities

We show that the equivalent energy of the system (P) is dissipative.

Lemma 3.

$$E'(t) = \frac{dE}{dt} = -\int_{\Gamma_1} e^{\varphi}(by_t + f(y_t)) y_t d\Gamma \le 0,$$

for all $t > 0$.

Proof. First, we have

$$\begin{aligned} De^{\varphi} &= \sum_i \left(\sum_j a_{ij}\frac{\partial e^{\varphi}}{\partial x_j}\right)\frac{\partial}{\partial x_i} \\ &= e^{\varphi}\sum_i \left(\sum_j a_{ij}\frac{\partial \varphi}{\partial x_j}\right)\frac{\partial}{\partial x_i} = e^{\varphi} D\varphi. \end{aligned}$$

If we use Green's formula we find

$$\begin{aligned} 0 &= \int_{\Omega} e^{\varphi}\left(y_{tt}-\Delta_g y-\langle D\varphi, Dy\rangle_g\right) y_t d\Omega \\ &= \left[\int_{\Omega} e^{\varphi} y_{tt} y_t d\Omega + \int_{\Omega} e^{\varphi}\langle Dy, Dy_t\rangle_g d\Omega\right] \\ &\quad - \int_{\Gamma} e^{\varphi}\frac{\partial y}{\partial n} y_t d\Gamma = \frac{dE}{dt} - \int_{\Gamma_1} e^{\varphi}\frac{\partial y}{\partial n} y_t d\Gamma, \end{aligned}$$

then

$$\begin{aligned} \frac{dE}{dt} &= \int_{\Gamma_1} e^{\varphi}\frac{\partial y}{\partial n} y_t d\Gamma \\ &= -\int_{\Gamma_1} e^{\varphi}\left(b y_t + f\left(y_t\right)\right) y_t d\Gamma. \end{aligned}$$

■

To prove the following inequality, we need to introduce a new differential multiplier $h(\varphi)\, y$, where $h = Dd$.

Lemma 4. *For all* $T - S > T_0$, T_0 *is sufficiently large,*

$$\begin{aligned} E(T) &\le C\int_S^T\int_{\Gamma_1} e^{\varphi}\left(\left|f\left(y_t\right)\right|^2 + \left|y_t\right|^2\right) d\Sigma \\ &\quad + \int_S^T\int_{\Omega} e^{\varphi}\left|y\right|^2 dQ, \end{aligned}$$

where $d\Sigma = d\Gamma dt$ *and* $dQ = d\Omega dt$.

Proof. First, we have by lemma 3

$$E(T) \le E(S) \le E(T) + C\int_S^T\int_{\Gamma_1} e^{\varphi}\left(\left|f\left(y_t\right)\right|^2 + \left|y_t\right|^2\right) d\Sigma. \tag{5}$$

On the other hand, we can see that

$$0 = \int_S^T\int_{\Omega} e^{\varphi} y_{tt} My + \int_S^T\int_{\Omega} e^{\varphi}\left(-\Delta_g y - \left(D\varphi, Dy\right)_g\right) My, \tag{6}$$

where $My = 2h(y) + \left(divh - m_0 + h(\varphi)\right) y$.

But the integration by part gives

$$\begin{aligned}\int_S^T\int_\Omega e^{\varphi}y_{tt}MydQ &= \int_\Omega e^{\varphi}y_tMyd\Omega\Big|_S^T - \int_S^T\int_\Omega e^{\varphi}y_tMy_tdQ \\ &= \int_\Omega e^{\varphi}y_tMyd\Omega\Big|_S^T - 2\int_S^T\int_{\Omega\Omega} e^{\varphi}h\left(y_t\right)y_tdQ \\ &\quad -\int_S^T\int_\Omega e^{\varphi}\left(divh - m_0 + h\left(\varphi\right)\right)\left|y_t\right|^2 dQ,\end{aligned}$$

but, by lemma 2, we have

$$\begin{aligned}\int_S^T\int_\Omega e^{\varphi}y_th\left(y_t\right)dQ &= \int_S^T\int_{\Gamma_1} e^{\varphi}\left\langle h,n\right\rangle_g\left|y_t\right|^2 d\Sigma - \int_S^T\int_\Omega y_tdiv\left(e^{\varphi}hy_t\right)dQ \\ &= \int_S^T\int_{\Gamma_1} e^{\varphi}\left\langle h,n\right\rangle_g\left|y_t\right|^2 d\Sigma - \int_S^T\int_\Omega e^{\varphi}y_th\left(y_t\right)dQ \\ &\quad -\int_S^T\int_\Omega e^{\varphi}\left(divh + h\left(\varphi\right)\right)\left|y_t\right|^2 dQ,\end{aligned}$$

then

$$\begin{aligned}2\int_S^T\int_\Omega e^{\varphi}y_th\left(y_t\right)dQ &= \int_S^T\int_{\Gamma_1} e^{\varphi}\left\langle h,n\right\rangle_g\left|y_t\right|^2 d\Sigma \\ &\quad -\int_S^T\int_\Omega e^{\varphi}\left(divh + h\left(\varphi\right)\right)\left|y_t\right|^2 dQ,\end{aligned}$$

so

$$\begin{aligned}\int_S^T\int_\Omega e^{\varphi}y_{tt}MydQ &= \int_\Omega e^{\varphi}y_tMyd\Omega_g\Big|_S^T \\ &\quad -\int_S^T\int_{\Gamma_1} e^{\varphi}\left\langle h,n\right\rangle_g\left|y_t\right|^2 d\Sigma \\ &\quad +m_0\int_S^T\int_\Omega e^{\varphi}\left|y_t\right|^2 dQ.\end{aligned}$$

If we use Green's formula we obtain

$$\begin{aligned}
&\int_S^T \int_\Omega e^{\varphi} \left(-\Delta_g y - \langle D\varphi, Dy\rangle_g\right) My dQ \\
&= -\int_S^T \int_\Gamma e^{\varphi} \frac{\partial y}{\partial n} My d\Sigma + \int_S^T \int_\Omega e^{\varphi} \langle Dy, D(My)\rangle_g \\
&= -\int_S^T \int_\Gamma e^{\varphi} \frac{\partial y}{\partial n} My d\Sigma
\end{aligned}$$

$$\begin{aligned}
&+2\int_S^T \int_\Omega e^{\varphi} \langle Dy, D(h(y))\rangle_g dQ + \int_S^T \int_\Omega e^{\varphi} \langle Dy, D(divh + h(\varphi))\rangle_g y dQ \\
&+ \int_S^T \int_\Omega e^{\varphi} (divh - m_0 + h(\varphi)) \|Dy\|_g^2 dQ.
\end{aligned}$$

By identity (4)

$$\begin{aligned}
&\int_S^T \int_\Omega e^{\varphi} \left(-\Delta_g y - \langle D\varphi, Dy\rangle_g\right) My dQ \\
&= -\int_S^T \int_\Gamma e^{\varphi} \frac{\partial y}{\partial n} My d\Sigma + 2\int_S^T \int_\Omega e^{\varphi} Dh(Dy, Dy) dQ \\
&+ \int_S^T \int_\Omega e^{\varphi} h\left(\|Dy\|_g^2\right) dQ + \int_S^T \int_\Omega e^{\varphi} \langle Dy, D(divh + h(\varphi))\rangle_g y dQ \\
&+ \int_S^T \int_\Omega e^{\varphi} (divh - m_0 + h(\varphi)) \|Dy\|_g^2 dQ.
\end{aligned}$$

Lemma 2 gives

$$\begin{aligned}
&\int_S^T \int_\Omega e^{\varphi} \left(-\Delta_g y - \langle D\varphi, Dy\rangle_g\right) My dQ \\
&= -\int_S^T \int_\Gamma e^{\varphi} \frac{\partial y}{\partial n} My d\Sigma + \int_S^T \int_\Gamma e^{\varphi} \langle h, n\rangle_g \|Dy\|_g^2 d\Sigma \\
&+2\int_S^T \int_\Omega e^{\varphi} Dh(Dy, Dy) dQ - m_0 \int_S^T \int_\Omega e^{\varphi} \|Dy\|_g^2 dQ \\
&+ \int_S^T \int_\Omega e^{\varphi} \langle Dy, D(divh + h(\varphi))\rangle_g y dQ.
\end{aligned}$$

If we replace in (6), we find

$$\begin{aligned}
& 2\int_S^T\int_\Omega e^{\varphi} Dh\left(Dy,Dy\right)dQ + m_0\int_S^T\int_\Omega e^{\varphi}\left(|y_t|^2-\|Dy\|_g^2\right)dQ \\
= & -\int_\Omega e^{\varphi} y_t My d\Omega_g\Big|_S^T + \int_S^T\int_{\Gamma_0} e^{\varphi}\left(2\frac{\partial y}{\partial n}h\left(y\right)-\langle h,n\rangle_g\|Dy\|_g^2\right)d\Sigma \\
& + \int_S^T\int_{\Gamma_1} e^{\varphi}\left(\frac{\partial y}{\partial n}My+\langle h,n\rangle_g\left(|y_t|^2-\|Dy\|_g^2\right)\right)d\Sigma \\
& - \int_S^T\int_\Omega e^{\varphi}\left\langle Dy, D\left(divh+h\left(\varphi\right)\right)\right\rangle_g y dQ.
\end{aligned}$$

If we use (3) and lemma 3 we obtain

$$\begin{aligned}
2m_0\left(T-S\right)E\left(T\right) & \leq 2m_0\int_S^T E\left(t\right) \\
& \leq I_\Omega + I_{\Sigma_0} + I_{\Sigma_1} + I_Q
\end{aligned}$$

where

$$I_\Omega = -\int_\Omega e^{\varphi} y_t My d\Omega\Big|_S^T,$$

$$I_{\Sigma_0} = \int_S^T\int_{\Gamma_0} e^{\varphi}\left(2\frac{\partial y}{\partial n}h\left(y\right)-\langle h,n\rangle_g\|Dy\|_g^2\right)d\Sigma,$$

$$I_{\Sigma_1} = \int_S^T\int_{\Gamma_1} e^{\varphi}\left(\frac{\partial y}{\partial n}My+\langle h,n\rangle_g\left(|y_t|^2-\|Dy\|_g^2\right)\right)d\Sigma$$

and

$$I_Q = -\int_S^T\int_\Omega e^{\varphi}\left\langle Dy, D\left(divh+h\left(\varphi\right)\right)\right\rangle_g y dQ.$$

But by (5)

$$\begin{aligned}
I_\Omega & = -\int_\Omega e^{\varphi} y_t My d\Omega\Big|_S^T \leq C\left(E\left(S\right)+E\left(T\right)\right) \\
& \leq CE\left(T\right) + C\int_S^T\int_{\Gamma_1} e^{\varphi}\left(|f\left(y_t\right)|^2+|y_t|^2\right)d\Sigma
\end{aligned}$$

and we have on Γ_0 (see $[LTY]$)

$$\|Dy\|_g^2 = \left|\frac{\partial y}{\partial n}\right|^2 \text{ and } h(y) = \langle h, Dy\rangle_g = \langle h, n\rangle_g \frac{\partial y}{\partial n},$$

so

$$I_{\Sigma_0} = \int_S^T \int_{\Gamma_0} e^{\varphi} \langle h, n\rangle_g \left|\frac{\partial y}{\partial n}\right|^2 d\Sigma \leq 0.$$

We have, for all $\varepsilon > 0$,

$$\begin{aligned}
I_{\Sigma_1} = & \int_S^T \int_{\Gamma_1} e^{\varphi} \left(\frac{\partial y}{\partial n}\left(2h(y) + (divh - m_0 + h(\varphi))\, y\right) + \langle h, n\rangle_g \left(|y_t|^2 - \|Dy\|_g^2\right)\right) d\Sigma \\
& \leq -2 \int_S^T \int_{\Gamma_1} e^{\varphi} \left(by_t + f(y_t)\right) \langle h, Dy\rangle_g \, d\Sigma \\
& \quad - \int_S^T \int_{\Gamma_1} e^{\varphi} \left(divh - m_0 + h(\varphi)\right)\left(by_t + f(y_t)\right) y d\Sigma \\
& \quad + \sup_{\Gamma_1} \langle h, n\rangle_g \int_S^T \int_{\Gamma_1} e^{\varphi} |y_t|^2 d\Sigma - h_0 \int_S^T \int_{\Gamma_1} \|Dy\|_g^2 d\Sigma \\
& \quad \leq c(\varepsilon) \int_S^T \int_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2\right) d\Sigma \\
& \quad + \left(\varepsilon \sup_{\Gamma_1} \|h\|_g^2 - h_0\right) \int_S^T \int_{\Gamma_1} e^{\varphi} \|Dy\|_g^2 d\Sigma + \varepsilon\delta \int_S^T E(t)\, dt,
\end{aligned}$$

where δ is the constant verifying $\int_{\Gamma_1} e^{\varphi} |y|^2 d\Gamma \leq \delta \int_{\Omega} e^{\varphi} \|Dy\|_g^2 d\Omega$.

Lemma 3 and (5) imply that

$$\begin{aligned}
I_{\Sigma_1} \leq \; & c(\varepsilon) \int_S^T \int_{\Sigma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2\right) d\Sigma \\
& + \left(\varepsilon \sup_{\Gamma_1} \|h\|_g^2 - h_0\right) \int_S^T \int_{\Sigma_1} e^{\varphi} \|Dy\|_g^2 d\Sigma \\
& + \varepsilon\delta (T - S) E(T).
\end{aligned}$$

On the other hand, by $(H4)$, lemma 3 and (5)

$$\begin{aligned} I_Q &= -\int_S^T\int_\Omega e^\varphi \langle Dy, D(divh + h(\varphi))\rangle_g y dQ \\ &\leq \varepsilon c(T-S)E(T) + \int_S^T\int_{\Gamma_1} e^\varphi\left(|f(y_t)|^2 + |y_t|^2\right)d\Sigma \\ &\quad + c(\varepsilon)\int_S^T\int_\Omega e^\varphi |y|^2 dQ \end{aligned}$$

with ε sufficiently small we obtain the result. ■

4 Absorption of the lower order term

To absorb the lower order term in lemma 4 we combine the idea in $[FF]$ with the one in $[LT]$.

Lemma 5. *For all $T - S > T_0$, where T_0 sufficiently large, we have*

$$\int_S^T\int_\Omega e^\varphi |y|^2 dQ \leq C\int_S^T\int_{\Gamma_1} e^\varphi\left(|f(y_t)|^2 + |y_t|^2\right)d\Sigma.$$

Proof. It is sufficient to prove (see $[FF]$) that, for some T_0 large enough, we have

$$\int_0^{T_0}\int_\Omega e^\varphi |y|^2 \leq C\int_0^{T_0}\int_{\Gamma_1} e^\varphi\left(|f(y_t)|^2 + |y_t|^2\right)d\Sigma.$$

We argue by contradiction. Let (y_k) be a sequence of solutions to (P) such that

$$\lim_{k\to\infty}\frac{\int_0^{T_0}\int_{\Gamma_1} e^\varphi\left(|f(y_{kt})|^2 + |y_{kt}|^2\right)d\Sigma}{\int_0^{T_0}\int_\Omega e^\varphi |y_k|^2 dQ} = 0$$

If we put $C_k = \left(\int_0^{T_0}\int_\Omega e^\varphi |y_k|^2 dQ\right)^{\frac{1}{2}}$ and $\overline{y_k} = \frac{y_k}{C_k}$ then we can see that $\overline{y_k}$ is solution of

$$\begin{cases} \overline{y_{ktt}} - \Delta_g\overline{y_k} - \langle D\varphi, D\overline{y_k}\rangle_g = 0 \text{ in }]0,T_0[\times\Omega, \\ \overline{y_k} = 0 \text{ on }]0,T_0[\times\Gamma_0, \\ \frac{\partial\overline{y_k}}{\partial n} + b\overline{y_k} + \frac{1}{C_k}f(y_{kt}) = 0 \text{ on }]0,T_0[\times\Gamma_1, \end{cases}$$

moreover, we have

$$\int_0^{T_0}\int_\Omega e^{\varphi}\left|\overline{y_k}\right|^2 dQ=1 \tag{7}$$

$$\lim_{k\to\infty}\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2+\left|\overline{y_{kt}}\right|^2\right)d\Sigma=0 \tag{8}$$

and

$$\overline{E_k}\left(t\right):=1/2\int_0^{T_0}\int_\Omega e^{\varphi}\left(\left|\overline{y_{kt}}\right|^2+\left\|D\overline{y_k}\right\|_g^2\right)dQ=\frac{E_k\left(t\right)}{C_k^2}$$

where E_k represents the energy of y_k and $\overline{E_k}$ the energy of $\overline{y_k}$.

We have from lemma 4

$$\overline{E_k}\left(T_0\right)\le C\left(\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2+\left|\overline{y_{kt}}\right|^2\right)d\Sigma+1\right). \tag{9}$$

On the other hand, by (5)

$$\overline{E_k}\left(0\right)=\frac{E_k\left(0\right)}{C_k^2}\le\overline{E_k}\left(T_0\right)+C\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2+\left|\overline{y_{kt}}\right|^2\right)d\Sigma.$$

If we use (9) we find

$$\overline{E_k}\left(0\right)\le C\left(\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2+\left|\overline{y_{kt}}\right|^2\right)d\Sigma+1\right)$$

From (8), we obtain that $\left(\overline{E_k}\left(0\right)\right)$ is bounded, then there exists a subsequence $\left(\overline{y_k}\right)$ denoted by the same symbol such that

$$\overline{y_k}\to\overline{y}\text{ weakly}^*\text{ in }L^\infty\left(0,T_0;H^1_{\Gamma_0}\left(\Omega\right)\right),$$

and

$$\overline{y_k}\to\overline{y}\text{ weakly in }L^2\left(]0,T_0[\times\Gamma\right).$$

We shall consider two cases

Case 1 $\overline{y}=0$.

Then $\lim\limits_{k\to\infty}\int_0^{T_0}\int_\Omega e^{\varphi}\left|\overline{y_k}\right|^2 dQ=0$, this contradicts (7).

Case 2 $\overline{y}\neq 0$.

First we have from (8)

$$\lim_{k\to\infty}\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left|\overline{y_{kt}}\right|^2 d\Sigma=\lim_{k\to\infty}\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2 d\Sigma=0,$$

then $z=\overline{y_t}$ is solution of

$$\begin{cases} z_{tt}-\Delta_g z-\langle D\varphi, Dz\rangle_g=0 \text{ in }]0,T_0[\times\Omega, \\ z=0 \text{ on }]0,T_0[\times\Gamma, \\ \frac{\partial z}{\partial n}=0 \text{ on }]0,T_0[\times\Gamma_1. \end{cases}$$

Then, for sufficiently large T_0, we find $z=0$ (see Theorem 8.1 in $[TY]$).

$$\begin{cases} -\Delta_g \overline{y}-\langle D\varphi, D\overline{y}\rangle_g=0 \text{ in }]0,T_0[\times\Omega, \\ \overline{y}=0 \text{ on }]0,T_0[\times\Gamma_0, \\ \frac{\partial \overline{y}}{\partial n}=0 \text{ on }]0,T_0[\times\Gamma_1. \end{cases}$$

If we multiply the first equation by $e^{\varphi}\overline{y}$, integrate over Ω and use Green's formula we find

$$0<\int_\Omega e^{\varphi}\left|\overline{y}\right|^2 d\Omega\leq C\int_\Omega e^{\varphi}\left\|D\overline{y}\right\|_g^2 d\Omega=0.$$

Contradiction. ■

5 Completion of the proof of main theorem

By combining the result of lemma 4 with the one of lemma 5 we obtain, for any value of $T-S>T_0$ where T_0 sufficiently large,

$$E(T)\leq C\int_S^T\int_{\Gamma_1} e^{\varphi}\left(\left|f(y_t)\right|^2+\left|y_t\right|^2\right)d\Sigma. \tag{10}$$

But

$$\begin{aligned} &\int_S^T\int_{\Gamma_1} e^{\varphi}\left(\left|f(y_t)\right|^2+\left|y_t\right|^2\right)d\Sigma \\ =&\int_{|y_t|\geq 1} e^{\varphi}\left(\left|f(y_t)\right|^2+\left|y_t\right|^2\right)d\Sigma+\int_{|y_t|\leq 1} e^{\varphi}\left(\left|f(y_t)\right|^2+\left|y_t\right|^2\right)d\Sigma \\ \leq&\left(c_1^{-1}+c_2\right)\int_{|y_t|\geq 1} e^{\varphi}f(y_t)\,y_t d\Sigma+e^{\varphi^*}\int_{|y_t|\leq 1} l\left(f(y_t)\,y_t\right)d\Sigma. \end{aligned}$$

If we use Jensen's inequality we find

$$\begin{aligned} &\int_S^T\int_{\Gamma_1} e^{\varphi}\left(\left|f(y_t)\right|^2+\left|y_t\right|^2\right)d\Sigma \\ \leq&\left(c_1^{-1}+c_2\right)\int_{|y_t|\geq 1} e^{\varphi}f(y_t)\,y_t d\Sigma \\ &+e^{\varphi^*}mes\Sigma_1 l\left(\frac{1}{mes\Sigma_1}\int_{|y_t|\leq 1} f(y_t)\,y_t d\Sigma\right), \end{aligned}$$

so

$$\int_S^T \int_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma \leq \frac{1}{CK} \left(cI + \widetilde{l} \right) \left(\int_S^T \int_{\Gamma_1} f(y_t)\, y_t d\Sigma \right),$$

where

$$\widetilde{l}(s) = l\left(\frac{s}{mes\Sigma_1} \right),\ K = \frac{1}{Ce^{\varphi^*} mes\Sigma_1} \text{ and } c = \frac{c_1^{-1} + c_2}{e^{\varphi^*} mes\Sigma_1}.$$

C represents the constant in (10).

If we replace in (10) we obtain

$$KE(T) \leq \left(cI + \widetilde{l} \right) \left(\int_S^T \int_{\Gamma_1} f(y_t)\, y_t d\Sigma \right).$$

Since $\left(cI + \widetilde{l} \right)$ is invertible for any positive value of a constant c, we obtain

$$\begin{aligned} \left(cI + \widetilde{l} \right)^{-1} (KE(T)) &\leq \int_S^T \int_{\Gamma_1} f(y_t)\, y_t d\Sigma \\ &\leq \frac{1}{e^{\varphi_*}} \int_S^T \int_{\Gamma_1} e^{\varphi} f(y_t)\, y_t d\Sigma \\ &= \frac{1}{e^{\varphi_*}} (E(S) - E(T)). \end{aligned}$$

If we put $p(s) = e^{\varphi_*} \left(cI + \widetilde{l} \right)^{-1} (Ks)$ then

$$p(E(T)) + E(T) \leq E(S)$$

Finally, the result follows from $[LT]$.

6 Application: The case of the second order hyperbolic equation with variable coefficients and a polynomial growth at the origin of the function feedback

Consider the second order hyperbolic equations with variable coefficients

$$\begin{cases} y_{tt} - \mathcal{A}y - \langle D\psi, Dy \rangle_g = 0 \text{ in } Q, \\ y = 0 \text{ on } \Sigma_0, \\ \frac{\partial y}{\partial v_{\mathcal{A}}} + by_t + \zeta(y_t) = 0 \text{ on } \Sigma_1, \\ y(0) = y_0, y_t(0) = y_1 \text{ in } \Omega. \end{cases} \qquad (P^*)$$

The following result is a consequence of theorem 1.

Theorem 6. *If $\zeta : \mathbb{R} \to \mathbb{R}$ is an increasing function of class C^1 such that for some positive constants M_1 and M_2 we have*

$$M_1 |s|^2 \leq \zeta(s)\, s \leq M_2 |s|^2 : |s| \geq 1 \text{ and } \zeta(s)\, s > 0 \text{ for all } s \neq 0,$$

and for some $\gamma \geq 1$

$$M_3 |s|^{\gamma} \leq |\zeta(s)| \leq M_4 |s|^{\frac{1}{\gamma}} : \ |s| \leq 1, \tag{11}$$

then

$$E(t) \leq Ce^{-\omega t} \text{ if } \gamma = 1$$

and

$$E(t) \leq Ct^{\frac{2}{1-\gamma}} \text{ if } \gamma > 1,$$

where $C, \omega > 0$.

Proof. (P^*) is equivalent to

$$\begin{cases} y_{tt} - \Delta_g y - \langle D\varphi, Dy \rangle_g = 0 \text{ in } Q, \\ y = 0 \text{ on } \Sigma_0, \\ \frac{\partial y}{\partial n} + \|v_{\mathcal{A}}\|_g^{-1} b y_t + f(y_t) = 0 \text{ on } \Sigma_1, \\ y(0) = y_0, y_t(0) = y_1 \text{ in } \Omega. \end{cases}$$

Where $\varphi = \psi - \frac{1}{2} \log \det(g_{ij})$ and $f(s) = \|v_{\mathcal{A}}\|_g^{-1} \zeta(s)$ for all $s \in \mathbb{R}$.

From (11) we have, for some positive constants M_5 and M_6, $M_5 |s|^{\gamma} \leq |f(s)| \leq M_6 |s|^{\frac{1}{\gamma}}$ for $|s| \leq 1$, then $l(s) = \alpha s^m$ with $\alpha = M_5^{\frac{-2}{\gamma+1}} + M_6^{\frac{2\gamma}{\gamma+1}}$ and $m = \frac{2}{\gamma+1}$. Repeating the proof of corollary 2 in $[TY]$ we find the desired result. ■

Remark 7. *Theorem 6 removes the assumption of smallness on $\|\varphi\|_\infty := \sup_{x \in \Omega} \|\nabla \varphi(x)\|$ made in $[Gue]$ to obtain the uniform stabilization of the wave equation defined on an Euclidean domain with constant coefficients.*

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