On existence of embeddings for point-line geometries

Anna Kasikova*

Abstract

We describe a construction of a point-line presheaf on a point-line geometry from a set of presheaves on subspaces of the geometry. Then we combine our construction with theorems of M. Ronan to give a new proof of the fact that all polar spaces of finite rank at least four, and several other Grassmann geometries of spherical buildings, are embeddable in projective spaces.

1 Introduction

Regarding projective embeddings of building Grassmann geometries the following is known. (1) For a number of spherical building geometries, there are constructions of embeddings, specific to each geometry. In particular, part of the proof of the classification theorem of spherical buildings in [18] consists in showing that every polar space of finite rank at least four embeds in a projective space. (2) If $\mathcal{B}$ is a building arising from a (B, N)-pair of a Chevalley group of a finite-dimensional semisimple Lie algebra L, and Γ is a Grassmann geometry of $\mathcal{B}$, then Γ embeds into the projective space of a highest weight module for L [2] (see also [3]).

In the present paper we offer a construction of projective embeddings for a point-line geometry Γ from a collection of embeddings for subspaces of Γ. When

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applied to Grassmannians of spherical buildings of rank at least four, our construction does not require existence of an associated Lie algebra, and does not use the classification of spherical buildings, except for the buildings of type A_n. As an application of our construction, in Section 8 we obtain new proofs of the following theorems. The diagrams are labeled as in [4]; the second subscript is $j \in I$ such that the points of the Grassmannian are the residues of the building of type $I - \{j\}$, where I denotes the type set of the building.

Theorem 1.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a nondegenerate polar space of finite rank at least four with thick lines. Then Γ is embeddable in a projective space.*

Theorem 1.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be one of the following building Grassmannians.*
(i) $\mathsf{F}_{4,1}$ *with embeddable symplecta.*
(ii) $\mathsf{D}_{5,5}$.
(iii) $\mathsf{E}_{n,n}$, $n \in \{6,7,8\}$; $\mathsf{E}_{6,2}$; $\mathsf{E}_{7,1}$.
Suppose that Γ has thick lines. Then Γ is embeddable in a projective space.

2 Presheaves and isomorphisms of presheaves

In this section we describe an abridged version of presheaves of Ronan [14].

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry, let $\mathcal{M}$ be the set of point-line flags of Γ, and let $\mathbb{D}$ be a division ring. A *point-line presheaf* $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ on Γ over $\mathbb{D}$, or just a $\mathbb{D}$-*presheaf* on Γ, is a set of 1-dimensional left vector spaces $\{\mathcal{F}_p | p \in \mathcal{P}\}$ over $\mathbb{D}$, a set of 2-dimensional left vector spaces $\{\mathcal{F}_L | L \in \mathcal{L}\}$ over $\mathbb{D}$, and a set of injective $\mathbb{D}$-linear connecting maps $\{\phi_{pL} : \mathcal{F}_p \to \mathcal{F}_L | (p, L) \in \mathcal{M}\}$. We require that, for every line $L \in \mathcal{L}$, the set $\{(\mathcal{F}_p)\phi_{pL} | p \in L\}$ be the set of all 1-spaces of $\mathcal{F}_L$.

Suppose $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ and $\mathcal{F}' = (\{\mathcal{F}'_p\}_p, \{\mathcal{F}'_L\}_L, \{\phi'_{pL}\}_{pL})$ are two $\mathbb{D}$-presheaves on a point-line geometry Γ. An *isomorphism* of presheaves on Γ, $\psi : \mathcal{F} \to \mathcal{F}'$, is a set of bijective $\mathbb{D}$-linear maps $\psi_z : \mathcal{F}_z \to \mathcal{F}'_z$, indexed by $z \in \mathcal{P} \cup \mathcal{L}$, that commute with the connecting maps. That is, for every point-line flag (p, L) of Γ, we have $\psi_p \circ \phi'_{pL} = \phi_{pL} \circ \psi_L$.

Lemma 2.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. Suppose $\mathbb{D}$ is a division ring, let $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ and $\mathcal{F}' = (\{\mathcal{F}'_p\}_p, \{\mathcal{F}'_L\}_L, \{\phi'_{pL}\}_{pL})$ be $\mathbb{D}$-presheaves on Γ, and let $\psi : \mathcal{F} \to \mathcal{F}'$ be an isomorphism of presheaves. Then the following statements hold.*

(i) ψ is uniquely determined by its point terms, that is for every $L \in \mathcal{L}$ and for every $u \in \mathcal{F}_L$ we have $(u)\psi_L = (u)[\phi_{pL}^{-1} \circ \psi_p \circ \phi'_{pL}]$, where $p \in L$ is such that $u \in \phi_{pL}(\mathcal{F}_p)$.

(ii) ψ is uniquely determined by its line terms, that is for every $p \in \mathcal{P}$, for every $u \in \mathcal{F}_p$ and for every line $L \in \mathcal{L}$ containing p, we have $(u)\psi_p = (u)[\phi_{pL} \circ \psi_L \circ (\phi'_{pL})^{-1}]$.

Proof. Let $L \in \mathcal{L}$ and let $u \in \mathcal{F}_L$. By the definition of a presheaf the images of the maps $\{\phi_{pL} | p \in L\}$ range through the set of all 1-spaces of $\mathcal{F}_L$, therefore there exists $p \in \mathcal{P}$ such that $u \in \phi_{pL}(\mathcal{F}_p)$. By the definition of a presheaf isomorphism $\psi_p \circ \phi'_{pL} = \phi_{pL} \circ \psi_L$, and the map ϕ_{pL} is injective. Therefore $(u)[\phi_{pL}^{-1} \circ \psi_p \circ \phi'_{pL}] =$

$(u)[\phi_{pL}^{-1} \circ \phi_{pL} \circ \psi_L] = (u)\psi_L$. This shows that (i) holds. The proof of (ii) is similar and we omit it. ∎

For a division ring $\mathbb{D}$, we denote $\mathrm{C}(\mathbb{D})$ the center of $\mathbb{D}$ and we denote $\mathrm{C}(\mathbb{D})^\circ$ the multiplicative group of $\mathrm{C}(\mathbb{D})$.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry, suppose $\mathbb{D}$ is a division ring, and let $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ be a presheaf on Γ over $\mathbb{D}$. Suppose $\alpha \in \mathbb{D}$. For every $z \in \mathcal{P} \cup \mathcal{L}$, let $\psi_z = \mathrm{id}_{\mathcal{F}_z}\alpha$ and let $\psi = \{\psi_z | z \in \mathcal{P} \cup \mathcal{L}\}$. We say that the indexed set of maps ψ is *multiplication of the presheaf $\mathcal{F}$ by α* and we write $\psi = \mathrm{id}_{\mathcal{F}}\alpha$; we write $\psi = \mathrm{id}_{\mathcal{F}}$ if $\alpha = 1$.

Lemma 2.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. Suppose $\mathbb{D}$ is a division ring and let $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ be a $\mathbb{D}$-presheaf on Γ. Then, for every $\alpha \in \mathrm{C}(\mathbb{D})^\circ$, $\mathrm{id}_{\mathcal{F}}\alpha$ is an automorphism of $\mathcal{F}$.*

Proof. Let $\alpha \in \mathrm{C}(\mathbb{D})^\circ$ and let $\psi = \{\mathrm{id}_{\mathcal{F}_z}\alpha | z \in \mathcal{P} \cup \mathcal{L}\}$ be multiplication of $\mathcal{F}$ by α. Since $\alpha \in \mathrm{C}(\mathbb{D})^\circ$, the constituent maps of ψ are $\mathbb{D}$-linear and bijective. Suppose (p, L) is a point-line flag of Γ. Since the map ϕ_{pL} is $\mathbb{D}$-linear, $(\mathrm{id}_{\mathcal{F}_p}\alpha) \circ \phi_{pL} = \phi_{pL} \circ (\mathrm{id}_{\mathcal{F}_L}\alpha)$. ∎

Lemma 2.3. *Let V be a left vector space over a division ring $\mathbb{D}$. Suppose that $f : V \to V$ is a bijective $\mathbb{D}$-linear map stabilizing all 1-dimensional subspaces of V. Then the following statements hold.*

(i) If $\dim(V) \geq 2$, then $f = \mathrm{id}_V\alpha$ for some $\alpha \in \mathrm{C}(\mathbb{D})^\circ$.

(ii) If $\dim(V) = 1$ then V can be identified with $\mathbb{D}$, regarded as a left vector space over itself under left multiplication, and f is right multiplication by an element of $\mathbb{D}$. ∎

Proposition 2.4. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. Suppose that $\mathcal{L} \neq \emptyset$ and suppose that the geometry Γ is connected. Let $\mathbb{D}$ be a division ring and suppose $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ is a $\mathbb{D}$-presheaf on Γ. Then the automorphisms of $\mathcal{F}$ are precisely the morphisms $\mathrm{id}_{\mathcal{F}}\alpha$, where $\alpha \in \mathrm{C}(\mathbb{D})^\circ$.*

Proof. By Lemma 2.2, for every $\alpha \in \mathrm{C}(\mathbb{D})^\circ$, multiplication by α is an automorphism of $\mathcal{F}$. To prove the other direction, suppose $\psi = \{\psi_z | z \in \mathcal{P} \cup \mathcal{L}\}$ is an automorphism of $\mathcal{F}$. Then the fact that $\psi = \mathrm{id}_{\mathcal{F}}\alpha$ for some $\alpha \in \mathrm{C}(\mathbb{D})^\circ$ follows from the following three statements and from the connectedness of Γ.

1. *For every line $L \in \mathcal{L}$, there is $\alpha_L \in \mathrm{C}(\mathbb{D})^\circ$ such that $\psi_L = \mathrm{id}_{\mathcal{F}_L}\alpha_L$.* Let $L \in \mathcal{L}$, let $p \in L$, and suppose $\langle v \rangle = (\mathcal{F}_p)\phi_{pL}$. Since ψ_p maps $\mathcal{F}_p$ to $\mathcal{F}_p$, the map ψ_L must map $\langle v \rangle$ to itself. This shows that ψ_L stabilizes all 1-spaces of $\mathcal{F}_L$. Therefore by Lemma 2.3(i) there is $\alpha_L \in \mathrm{C}(\mathbb{D})^\circ$ such that $\psi_L = \mathrm{id}_{\mathcal{F}_L}\alpha_L$.

2. *If $L \in \mathcal{L}$ and $p \in L$, then $\psi_p = \mathrm{id}_{\mathcal{F}_p}\alpha_L$.* This follows from (1) and Lemma 2.1(ii). For $p \in \mathcal{P}$, if $\psi_p = \mathrm{id}_{\mathcal{F}_p}\alpha$ for some $\alpha \in \mathrm{C}(\mathbb{D})^\circ$, then we let $\alpha_p = \alpha$.

3. *If L and M are intersecting lines of Γ, then $\alpha_L = \alpha_M$.* Let L and M be two intersecting lines of Γ and let $p \in L \cap N$. By Step 2 $\alpha_L = \alpha_p = \alpha_M$. ∎

We now define multiplication of a presheaf isomorphism by an element of $\mathrm{C}(\mathbb{D})^\circ$. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. Suppose $\mathbb{D}$ is a division ring, and let $\mathcal{F}$ and $\mathcal{F}'$ be $\mathbb{D}$-presheaves on Γ. Suppose that $\psi : \mathcal{F} \to \mathcal{F}'$ is a presheaf isomorphism and let $\alpha \in \mathbb{D}$. We define the *product* $\psi\alpha$ as the composition of the presheaf isomorphism ψ with $\mathrm{id}_{\mathcal{F}}\alpha$; we call this operation *multiplication of the presheaf isomorphism* ψ by α. We have $\psi\alpha = \{\psi_z\alpha | z \in \mathcal{P} \cup \mathcal{L}\}$. The following is immediate from Lemma 2.2.

Lemma 2.5. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. Suppose $\mathbb{D}$ is a division ring, let $\mathcal{F}$ and $\mathcal{F}'$ be $\mathbb{D}$-presheaves on Γ, and suppose $\psi : \mathcal{F} \to \mathcal{F}'$ is a presheaf isomorphism. Then, for every $\alpha \in \mathrm{C}(\mathbb{D})^\circ$, the product $\psi\alpha$ is a presheaf isomorphism $\mathcal{F} \to \mathcal{F}'$.* ∎

We have the following corollary of Proposition 2.4.

Corollary 2.6. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. Suppose that $\mathcal{L} \neq \emptyset$ and suppose that Γ is connected. Let $\mathbb{D}$ be a division ring, let $\mathcal{F}$ and $\mathcal{F}'$ be $\mathbb{D}$-presheaves on Γ, and suppose $\psi : \mathcal{F} \to \mathcal{F}'$ and $\psi' : \mathcal{F} \to \mathcal{F}'$ are presheaf isomorphisms. Then there exists $\alpha \in \mathrm{C}(\mathbb{D})^\circ$ such that $\psi' = \psi\alpha$.*

Proof. By Proposition 2.4 $\psi' \circ \psi^{-1} = \mathrm{id}_{\mathcal{F}}\alpha$ for some $\alpha \in \mathrm{C}(\mathbb{D})^\circ$. Therefore by the linearity of the constituent maps of ψ we have $\psi' = \psi\alpha$. ∎

3 Additional definitions

Let S be a set. Suppose $Y \subseteq S$ and suppose $\mathcal{X}$ is a set of subsets of S. We define $\mathcal{X}_Y = \{X \in \mathcal{X} | X \cap Y \neq \emptyset\}$ and we say that $\mathcal{X}_Y$ is the $\mathcal{X}$-*shadow* of Y.

Let $G = (V, E)$ be a graph. Suppose $S \subseteq V$. We denote $G|S$ or just S the subgraph of G induced on S. For $v \in V$ we denote $G(v)$ the set of all neighbors of v in G; we have $v \notin G(v)$. A *walk of length n* in G is a sequence of vertices $(v_0, \ldots, v_n)$ such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, \ldots, n-1\}$; the walk w is *circular* if $v_0 = v_n$. We denote $\mathrm{C}(G)$ the set of all circular walks of G. A *path* in a graph is a walk all of whose vertices are pairwise distinct. A *circuit* is a circular walk all of whose vertices, except its initial and its terminal vertices, are pairwise distinct; circuits of length 3 are called *triangles*. Walks of length 1 in G will be called *arcs*. For a walk $w = (v_0, \ldots, v_n)$ we define the *inverse walk* of w to be the walk $w^{-1} = (v_n, \ldots, v_0)$. A *backtrack* is a circular walk of the form $w \circ w^{-1}$, where w is a walk. If $u = (u_0, \ldots, u_m)$ and $v = (v_0, \ldots, v_n)$, are walks in G, and $u_m = v_0$, then the *concatenation* of u and v is the walk $u \circ v = (u_0, \ldots, u_m, v_1, \ldots, v_n)$. Suppose u and v are walks. A *segment* of v is a decomposition $v = v' \circ u \circ v''$, where v' and v'' are walks; we also say that the walk u is a segment of v. For a walk w in G we denote its set of vertices by $\mathrm{supp}_V(w)$. For $x, y \in V$ we denote $\mathrm{d}_G(x, y)$ the length of a shortest walk from x to y in G.

Most of our point-line terminology can be found in Shult [15]. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. We identify every line $L \in \mathcal{L}$ with the subset of $\mathcal{P}$ consisting of the points incident with L. A point-line geometry Γ has *thick lines* if every line of Γ contains at least three points. A *subgeometry* of Γ is a point line geometry $\Gamma' = (\mathcal{P}', \mathcal{L}')$, such that $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, and the graph of Γ' is a subgraph of Γ; we denote $\mathcal{P}(\Gamma') = \mathcal{P}'$ and $\mathcal{L}(\Gamma') = \mathcal{L}'$. Intersection of subgeometries is defined as the intersection of subgraphs. A subgeometry is *full* if, for every $L \in \mathcal{L}'$ and for every $p \in \mathcal{P}$ incident with L in Γ, $p \in \mathcal{P}'$ and p is incident with L in Γ'. A *subspace* S of Γ is a subset of $\mathcal{P}$ such that, for every $L \in \mathcal{L}$, either $|L \cap S| \leq 1$ or $L \subseteq S$. For a subspace S of Γ, we denote $\Gamma|S = (S, \mathcal{L}|S)$ the geometry of all points and lines of Γ contained entirely in S, with the incidence inherited from Γ; we have $\mathcal{L}(\Gamma|S) = \mathcal{L}|S$. We say that $\Gamma|S$ is the subgeometry of Γ induced on S; $\Gamma|S$ is a full subgeometry of Γ. Let $\mathcal{M}$ be the set of all point-line flags of Γ and suppose S is a subgeometry of Γ. We denote $\mathcal{M}(S)$ the set of the point-line flags of S. Suppose $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ is a presheaf on Γ over a division ring $\mathbb{D}$, and let S be a full subgeometry of Γ. We denote $\mathcal{F}|S$ the presheaf $(\{\mathcal{F}_p\}_{p\in\mathcal{P}(S)}, \{\mathcal{F}_L\}_{L\in\mathcal{L}(S)}, \{\phi_{pL}\}_{(p,L)\in\mathcal{M}(S)})$ on $\Gamma|S$. If S is a subspace of Γ, then we write $\mathcal{M}(S)$ for $\mathcal{M}(\Gamma|S)$, and $\mathcal{F}|S$ for $\mathcal{F}|(\Gamma|S)$.

4 The presheaf construction

The purpose of this section is to prove Theorem 4.2, in which we construct a presheaf on a point-line geometry from a collection of presheaves on its subspaces.

4.1 Main hypothesis

We consider the following situation that will be referred to as **Hypothesis (PSh)**.

$\Gamma = (\mathcal{P}, \mathcal{L})$ *is a point-line geometry.*

$\mathcal{S}$ *is a family of full subgeometries of* Γ*, such that every point and every line of* Γ *lies in at least one member of* $\mathcal{S}$*.*

$\mathcal{G} = (\mathcal{S}, \mathcal{E})$ *is a graph with the vertex set* $\mathcal{S}$*, the edge set* $\mathcal{E}$*, and the arc set* $\mathcal{A}$*;* $\mathbb{D}$ *is a division ring and* $\{\mathcal{F}_S | S \in \mathcal{S}\}$ *is a set of* $\mathbb{D}$*-presheaves on the geometries* S*.*

$\Psi = \{\psi_{S,T} | (S,T) \in \mathcal{A}\}$ *is a set of presheaf isomorphisms* $\psi_{S,T} : \mathcal{F}_S|(S \cap T) \to \mathcal{F}_T|(S \cap T)$ *satisfying the following condition*

(PSh-inv) For every arc $(S,T) \in \mathcal{A}$*,* $\psi_{T,S} = \psi_{S,T}^{-1}$

Suppose that hypothesis (PSh) holds. We denote by $\mathcal{M}$ the set of all point-line flags of Γ and by G the point-collinearity graph of Γ. Suppose $S \in \mathcal{S}$. Then we write $\mathcal{F}_S = (\{\mathcal{F}_{S,p}\}_{p\in\mathcal{P}(S)}, \{\mathcal{F}_{S,L}\}_{L\in\mathcal{L}(S)}, \{\phi_{S,pL}\}_{(p,L)\in\mathcal{M}(S)})$. We denote $\mathcal{F}_{S,L,p}$ the 1-dimensional subspace $(\mathcal{F}_{S,p})\phi_{S,pL}$ of $\mathcal{F}_{S,L}$.

For $p \in \mathcal{P}$ we denote $\mathcal{S}(p)$ the set $\{S \in \mathcal{S} | p \in S\}$, and for $X \subseteq \mathcal{P}$ we denote $\mathcal{S}(X)$ the set $\{S \in \mathcal{S} | X \subseteq S\}$. Suppose $(S,T) \in \mathcal{A}$ and suppose $z \in \mathcal{P}(S \cap T) \cup \mathcal{L}(S \cap T)$. Then the term $\mathcal{F}_{S,z} \to \mathcal{F}_{T,z}$ of the presheaf isomorphism $\psi_{S,T}$ will be denoted $\psi_{S,T,z}$. For $\mathcal{X} \subseteq \mathcal{S}$, we let $\mathcal{P}_\mathcal{X} = \cap_{S\in\mathcal{X}}\mathcal{P}(S)$ and we let $\mathcal{P}^\mathcal{X} = \cup_{S\in\mathcal{X}}\mathcal{P}(S)$; we denote $\mathcal{A}_\mathcal{X}$ the set of arcs of the graph $\mathcal{G}|\mathcal{X}$.

Suppose $w = (S_0, \ldots, S_n)$ is a walk in $\mathcal{G}$, let $S = S_0$, and let $\mathcal{X} = \mathrm{supp}_\mathcal{S}(w)$. We let $\mathcal{P}_w = \mathcal{P}_\mathcal{X}$ and $\mathcal{P}^w = \mathcal{P}^\mathcal{X}$. Further, we denote $\Gamma_w = (\mathcal{P}_w, \mathcal{L}_w)$ the geometry $\cap_{S\in\mathcal{X}} S$, and we denote $\mathcal{F}_w$ the presheaf $\mathcal{F}_S|\Gamma_w$. Suppose $z \in \mathcal{P}_w \cup \mathcal{L}_w$. If $n = 0$, we define $\psi_{w,z} = \mathrm{id}_{\mathcal{F}_{S,z}}$. If $n > 0$, then we define $\psi_{w,z} = \psi_{S_0,S_1,z} \circ \cdots \circ \psi_{S_{n-1},S_n,z}$. We let $\psi_w = \{\psi_{w,z} | z \in \mathcal{P}_w \cup \mathcal{L}_w\}$. Suppose that w is a circular walk and that $\mathcal{P}_w \neq \emptyset$. Then ψ_w is an automorphism of the presheaf $\mathcal{F}_w$. For a set of walks $\mathcal{W}$ in $\mathcal{G}$ and for $p \in \mathcal{P}$, we denote $\mathcal{W}_p$ the set of all walks in $\mathcal{W}$ that lie in the subgraph $\mathcal{G}|\mathcal{S}(p)$.

4.2 Additional conditions

Suppose that hypotheses (PSh) holds. We introduce two further conditions.

Suppose $X \in \mathcal{P}$ or $X \subseteq \mathcal{P}$.

(Con$_X$) The graph $\mathcal{G}|\mathcal{S}(X)$ is nonempty and connected.

If (Con$_p$) holds for every $p \in \mathcal{P}$, then we say that (Con$_\mathcal{P}$) holds; if (Con$_L$) holds for every $L \in \mathcal{L}$, then we say that (Con$_\mathcal{L}$) holds.

Suppose that $\mathcal{C}$ is a set of circular walks in $\mathcal{G}$.

(Id$_\mathcal{C}$) For every $w \in \mathcal{C}$, we have $\mathcal{P}_w \neq \emptyset$ and $\psi_w = \mathrm{id}_{\mathcal{F}_w}$.

Let $\mathcal{Z} \in \{\mathcal{P}, \mathcal{L}\}$. If, for every $z \in \mathcal{Z}$, condition (Id$_\mathcal{C}$) holds with $\mathcal{C} = \mathrm{C}(\mathcal{G}|\mathcal{S}(z))$, then we say that condition (Id$_\mathcal{Z}$) holds.

Lemma 4.1. *Suppose that hypothesis (PSh) holds. If condition (Id$_\mathcal{P}$) holds, then condition (Id$_\mathcal{L}$) holds.*

Proof. Suppose $L \in \mathcal{L}$ and suppose w is a circular walk in $\mathcal{G}|\mathcal{S}(L)$. Let $p \in L$. Then $\mathcal{S}(L) \subseteq \mathcal{S}(p)$ and w is a circular walk in $\mathcal{G}|\mathcal{S}(p)$. Therefore, $\psi_w = \mathrm{id}_{\mathcal{F}_w}$. ∎

4.3 The construction

Suppose that hypothesis (PSh) holds. Let $z \in \mathcal{P} \cup \mathcal{L}$. We define $V'_z = \bigoplus\{\mathcal{F}_{S,z} | S \in \mathcal{S}(z)\}$. Let V''_z be the subspace of V'_z spanned by the set of vectors of the form $v - (v)\psi_{S,T,z}$, where (S, T) runs through the arcs of $\mathcal{G}|\mathcal{S}(z)$ and, for each arc (S, T), v runs through the vectors of $\mathcal{F}_{S,z}$. We define $V_z = V'_z / V''_z$ and we denote λ_z the corresponding quotient map $V'_z \to V_z$.

Theorem 4.2. *Suppose that hypothesis (PSh) and conditions (Con$_\mathcal{P}$), (Con$_\mathcal{L}$), and (Id$_\mathcal{P}$) hold. Then the following statements are true.*

(i) For every $p \in \mathcal{P}$, $\dim(V_p) = 1$. For every $L \in \mathcal{L}$, $\dim(V_L) = 2$.

(ii) For every $z \in \mathcal{P} \cup \mathcal{L}$ and for every $S \in \mathcal{S}(z)$, the map $(\lambda_z|\mathcal{F}_{S,z}) : \mathcal{F}_{S,z} \to V_z$ is bijective.

(iii) There exists a $\mathbb{D}$-presheaf $\mathcal{F} = (\{V_p\}_{p\in\mathcal{P}}, \{V_L\}_{L\in\mathcal{L}}, \{\phi_{pL}\}_{(p,L)\in\mathcal{M}})$ on Γ such that, for every $S \in \mathcal{S}$,

$$(\mathcal{F}|S) \cong \mathcal{F}_S$$

More precisely, for every $S \in \mathcal{S}$ and for every flag $(p, L) \in \mathcal{M}(S)$

$$\phi_{pL} = (\lambda_p|\mathcal{F}_{S,p})^{-1} \circ \phi_{S,pL} \circ (\lambda_L|\mathcal{F}_{S,L})$$

Therefore, for every $S \in \mathcal{S}$, the set of maps $\{(\lambda_z|\mathcal{F}_{S,z})|z \in \mathcal{P}(S) \cup \mathcal{L}(S)\}$ is an isomorphism of presheaves $\mathcal{F}_S \to (\mathcal{F}|S)$.

Remark 4.3. For a point-line geometry uniqueness, up to an isomorphism, of an embedding presheaf is equivalent to the existence of an absolutely universal embedding (see Remark 8.8 of Subsection 8.2). When is the presheaf $\mathcal{F}$ of Theorem 4.2 unique? Consider the following hypothesis.

(R1) Suppose the hypothesis of Theorem 4.2 holds. Suppose further that, for every $S \in \mathcal{S}$, $\mathcal{F}_S$ is the embedding presheaf on S unique up to multiplication by elements of

$\mathrm{C}(\mathbb{D})$, *and that the isomorphisms* $\psi_{S,T}$ *are unique up to multiplication by elements of* $\mathrm{C}(\mathbb{D})$.

Let $\mathcal{D}$ be the set of all circular walks of $\mathcal{G}$ lying in the various subgraphs $\mathcal{G}|\mathcal{S}(p)$ and suppose $g \in \mathrm{Hom}_{\mathbb{Z}}(F\mathcal{A}^*, \mathrm{C}(\mathbb{D}))$ is such that

(R2) $(([D]^*)\partial_{[\mathcal{D}]^*})g = 1$ *for all* $D \in \mathcal{D}$.

Define $\Psi' = \{\psi_a\,(a^*)g | a \in \mathcal{A}\}$. Since Ψ satisfies $(\mathrm{Id}_{\mathcal{P}})$ and g satisfies (R2), Ψ' satisfies $(\mathrm{Id}_{\mathcal{P}})$. Denote $\mathcal{F}$ and $\mathcal{F}'$ the presheaves on Γ obtained from Ψ and Ψ'. The presheaves $\mathcal{F}$ and $\mathcal{F}'$ will be isomorphic if and only if

(R3) there exists $h : \mathcal{S} \to \mathrm{C}(\mathbb{D})^\circ$ *such that, for every arc* $a = (x, y)$ *of* $\mathcal{G}$, $(a^*)g = [(x)h]^{-1}(y)h$.

Therefore, under hypothesis (R1), the presheaf $\mathcal{F}$ is unique up to an isomorphism if and only if

(R4) for every $g \in \mathrm{Hom}_{\mathbb{Z}}(F\mathcal{A}^*, \mathrm{C}(\mathbb{D}))$ *condition (R2) implies (R3).*

(Consider the chain $F[\mathcal{D}]^* \to F\mathcal{A}^* \to F\mathcal{S}$ with the arrows being the boundary maps $\partial_{[\mathcal{D}]^*}$ and $\partial_{\mathcal{A}^*}$, where, for an arc $(x, y) \in \mathcal{A}$, $((x, y)^*)\partial_{\mathcal{A}^*} = y - x$. Then (R4) says that every cocycle of $F\mathcal{A}^*$ is a coboundary.) Condition (R4) holds if and only if

(R5) for every circular walk $C \in \mathrm{C}(\mathcal{G})$, $([C]^*)\partial_{[\mathrm{C}(\mathcal{G})]^*} = (\sum_i D_i)\partial_{[\mathrm{C}(\mathcal{G})]^*}$, *where* $D_i \in \mathcal{D}$.

In [10], Theorem 1 Case 1 (with condition (T1) holding) essentially states that, for an embeddable geometry, (R1) and (R5) together imply existence of an absolutely universal embedding (see also Subsection 6.1).

4.4 Proof of Theorem 4.2

First, we prove the following lemma that gives sufficient conditions for the map $(\lambda_z|\mathcal{F}_{S,z}) : \mathcal{F}_{S,z} \to V_z$ to be an epimorphism or a monomorphism.

Lemma 4.4. *Suppose* $G = (X, E)$ *is a graph with the arc set* A, *and suppose that* $\{V_\alpha | \alpha \in X\}$ *is a set of left vector spaces over a division ring* $\mathbb{D}$. *Suppose further that* $\{\psi_{\alpha,\beta} : V_\alpha \to V_\beta | (\alpha, \beta) \in A\}$ *is a set of bijective* $\mathbb{D}$*-linear maps, such that* $\psi_{\alpha,\beta} = \psi^{-1}_{\beta,\alpha}$ *for all* $(\alpha, \beta) \in A$. *Let* $V' = \bigoplus\{V_\alpha | \alpha \in X\}$, *let* $V'' = \langle\{v - (v)\psi_{\alpha,\beta} | (\alpha, \beta) \in A, v \in V_\alpha\}\rangle_{V'}$, *and let* $V = V'/V''$; *we denote* λ *the quotient map. The following statements hold.*

(i) Suppose that the graph G *is connected. Then, for every* $\alpha \in X$, *the map* $(\lambda|V_\alpha) : V_\alpha \to V$ *is surjective and* $\dim(V) \le \dim V_\alpha$.

(ii) Suppose that, for every circular walk $(\alpha_0, \ldots, \alpha_n)$ *of* G, *we have* $\psi_{\alpha_0,\alpha_1} \circ \cdots \circ \psi_{\alpha_{n-1},\alpha_n} = \mathrm{id}_U$, *where* $U = V_{\alpha_0}$. *Then, for every* $\alpha \in X$, *the map* $(\lambda|V_\alpha) : V_\alpha \to V$ *is injective and* $\dim(V) \ge \dim(V_\alpha)$.

Proof. For a walk $w = (x_0, \ldots, x_n)$ we let $\psi_w = \psi_{x_0,x_1} \cdots \psi_{x_{n-1},x_n}$.

(i) Let $\alpha \in X$ and let $u \in V$. We show that there is a vector $v \in V_\alpha$, such that $\lambda(v) = u$. By the definition of the space V, there are $\beta \in X$ and $v' \in V_\beta$, such that $u = (v')\lambda$. By hypothesis the graph G is connected, therefore there is a walk w from β to α in G. Let $v = (v')\psi_w$. Then $v \in V_\alpha$ and $(v)\lambda = (v')\lambda = u$.

(ii) It suffices to consider the case when G is connected. Let $\alpha \in X$ and suppose that $u, v \in V_\alpha$ are distinct. We show that there exists $f \in \mathrm{Hom}_{\mathbb{D}}(V, \mathbb{D})$ such that $((u)\lambda)f \ne ((v)\lambda)f$.

Suppose $f_\alpha \in \mathrm{Hom}_{\mathbb{D}}(V_\alpha, \mathbb{D})$ is such that $(u)f_\alpha \neq (v)f_\alpha$. First, we define $f' \in \mathrm{Hom}_{\mathbb{D}}(V', \mathbb{D})$ by the following rule. Let $\beta \in X$ and let $x \in V_\beta$. Since G is connected, there exists a walk w in G from β to α. Let $x_\alpha = (x)\psi_w$. Then $x_\alpha \in V_\alpha$ and we define $(x)f' = (x_\alpha)f_\alpha$. Suppose w' is another walk from β to α in G, and let $x'_\alpha = (x)\psi_{w'}$. Then $w' \circ w^{-1}$ is a circular walk and by hypothesis $(x)(\psi_{w'} \circ \psi_{w^{-1}}) = x$. Therefore $x'_\alpha = (x)\psi_{w'} = (x)\psi_w = x_\alpha$. This shows that f' is well defined on the elements of $\cup\{V_\alpha | \alpha \in X\}$ and we can extend it to V' by $\mathbb{D}$-linearity. We claim that $V'' \subseteq \mathrm{Ker}(f')$. Let $\beta \in X$, let $x \in V_\beta$, and suppose (β, β') is an arc of G. Then there is a walk from α to β' in G of the from $w \circ (\beta, \beta')$, where w is a walk from α to β in G. Therefore $((x)\psi_{\beta,\beta'})f' = (x)f'$. This shows that $x - (x)\psi_{\beta,\beta'} \in \mathrm{Ker}(f')$ and proves the claim. Since $V'' \subseteq \mathrm{Ker}(f')$, f' induces a map $f \in \mathrm{Hom}_{\mathbb{D}}(V, \mathbb{D})$ and, for every $x \in V_\alpha$, $((x)\lambda)f = (x)f' = (x)f_\alpha$. In particular, $((u)\lambda)f = (u)f_\alpha \neq (v)f_\alpha = ((v)\lambda)f$. ■

Corollary 4.5. *Suppose that hypothesis (PSh) holds. Let $\mathcal{Z} \in \{\mathcal{P}, \mathcal{L}\}$. If $\mathcal{Z} = \mathcal{P}$ then let $k = 1$; if $\mathcal{Z} = \mathcal{L}$ then let $k = 2$.*

(i) Suppose condition (Con$_\mathcal{Z}$) holds. Then, for every $z \in \mathcal{Z}$ and for every $S \in \mathcal{S}(z)$, the map $(\lambda_z|\mathcal{F}_{S,z}) : \mathcal{F}_{S,z} \to V_z$ is surjective and $\dim(V_z) \leq k$.

(ii) Suppose condition (Id$_\mathcal{Z}$) holds. Then, for every $z \in \mathcal{Z}$ and for every $S \in \mathcal{S}(z)$, the map $(\lambda_z|\mathcal{F}_{S,z}) : \mathcal{F}_{S,z} \to V_z$ is injective, and $\dim(V_z) \geq k$. ■

Proof of Theorem 4.2. Parts (i) and (ii) are immediate from Corollary 4.5. To prove (iii) we define a $\mathbb{D}$-presheaf $\mathcal{F} = (\{V_p\}_{p\in\mathcal{P}}, \{V_L\}_{L\in\mathcal{L}}, \{\phi_{pL}\}_{(p,L)\in\mathcal{M}})$ on Γ as follows.

The spaces V_z, $z \in \mathcal{P} \cup \mathcal{L}$, have been defined. Suppose $(p, L) \in \mathcal{M}$. To define ϕ_{pL} we choose $S \in \mathcal{S}(L)$; since S is a full subgeometry, $(p, L) \in \mathcal{M}(S)$. By part (ii) of the present theorem the map $(\lambda_p|\mathcal{F}_{S,p}) : \mathcal{F}_{S,p} \to V_p$ is a bijection, therefore we can define an injective $\mathbb{D}$-linear map $\phi_{pL} : V_p \to V_L$ by

$$\phi_{pL} = (\lambda_p|\mathcal{F}_{S,p})^{-1} \circ \phi_{S,pL} \circ (\lambda_L|\mathcal{F}_{S,L,p}) \tag{4.1}$$

We claim that the map ϕ_{pL} is independent of the choice of $S \in \mathcal{S}(L)$. Let $T \in \mathcal{S}(L)$. By hypothesis the graph $\mathcal{G}|\mathcal{S}(L)$ is connected, therefore it suffices to consider T adjacent to S. Since $\psi_{T,S}$ is a presheaf isomorphism, using the definitions of V_p, V_L, and ϕ_{pL}, we obtain

$$\begin{aligned}
\phi_{T,pL} \circ (\lambda_L|\mathcal{F}_{T,L,p}) &= \psi_{T,S,p} \circ \phi_{S,pL} \circ \psi^{-1}_{T,S,L} \circ (\lambda_L|\mathcal{F}_{T,L,p}) \\
&= \psi_{T,S,p} \circ \phi_{S,pL} \circ (\lambda_L|\mathcal{F}_{S,L,p}) \\
&= \psi_{T,S,p} \circ (\lambda_p|\mathcal{F}_{S,p}) \circ \phi_{pL} \\
&= (\lambda_p|\mathcal{F}_{T,p}) \circ \phi_{pL}
\end{aligned}$$

Therefore $(\lambda_p|\mathcal{F}_{T,p})^{-1} \circ \phi_{T,pL} \circ (\lambda_L|\mathcal{F}_{T,L}) = \phi_{pL}$. ■

5 Conditions implying (Id$_\mathcal{P}$)

In this section we prove Theorem 5.4 and its Corollary 5.5 that describe sufficient conditions under which the set of presheaf isomorphisms Ψ in hypothesis (PSh) can be replaced with a set Ψ' satisfying condition (Id$_\mathcal{P}$). Then we combine Corollary 5.5 with Theorem 4.2 to obtain a criterion for existence of presheaves – Corollary 5.6.

5.1 Associates and nonpointed circular walks. $\mathcal{C}$-homotopy in graphs

In this subsection we mostly follow [15]. Let $G = (V, \mathcal{E})$ be a graph.

Suppose $C = (v_0, \ldots, v_n)$ is a circular walk in G. An *associate* of C is any walk $C_i = (v_i, v_{i+1}, \ldots, v_{n-1}, v_n, v_1, \ldots, v_{i-1}, v_i)$, where $i \in \{0, \ldots, n-1\}$ and the indices are added mod n. We call the set of circular walks $[C] = \{C_0, \ldots, C_{n-1}\}$ the *nonpointed circular walk* corresponding to C or the *oriented association class* of C. We let $\mathrm{supp}_V([C]) = \mathrm{supp}_V(C)$. For a set $\mathcal{C}$ of circular walks in G, we denote $[\mathcal{C}]$ the set of nonpointed circular walks corresponding to the walks in $\mathcal{C}$.

Suppose (w, z) is a pair of walks in G, such that w and z have common initial and terminal vertices, and suppose $w = w_1 \circ w_2 \circ w_3$ and $z = w_1 \circ z_2 \circ w_3$. Then we say that the ordered pair (w, z) is a *deformation* of w by the circular walk $w_2^{-1} \circ z_2$, and that the circular walk $w_2^{-1} \circ z_2$ *corresponds to the deformation* (w, z). The circular walk $w_2^{-1} \circ z_2$ corresponding to the deformation (w, z) is not unique.

Assume now that the graph G is connected.

Suppose that $\mathcal{C}$ is a set of circular walks in G, closed under orientation reversal and taking associates. Let (w, z) be a pair of walks in G such that $w = w_1 \circ w_2 \circ w_3$ and $z = w_1 \circ z_2 \circ w_3$. If $w_2^{-1} \circ z_2 \in \mathcal{C}$ or $w_2^{-1} \circ z_2$ is a backtrack, then we say that (w, z) is an *elementary $\mathcal{C}$-homotopy*. Two walks w and z in $\mathcal{G}$ having common initial and terminal vertices are said to be *$\mathcal{C}$-homotopic* if there is a sequence of elementary $\mathcal{C}$-homotopies $(w_0, w_1), \ldots, (w_{p-1}, w_p)$ with $w_0 = w$ and $w_p = z$. If a circular walk w in G is $\mathcal{C}$-homotopic to the walk of length 0 beginning at its initial vertex, then we say that w is *$\mathcal{C}$-contractible*.

Suppose that F is a subgraph of G and suppose every circular walk in F is $\mathcal{C}$-homotopic in G to a walk of length 0. Then we say that F is *$\mathcal{C}$-contractible* in G. If G itself is $\mathcal{C}$-contractible, then we say that G is *$\mathcal{C}$-simply connected*. If $\mathcal{C}$ is the set of all triangles of G then we omit $\mathcal{C}$ and just say "contractible" and "simply connected".

There is some flexibility in the choice of the definition of $\mathcal{C}$-homotopy. However, in any definition of $\mathcal{C}$-homotopy one has to allow the deformations by backtracks to be $\mathcal{C}$-homotopies since this has the following desired consequence. Suppose $v \in V$ and suppose that all circular walks in G with initial vertex v are $\mathcal{C}$-contractible. Then, using deformations by backtracks, one can show that every circular walk in G is $\mathcal{C}$-contractible (see Section 13.1 of [15]).

5.2 Boundary in graphs

Let $\mathcal{X}$ be a set. We denote $\mathrm{F}\mathcal{X}$ the free $\mathbb{Z}$-module with base $\mathcal{X}$. Suppose that $i : \mathcal{X} \to \mathcal{X}$ is an involution on $\mathcal{X}$, that is i is a bijection and $i^2 = \mathrm{id}_{\mathcal{X}}$. Let M be the submodule of $\mathrm{F}\mathcal{X}$ generated by the set $\{X + (X)i | X \in \mathcal{X}\}$. We define $\mathrm{F}\mathcal{X}^* = \mathrm{F}\mathcal{X}/M$. For $X \in \mathcal{X}$, we denote X^* the image of X in $\mathrm{F}\mathcal{X}^*$.

Let $G = (V, E)$ be a graph with the arc set $\mathcal{A}$.

The map $a \mapsto a^{-1}$ is an involution on $\mathcal{A}$. We let $\mathrm{F}\mathcal{A}^*$ be the corresponding $\mathbb{Z}$-module defined as in the preceding paragraph and, for $a \in \mathcal{A}$, we denote a^* the image of a in $\mathrm{F}\mathcal{A}^*$ under the defining quotient map. For a walk W in G and for $a \in \mathcal{A}$, denote $\mu(W, a)$ the number of segments of W equal to a. Suppose $\mathcal{W}$ is a set of walks in G and consider the module $F\mathcal{W}$. We define the boundary operator

$$\partial_{\mathcal{W}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{W}, \mathrm{F}\mathcal{A}^*)$$

by the rule that, for $X = \Sigma_{W\in\mathcal{W}}\lambda_W W$, $(X)\partial_{\mathcal{W}} = \Sigma_{a\in\mathcal{A}}\mu_a a^*$, where $\mu_a = \Sigma_{W\in\mathcal{W}}\lambda_W\mu(W,a)$.

Let $\mathcal{C}$ be a set of circular walks in G closed under taking inverses and associates. The map $C \mapsto C^{-1}$ is an involution on $\mathcal{C}$ and it induces an involution $i : [C] \mapsto [C^{-1}]$ on the set of nonpointed circular walks $[\mathcal{C}]$. We let $\mathrm{F}[\mathcal{C}]^*$ be the $\mathbb{Z}$-module corresponding to $[C]$ and i. Since $\mathrm{F}[\mathcal{C}] \cong \mathrm{F}\mathcal{C}/M$, where $M = \langle C - C' | C, C' \in \mathcal{C}$ and $[C] = [C']\rangle_{\mathrm{F}\mathcal{C}}$, the map $\partial_{\mathcal{C}}$ induces a map

$$\partial_{[\mathcal{C}]^*} \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathcal{C}]^*, \mathrm{F}\mathcal{A}^*)$$

that can be described by the rule that, for $X = \Sigma_{C\in\mathcal{C}}\lambda_C[C]^*$, $(X)\partial_{[\mathcal{C}]^*} = \Sigma_{a\in\mathcal{A}}\mu_a a^*$ (note that the representation of X is not unique, since it is allowed to contain summands corresponding to walks $C \in \mathcal{C}$ that are associates or inverses of each other). The map $\partial_{[\mathcal{C}]^*}$ is never surjective since, in every circular walk, every vertex is incident with an even number of arcs, while in a single arc a vertex is incident with exactly one arc – an odd number. We denote $\mathrm{F}\mathcal{A}^*_{[\mathcal{C}]^*}$ the image of $\mathrm{F}[\mathcal{C}]^*$ in $\mathrm{F}\mathcal{A}^*$ under $\partial_{[\mathcal{C}]^*}$.

Lemma 5.1. *Let $G = (V, E)$ be a graph and suppose $\mathcal{C}$ is a set of circular walks in G closed under taking inverses and associates. Let $C \in \mathcal{C}$ and suppose that C is $\emptyset$-contractible. Then $([C]^*)\partial_{[\mathcal{C}]^*} = 0$.*

Proof. By hypothesis C is contractible by backtracks. If B is a backtrack then, for every arc a of G, $\mu(B,a) = \mu(B,a^{-1})$. ∎

Lemma 5.2. *Let $G = (V, E)$ be a graph and let $\mathcal{C}$ be a set of circular walks in G closed under taking inverses and associates. Suppose (C_1, C_2) is a $\mathcal{C}$-homotopy corresponding to a walk $C \in \mathcal{C}$. Then $([C_1]^* + [C]^*)\partial_{[\mathcal{C}]^*} = ([C_2]^*)\partial_{[\mathcal{C}]^*}$.* ∎

Let G be a graph and let $\mathcal{C}$ be a set of circular walks in G closed under taking inverses and associates. We extend the definition of supp_V of Section 3 from walks of G to the elements of $\mathrm{F}[\mathcal{C}]^*$. Let $X \in \mathrm{F}[\mathcal{C}]^*$. Then X can be written in the form $X = \Sigma_{C\in\mathcal{X}}\lambda_C[C]^*$, where $\mathcal{X} \subseteq \mathcal{C}$ satisfies the following conditions.

(Supp1) For every $C \in \mathcal{C}$, at most one associate of C is in $\mathcal{X}$.

(Supp2) For every $C \in \mathcal{X}$, $C^{-1} \notin \mathcal{X}$.

(Supp3) For every $C \in \mathcal{X}$, $\lambda_C > 0$.

Any set $\mathcal{X}$ satisfying (Supp1)-(Supp3) will be denoted $\mathrm{supp}_{\mathcal{C}}(X)$, and the sum $\Sigma_{C\in\mathcal{X}}\lambda_C[C]^*$ will be called the *positive expansion of* X. Since $\mathcal{C}$ is a basis for $\mathrm{F}\mathcal{C}$, and since $\lambda_C > 0$, $\mathrm{supp}_{\mathcal{C}}(X)$ is determined by X up to replacing a walk $C \in \mathrm{supp}_{\mathcal{C}}(X)$ by an associate, and the coefficients λ_C are uniquely determined by X. We define $\mathrm{supp}_V(X) = \cup\{\mathrm{supp}_V(C) | C \in \mathrm{supp}_{\mathcal{C}}(X)\}$. If $C, D \in \mathcal{C}$ are associates of each other then $\mathrm{supp}_V(D) = \mathrm{supp}_V(C)$, therefore $\mathrm{supp}_V(X)$ is independent of the choice of the set $\mathrm{supp}_{\mathcal{C}}(X)$.

5.3 Existence of a set of isomorphisms Ψ' satisfying (Id$_{\mathcal{C}}$)

Suppose hypothesis (PSh) holds. Recall that, for a set of circular walks $\mathcal{C}$ in $\mathcal{G}$ and for $p \in \mathcal{P}$, we denote $\mathcal{C}_p = \mathcal{C} \cap \mathrm{C}(\mathcal{G}|\mathcal{S}(p))$. We consider the following additional hypothesis that will be referred to as

Hypothesis (Crc). *There exists a set of circular walks $\mathcal{C}$ in $\mathcal{G}$ closed under taking inverses and associates and satisfying the following conditions.*

(Crc-con) The graph $\mathcal{G}$ is $\mathcal{C}$-simply connected.

(Crc-pnt) For every $p \in \mathcal{P}$, the graph $\mathcal{G}|\mathcal{S}(p)$ is $\mathcal{C}_p$-simply connected.

(Crc-cir) For every $C \in \mathcal{C}$, we have $\mathcal{P}_C \neq \emptyset$ and $\psi_C = \mathrm{id}_{\mathcal{F}_C}\alpha_C$, where $\alpha_C \in \mathrm{C}(\mathbb{D})^\circ$.

(Crc-ker) Let $f_{\Psi,[\mathcal{C}]^*} \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathcal{C}]^*, \mathrm{C}(\mathbb{D})^\circ)$ be defined by $([C]^*)f_{\Psi,[\mathcal{C}]^*} = \alpha_C$ for every $C \in \mathcal{C}$. Then $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f_{\Psi,[\mathcal{C}]^*})$.

Existence of the function $f_{\Psi,[\mathcal{C}]^*}$ in condition (Crc-ker) follows from condition (Crc-cir) and from the following lemma.

Lemma 5.3. *Assume that hypothesis (PSh) holds. Suppose C is a circular walk in $\mathcal{G}$ such that $\mathcal{P}_C \neq \emptyset$, let $p \in \mathcal{P}_C$, and suppose $\psi_{C,p} = \mathrm{id}_V\alpha$ for some $\alpha \in \mathrm{C}(\mathbb{D})^\circ$, where $V = \mathcal{F}_{C,p}$. Then the following statements hold.*

(i) For every associate C' of C, $\psi_{C',p} = \mathrm{id}_W\alpha$, where $W = \mathcal{F}_{C',p}$.

(ii) $\psi_{C^{-1},p} = \mathrm{id}_V\alpha^{-1}$.

Proof. First, we observe that if $C' = C^{-1}$ or $C' \in [C]$, then $p \in \mathcal{P}_{C'}$.

(i) Suppose that $C = (S_0, \dots, S_n)$, where $S_0 = S_n$. Let $C' = (S_i, S_{i+1}, \dots, S_{i-1})$ be an associate of C and let $w = (S_0, \dots, S_i)$. Since the map $\psi_{w,p}$ is $\mathbb{D}$-linear, we obtain

$$\begin{aligned}
\psi_{C',p} &= \psi_{w^{-1},p} \circ \psi_{w,p} \circ \psi_{C',p} \\
&= \psi_{w^{-1},p} \circ \psi_{C,p} \circ \psi_{w,p} \\
&= \psi_{w^{-1},p} \circ (\mathrm{id}_V\alpha) \circ \psi_{w,p} \\
&= (\psi_{w^{-1},p} \circ \psi_{w,p})\alpha \\
&= \mathrm{id}_W\alpha
\end{aligned}$$

(ii) We have $\psi_{C^{-1},p} = (\psi_{C,p})^{-1} = [\mathrm{id}_V\alpha]^{-1} = \mathrm{id}_V\alpha^{-1}$. ∎

Theorem 5.4. *Suppose that hypotheses (PSh) and (Crc) hold, except possibly for the condition (Crc-pnt). Then there exists a set of scalars $\{\alpha_a \in \mathrm{C}(\mathbb{D})^\circ | a \in \mathcal{A}\}$ such that the set of presheaf isomorphisms $\Psi' = \{\psi_a\alpha_a | a \in \mathcal{A}\}$ satisfies condition (PSh-inv) of hypothesis (PSh), and satisfies condition (Id$_{\mathcal{C}}$).*

Corollary 5.5. *Suppose that hypotheses (PSh) and (Crc) hold. Then the set Ψ' of the conclusion of Theorem 5.4 satisfies condition (Id$_{\mathcal{P}}$).*

Theorem 5.4 and Corollary 5.5 will be proved in Subsection 5.5. Combining Corollary 5.5 with Theorem 4.2, and observing that (Crc-pnt) implies (Con$_{\mathcal{P}}$), we immediately obtain the following.

Corollary 5.6. *Suppose that hypotheses (PSh), (Con$_{\mathcal{L}}$), and (Crc) hold. Then there exists a $\mathbb{D}$-presheaf $\mathcal{F}$ on Γ such that $\mathcal{F}|\mathcal{P}_S \cong \mathcal{F}_S$ for every $S \in \mathcal{S}$.* ∎

5.4 Boundary and functionals

The main purpose of this subsection is to prove the following.

Proposition 5.7. *Let $G = (V, E)$ be a connected graph and let $\mathcal{C}$ be a set of circular walks in G closed under taking inverses and associates, and suppose that G is $\mathcal{C}$-simply connected. Let A be an abelian group and suppose $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathcal{C}]^*, A)$. Then $f = \partial_{[\mathcal{C}]^*} \circ g$ for some $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}^*, A)$ if and only if $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$.*

First, we prove three lemmas, the first of which is meant for use in Subsection 6.4.

Lemma 5.8. *Let $G = (V, E)$ be a graph with the set of arcs $\mathcal{A}$, and let $\mathcal{C}$ be a set of circular walks in G closed under taking inverses and associates.*

(i) Suppose $X \in \mathrm{F}[\mathcal{C}]^*$ and $X = \Sigma_{C\in\mathcal{C}}\lambda_C[C]^*$. Then $X \in \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$ if and only if $\Sigma_{C\in\mathcal{C}}\lambda_C\mu(C, a) = \Sigma_{C\in\mathcal{C}}\lambda_C\mu(C, a^{-1})$ for every $a \in \mathcal{A}$.

(ii) Suppose A is an abelian group and suppose $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathcal{C}]^*, A)$. Then $f = \partial_{[\mathcal{C}]^*} \circ g$ for some $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}^*_{[\mathcal{C}]^*}, A)$ if and only if $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$.

Proof. The proof of (i) is a straightforward verification and we omit it. If $f = \partial_{[\mathcal{C}]^*} \circ g$ for some $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}^*_{[\mathcal{C}]^*}, A)$ then $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$. Suppose $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$. Let $g : \mathrm{F}\mathcal{A}^*_{[\mathcal{C}]^*} \to A$ be defined by the rule that, for every $x \in \mathrm{F}\mathcal{A}^*_{[\mathcal{C}]^*}$, $(x)g = (X)f$, where $X \in \mathrm{F}[\mathcal{C}]^*$ is such that $(X)\partial_{[\mathcal{C}]^*} = x$. Since $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$, the map g is well defined. We have $f = \partial_{[\mathcal{C}]^*} \circ g$. Since f and $\partial_{[\mathcal{C}]^*}$ are both $\mathbb{Z}$-linear, g is $\mathbb{Z}$-linear. ■

Lemma 5.9. *Let $G = (V, E)$ be a connected graph with the set of arcs $\mathcal{A}$. Then there is a map $\gamma : \mathcal{A} \to \mathrm{C}(G)$, taking each $a \in \mathcal{A}$ to a circular walk $C(a)$, whose initial vertex is the terminal vertex of a, such that,*

(i) for every $a \in \mathcal{A}$, $[C(a^{-1})] = [C(a)^{-1}]$;

(ii) for every $C \in \mathrm{C}(G)$ with the sequence of arcs $(a_1, \ldots, a_n)$, the walk $a_1 \circ C(a_1) \circ \cdots \circ a_n \circ C(a_n)$ is $\emptyset$-contractible.

Proof. Choose a vertex $u \in V$. For every vertex $v \in V$, choose a walk $\mathrm{w}(u, v)$ from u to v in G; the latter is possible by the connectedness of G. For every arc $a = (x, y) \in \mathcal{A}$, let

$$C(a) = (y, x) \circ \mathrm{w}(u, x)^{-1} \circ \mathrm{w}(u, y)$$ ■

Lemma 5.10. *Let $G = (V, E)$ be a connected graph with the set of arcs $\mathcal{A}$, let A be an abelian group, and suppose $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathrm{C}(G)]^*, A)$ is such that $\mathrm{Ker}(\partial_{[\mathrm{C}(G)]^*}) \subseteq \mathrm{Ker}(f)$. Then there exists $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}^*, A)$ such that $f = \partial_{[\mathrm{C}(G)]^*} \circ g$.*

Proof. We write the group A additively. For every $a \in \mathcal{A}$, let $C(a)$ be as in Lemma 5.9. Then $[C(a^{-1})] = [C(a)^{-1}]$, therefore $([C(a^{-1})]^*)f = ([C(a)^{-1}]^*)f = -([C(a)]^*)f$. We define g by $(a^*)g = ([C(a)]^*)f$ for all $a \in \mathcal{A}$ and we claim that $f = \partial_{[\mathrm{C}(G)]^*} \circ g$.

Let $C \in \mathrm{C}(G)$ be arbitrary. Then by Lemma 5.9(ii) and by Lemma 5.1 $([C]^*)\partial_{[\mathrm{C}(G)]^*} = (\sum_a [C(a)]^*)\partial_{[\mathrm{C}(G)]^*}$, where a runs through the arcs of C. By hypothesis $\mathrm{Ker}(\partial_{[\mathrm{C}(G)]^*}) \subseteq \mathrm{Ker}(f)$, therefore

$$([C]^*)f = (\sum_a [C(a)]^*)f = \sum_a ([C(a)]^*)f = \sum_a (a^*)g$$ ■

Proof of Proposition 5.7. We write the group A additively and we view the module $\mathrm{F}[\mathcal{C}]^*$ as a submodule of $\mathrm{F}[\mathrm{C}(G)]^*$. If $f = \partial_{[\mathcal{C}]^*} \circ g$ for some $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}^*, A)$, then $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$. Suppose that $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$. We need to construct $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}^*, A)$ such that $f = \partial_{[\mathcal{C}]^*} \circ g$. First, we prove the following claim.

There exists $f' \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathrm{C}(G)]^*, A)$ such that $f'|\mathrm{F}[\mathcal{C}]^* = f$ and $\mathrm{Ker}(\partial_{[\mathrm{C}(G)]^*}) \subseteq \mathrm{Ker}(f')$.

We define $([\mathrm{C}]^*)f'$ for all $C \in \mathrm{C}(G)$ and extend the definition to $\mathrm{F}[\mathrm{C}(G)]^*$ by $\mathbb{Z}$-linearity. First, we define $f'' : [\mathrm{C}(G)] \to A$ as follows. Let $C \in \mathrm{C}(G)$ and let v denote the initial vertex of C. By hypothesis the graph G is $\mathcal{C}$-simply connected, therefore there exists a sequence of elementary $\mathcal{C}$ homotopies s that begins with (v) and ends with C. Let $(C_1, \dots, C_n)$ be a sequence of circular walks in $\mathcal{C}$ corresponding to the homotopies in s. We define

$$([\mathrm{C}])f'' = ([C_1]^*)f + \cdots + ([C_n]^*)f \tag{5.1}$$

Suppose C' is an associate of C, let s' be a sequence of elementary $\mathcal{C}$-homotopies beginning with a walk of length 0 and ending with C', and suppose $(D_1, \dots, D_m)$ is a sequence of circular walks in $\mathcal{C}$ corresponding to s'. Then by Lemma 5.2 $([C_1]^* + \cdots + [C_n]^*)\partial_{[\mathcal{C}]^*} = ([\mathrm{C}]^*)\partial_{[\mathcal{C}]^*} = ([\mathrm{C}']^*)\partial_{[\mathcal{C}]^*} = ([D_1]^* + \cdots + [D_m]^*)\partial_{[\mathcal{C}]^*}$. By hypothesis $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$, therefore $([C_1]^* + \dots [C_n]^*)f = ([D_1]^* + \dots [D_m]^*)f$. Since f is $\mathbb{Z}$-linear, this implies that $([C_1]^*)f + \cdots + ([C_n]^*)f = ([D_1]^*)f + \cdots + ([D_m]^*)f$. This shows that f'' is well defined. By equation 5.1, for every $C \in \mathrm{C}(G)$, $([C^{-1}])f'' = -([\mathrm{C}])f''$, therefore f'' induces a map $[\mathrm{C}(G)]^* \to A$. We define $f' \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathrm{C}(G)]^*, A)$ as the $\mathbb{Z}$-linear extension of the last map. We have $f'|\mathrm{F}[\mathcal{C}]^* = f$.

To finish the proof of the claim it remains to show that $\mathrm{Ker}(\partial_{[\mathrm{C}(G)]^*}) \subseteq \mathrm{Ker}(f')$. Let $X \in \mathrm{Ker}(\partial_{[\mathrm{C}(G)]^*})$ and suppose that $X = \sum_{i \in I} \lambda_i [C_i]^*$, where $C_i \in \mathrm{C}(G)$ and I is a finite set. By hypothesis the graph G is $\mathcal{C}$-simply connected. Therefore, for every $\in I$, there is a sequence of elementary $\mathcal{C}$-homotopies starting with a walk of length 0 and ending with C_i; let $\{C_{i,j} \in \mathcal{C} | j \in J_i\}$ be a corresponding set of circular walks in $\mathcal{C}$. By Lemma 5.2, for every $i \in I$, $([C_i]^*)\partial_{[\mathrm{C}(G)]^*} = (\sum_{j \in J_i} [C_{i,j}]^*)\partial_{[\mathrm{C}(G)]^*}$. Since $X \in \mathrm{Ker}(\partial_{[\mathrm{C}(G)]^*})$, $\sum_{i,j} \lambda_i [C_{i,j}]^* \in \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$. Since $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f)$, using the $\mathbb{Z}$-linearity of f' and f we obtain

$$(X)f' = \sum_i \lambda_i([C_i]^*)f' = \sum_{i,j} \lambda_i([C_{i,j}]^*)f = (\sum_{i,j} \lambda_i [C_{i,j}]^*)f = 0$$

Let f' be as in the claim. By Lemma 5.10 there exists $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}^*, A)$ such that $f' = \partial_{[\mathrm{C}(G)]^*} \circ g$. Therefore $f = f'|\mathrm{F}[\mathcal{C}]^* = (\partial_{[\mathrm{C}(G)]^*} \circ g)|\mathrm{F}[\mathcal{C}]^* = \partial_{[\mathcal{C}]^*} \circ g$. ■

5.5 Proof of Theorem 5.4

Proof of Theorem 5.4. We write the abelian group $\mathrm{C}(\mathbb{D})^\circ$ multiplicatively. We define the set $\{\alpha_a | a \in \mathcal{A}\}$ as follows. By hypothesis the graph $\mathcal{G}$ is $\mathcal{C}$-simply connected and by (Crc-ker) $\mathrm{Ker}(\partial_{[\mathcal{C}]^*}) \subseteq \mathrm{Ker}(f_{\Psi,[\mathcal{C}]^*})$. Therefore by Proposition 5.7 there exists $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}^*, \mathrm{C}(\mathbb{D})^\circ)$ such that

$$f_{\Psi,[\mathcal{C}]^*} = \partial_{[\mathcal{C}]^*} \circ g \tag{5.2}$$

For every $a \in \mathcal{A}$, define $\alpha_a = [(a)g]^{-1}$. We claim that the set $\Psi' = \{\psi_a \alpha_a | a \in \mathcal{A}\}$ satisfies condition (Id$_{\mathcal{C}}$).

For $C \in \mathrm{C}(\mathcal{G})$, let $\psi'_C = \{\psi'_{C,z} | z \in \mathcal{P}_C \cup \mathcal{L}_C\}$, where $\psi'_{C,z}$ are defined similarly to $\psi_{C,z}$ but with the set Ψ replaced with Ψ'. Let $C \in \mathcal{C}$ and suppose that C has length $n \geq 2$ and the sequence of arcs $a_0, \ldots, a_{n-1}$. For all $i \in \{0, \ldots, n-1\}$, let $\alpha_i = \alpha_{a_i}$ and, for $p \in \mathcal{P}_C$, let $\psi_{i,p} = \psi_{a_i,p}$; then $\psi'_{C,p} = \psi_{1,p}\alpha_1 \circ \cdots \circ \psi_{n,p}\alpha_n$. By condition (Crc-cir) $\mathcal{P}_C \neq \emptyset$. By Lemma 2.1(i) to show that $\psi'_C = \mathrm{id}_{\mathcal{F}_C}$ it suffices to show that, for every $p \in \mathcal{P}_C$, $\psi'_{C,p} = \mathrm{id}_{\mathcal{F}_{C,p}}$. Let $p \in \mathcal{P}_C$ and let $V = \mathcal{F}_{C,p}$. By condition (Crc-cir) and by equation 5.2 we have $\psi_{C,p} = \mathrm{id}_V \Pi_{i=0}^{n-1} \alpha_i^{-1}$. Therefore by the $\mathbb{D}$-linearity of the maps ψ_i we obtain

$$\begin{aligned} \psi'_{C,p} &= \psi_{0,p}\,\alpha_0 \circ \cdots \circ \psi_{n-1,p}\,\alpha_{n-1} \\ &= \psi_{C,p} \Pi_{i=0}^{n-1} \alpha_i \\ &= (\mathrm{id}_V \Pi_{i=0}^{n-1} \alpha_i^{-1}) \Pi_{i=0}^{n-1} \alpha_i \\ &= \mathrm{id}_V \end{aligned}$$ ∎

When the conclusion of Theorem 5.4 holds, for $C \in \mathrm{C}(\mathcal{G})$, we let ψ'_C be defined as in the proof of Theorem 5.4. To prove Corollary 5.5 we use the following lemma.

Lemma 5.11. *Suppose that hypothesis (PSh) holds. Let $p \in \mathcal{P}$, let $G = \mathcal{G}|\mathcal{S}(p)$, an let C and D be circular walks in G. Suppose that (C, D) is a deformation by a circular walk E, and suppose that $\psi_{C,p} = \mathrm{id}_{\mathcal{F}_{C,p}} \alpha_{C,p}$ and $\psi_{E,p} = \mathrm{id}_{\mathcal{F}_{E,p}} \alpha_{E,p}$, where $\alpha_{C,p}, \alpha_{E,p} \in \mathrm{C}(\mathbb{D})^{\circ}$. Then $\psi_{D,p} = \mathrm{id}_{\mathcal{F}_{D,p}}(\alpha_{C,p}\alpha_{E,p})$.*

Proof. Suppose that $C = C_1 \circ C_2 \circ C_3$, $D = C_1 \circ C'_2 \circ C_3$, and $E = C_2^{-1} \circ C'_2$. For $i \in \{1, 2, 3\}$ let $\psi_i = \psi_{C_i,p}$ and let $\psi'_2 = \psi_{C'_2,p}$. We have

$$\begin{aligned} \psi_{D,p} &= \psi_1 \circ \psi'_2 \circ \psi_3 \\ &= \psi_1 \circ \psi_2 \circ \psi_{C_2^{-1},p} \circ \psi'_2 \circ \psi_3 \\ &= \psi_1 \circ \psi_2 \circ \psi_{E,p} \circ \psi_3 \\ &= \psi_1 \circ \psi_2 \circ (\mathrm{id}_{\mathcal{F}_{E,p}} \alpha_{E,p}) \circ \psi_3 \\ &= (\psi_1 \circ \psi_2 \circ \psi_3) \alpha_{E,p} \\ &= \psi_{C,p} \alpha_{E,p} \\ &= \mathrm{id}_{\mathcal{F}_{C,p}} \alpha_{C,p} \alpha_{E,p} \end{aligned}$$ ∎

Proof of Corollary 5.5. Suppose $C \in \mathrm{C}(\mathcal{G})$ and $\mathcal{P}_C \neq \emptyset$. We need to show that $\psi'_C = \mathrm{id}_{\mathcal{F}_C}$. By Lemma 2.1(i) it suffices to show that, for every $p \in \mathcal{P}_C$, $\psi'_{C,p} = \mathrm{id}_V$ where $V = \mathcal{F}_{C,p}$.

Let $p \in \mathcal{P}_C$ and let $V = \mathcal{F}_{C,p}$. By hypothesis the graph $\mathcal{G}|\mathcal{S}(p)$ is $\mathcal{C}_p$-simply connected, therefore there exists a sequence s of elementary $\mathcal{C}_p$-homotopies beginning with a walk of length 0 and ending with C. Let $(C_1, \ldots, C_n)$ be a sequence of circular walks corresponding to the homotopies in s. By the definition of Ψ', for every $D \in \{C_1, \ldots, C_n\}$, $\psi_{D,p} = \mathrm{id}_W$, where $W = \mathcal{F}_{D,p}$. Therefore by Lemma 5.11 $\psi'_{C,p} = \mathrm{id}_V$. ∎

6 Conditions implying (Crc-cir) and (Crc-ker)

Suppose hypothesis (PSh) holds. In this section we state conditions which imply conditions (Crc-cir) and (Crc-ker).

6.1 Statement of results

Suppose that hypothesis (PSh) holds. Let $\mathcal{X} \subseteq \mathcal{S}$. We consider the following conditions.

(PX-l) The geometry $\Gamma|\mathcal{P}_{\mathcal{X}}$ *contains a line and is connected.*

(PX-e) For every arc $(S,T) \in \mathcal{A}_{\mathcal{X}}$*, the isomorphism of presheaves* $\mathcal{F}_S|(S \cap T) \to \mathcal{F}_T|(S \cap T)$ *is unique up to multiplication by an element of* $\mathrm{C}(\mathbb{D})^{\circ}$*, and the geometry* $\Gamma|\mathcal{P}^{\mathcal{X}}$ *has an embedding such that, for every* $S \in \mathcal{X}$*, the corresponding embedding presheaf is isomorphic to* $\mathcal{F}_S$*.*

Proposition 6.1. *Suppose that hypotheses (PSh) holds. Let* C *be a circular walk in* $\mathcal{G}$ *such that* $\mathcal{P}_C \neq \emptyset$*.*

(i) If $\mathrm{supp}_{\mathcal{S}}(C)$ *satisfies (PX-l), then* $\psi_C = \mathrm{id}_{\mathcal{F}_C}\alpha_C$*, where* $\alpha_C \in \mathrm{C}(\mathbb{D})^{\circ}$*.*

(ii) If $\mathrm{supp}_{\mathcal{S}}(C)$ *satisfies (PX-e), then* $\psi_C = \mathrm{id}_{\mathcal{F}_C}\alpha_C$*, where* $\alpha_C \in \mathrm{C}(\mathbb{D})^{\circ}$*.*

To state the second result we need a definition. Suppose that hypothesis (PSh) holds. Let $\mathcal{C}$ be a set of circular walks in $\mathcal{G}$, closed under taking associates and orientation reversal. For $X \in \mathrm{F}[\mathcal{C}]^*$, we define

$$\mathcal{P}_X = \cap\{S|S \in \mathrm{supp}_{\mathcal{S}}(X)\} \qquad \mathcal{P}^X = \cup\{S|S \in \mathrm{supp}_{\mathcal{S}}(X)\}$$

Proposition 6.2. *Suppose that hypotheses (PSh) holds, and suppose* $\mathcal{C}$ *is a set of circular walks in* $\mathcal{G}$*, closed under orientation reversal and taking associates, and satisfying (Crc-cir). Let* $f_{\Psi,[\mathcal{C}]^*} \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathcal{C}]^*, \mathrm{C}(\mathbb{D})^{\circ})$ *be defined as in (Crc-ker) and let* $X \in \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$*.*

(i) If $\mathcal{P}_X \neq \emptyset$ *and (Crc-pnt) holds, then* $X \in \mathrm{Ker}(f_{\Psi,[\mathcal{C}]^*})$*.*

(ii) If $\mathrm{supp}_{\mathcal{S}}(X)$ *satisfies (PX-e), then* $X \in \mathrm{Ker}(f_{\Psi,[\mathcal{C}]^*})$*.*

Propositions 6.1 and 6.2 will be proved in Section 6.4 after some preparation.

6.2 Lemma for the case $\mathcal{P}_X \neq \emptyset$

We consider the following **Hypothesis (Ctg)**:

$G = (V, E)$ is a connected graph with the set of arcs $\mathcal{A}$, V is a set of objects of a category $\mathcal{X}$, and f is a function, assigning to each arc $(x, y) \in \mathcal{A}$ an isomorphism $x \to y$ in $\mathcal{X}$, and satisfying condition (Ctg-inv):

(Ctg-inv) For all $a \in \mathcal{A}$, $(a^{-1})f = [(a)f]^{-1}$.

Suppose (Ctg) holds, except possibly for (Ctg-inv), and let w be a walk in G with the sequence of arcs $a_1, \ldots, a_n$. We define $(w)f = (a_1)f \circ \cdots \circ (a_n)f$. Since the composition is associative, $(w)f$ is well defined and, for any triple of walks w, w_1, w_2 in G such that $w = w_1 \circ w_2$, we have $(w)f = (w_1)f \circ (w_2)f$.

Lemma 6.3. *Suppose hypothesis (Ctg) holds. Then there exists a function g, assigning to each arc $(x, y) \in \mathcal{A}$ an automorphism of y in $\mathcal{X}$, and having the following properties.*

(i) For every $a \in \mathcal{A}$, $(a)g = (C(a))f$, where $C(a)$ is a circular walk in G beginning at the terminal vertex of a, and $[C(a^{-1})] = [C(a)^{-1}]$;

(ii) For every $a \in \mathcal{A}$, define $(a)f' = (a)f \circ (a)g$. Then, for every circular walk C in G, $(C)f' = \mathrm{id}_x$, where x denotes the initial vertex of C.

Proof. For every $a \in \mathcal{A}$, let $C(a)$ be as in the conclusion of Lemma 5.9. Define g by $(a)g = (C(a))f$ for all $a \in \mathcal{A}$ and let f' be defined as in (ii). Then part (i) holds by the definition of $C(a)$ (see Lemma 5.9(i)). To prove part (ii) let $C \in \mathrm{C}(G)$ and suppose $(a_1, \ldots, a_n)$ is the sequence of arcs of C. Let $C' = a_1 \circ C(a_1) \circ \cdots \circ a_n \circ C(a_n)$ and let $(b_1, \ldots, b_m)$ be the sequence of arcs of C'. Then

$$(C)f' = (a_1)f \circ (C(a_1))f \circ \cdots \circ (a_n)f \circ (C(a_n))f = (b_1)f \circ \cdots \circ (b_m)f \qquad (6.1)$$

By the definition of the walks $C(a)$ the walk C' is $\emptyset$-contractible (see Lemma 5.9(ii)), and by hypothesis f satisfies (Ctg-inv). Therefore $(b_1)f \circ \cdots \circ (b_m)f = \mathrm{id}_x$, where x is the initial vertex of C. ■

6.3 Lemma for the case (PX-e)

Lemma 6.4. *Suppose that hypothesis (PSh) holds and suppose that $\mathcal{X} \subseteq \mathcal{S}$ satisfies condition (PX-e). Then there exists a map $g : \mathcal{A}_{\mathcal{X}} \to \mathrm{C}(\mathbb{D})^\circ$ with the following properties.*

(i) For every $a \in \mathcal{A}_{\mathcal{X}}$, $(a^{-1})g = ((a)g)^{-1}$.

(ii) For every circular walk C in $\mathcal{G}|\mathcal{X}$ with the sequence of arcs $a_1, \ldots, a_n$ and such that $\mathcal{P}_C \neq \emptyset$, we have $\psi_C = \mathrm{id}_{\mathcal{F}_C}[\Pi_{i=1}^n (a_i)g]$.

Proof. We construct the map $g : \mathcal{A}_{\mathcal{X}} \to \mathrm{C}(\mathbb{D})^\circ$ as follows. By condition (PX-e) the geometry $\Gamma|\mathcal{P}^{\mathcal{X}}$ has an embedding presheaf $\mathcal{F}'$ such that, for every $S \in \mathcal{X}$, there is a presheaf isomorphism $\theta_S : \mathcal{F}_S \to \mathcal{F}'_S$. For every arc $(S, T) \in \mathcal{A}_{\mathcal{X}}$, let the presheaf isomorphism $\psi'_{S,T} : \mathcal{F}_S|(S \cap T) \to \mathcal{F}_T|(S \cap T)$ be defined by the rule that, for every $z \in (S \cap T) \cup [\mathcal{L}|(S \cap T)]$,

$$\psi'_{S,T,z} = \theta_{S,z} \circ (\theta_{T,z})^{-1}$$

By condition (PX-e) there is $\alpha_{S,T} \in \mathrm{C}(\mathbb{D})^\circ$ such that $\psi_{S,T} = \psi'_{S,T}\alpha_{S,T}$. We let the map $g : \mathcal{A}_{\mathcal{X}} \to \mathrm{C}(\mathbb{D})^\circ$ be defined by $(a)g = \alpha_a$ for every $a \in \mathcal{A}_{\mathcal{X}}$.

By (PSh-inv), for every $a \in \mathcal{A}_{\mathcal{X}}$, $\psi_{a^{-1}} = \psi_a^{-1}$. By the definition of ψ'_a we have $\psi'_{a^{-1}} = (\psi'_a)^{-1}$. Therefore $\alpha_{a^{-1}} = (\alpha_a)^{-1}$, that is (i) holds. Let $C = (S_0, \ldots, S_n)$ be as in (ii) and let $p \in \mathcal{P}_C$. Then

$$\psi'_{C,p} = (\theta_{S_0,p} \circ \theta_{S_1,p}^{-1}) \circ \cdots \circ (\theta_{S_{n-1},p} \circ \theta_{S_n,p}^{-1}) = \theta_{S_0,p} \circ \theta_{S_n,p}^{-1} = \mathrm{id}_{\mathcal{F}_{C,p}}$$

Therefore by the $\mathbb{Z}$-linearity of the maps $\psi_{S,T,z}$, we obtain

$$\psi_C = \psi'_C[\Pi_{i=1}^n (a_i)g] = \mathrm{id}_{\mathcal{F}_C}[\Pi_{i=1}^n g(a_i)]$$

■

6.4 Proofs of Propositions 6.1 and 6.2

Proof of Proposition 6.1. Let $\mathcal{X} = \mathrm{supp}_{\mathcal{S}}(C)$. If $\mathcal{X}$ satisfies condition (PX-l), then the conclusion follows from Proposition 2.4. If $\mathcal{X}$ satisfies (PX-e), then the conclusion follows from Lemma 6.4(ii). ∎

Proof of Proposition 6.2. We write the group $\mathrm{C}(\mathbb{D})^\circ$ multiplicatively. Let $\mathcal{X} = \mathrm{supp}_{\mathcal{S}}(X)$, let $G' = \mathcal{G}|\mathcal{X}$, and let $\mathcal{C}_{\mathcal{X}} = \mathcal{C} \cap \mathrm{C}(G')$. Since $X \in \mathrm{Ker}(\partial_{[\mathcal{C}_{\mathcal{X}}]^*})$, by Lemma 5.8(ii) to prove that $(X)f_{\Psi,[\mathcal{C}]^*} = 1$ it suffices to construct a map $g : \mathcal{A}_{\mathcal{X}} \to \mathrm{C}(\mathbb{D})^\circ$ such that (1) for every $a \in \mathcal{A}_{\mathcal{X}}$, $(a^{-1})g = (a)g^{-1}$ and (2) for every $C \in \mathcal{X}$ with the sequence of arcs $a_1, \ldots, a_n$, $\psi_C = \mathrm{id}_{\mathcal{F}_C}[\Pi_{i=1}^n(a_i)g]$. In case (ii) the map g exists by Lemma 6.4. To prove existence of the map g in case (i) we use Lemma 6.3.

Choose $p \in \mathcal{P}_X$. Let $G = \mathcal{G}|\mathcal{S}(p)$, let $\mathcal{A}_G$ be the set of arcs of G, and let $\Psi_G = \{\psi_{a,p}|a \in \mathcal{A}_G\}$. We apply Lemma 6.3 to G and Ψ_G. For every $a \in \mathcal{A}_G$, let $C(a)$, be as in the conclusion of the lemma, but denote the function of the conclusion defined on $\mathcal{A}_G$ by g' instead of g. For every $a \in \mathcal{A}_G$, we define $\psi'_{a,p} = \psi_{a,p} \circ (a)g'$. We claim that, for every $a \in \mathcal{A}_G$, $(a)g' = \mathrm{id}_{\mathcal{F}_{C(a),p}}\alpha_a$ for some $\alpha_a \in \mathrm{C}(\mathbb{D})^\circ$. By condition (Crc-cir), for every $C \in \mathcal{C}$, we have $\psi_{C,p} = \mathrm{id}_{\mathcal{F}_{C,p}}\alpha_C$. By condition (Crc-pnt) every $C \in \mathrm{C}(G)$ is $\mathcal{C}_p$-homotopic to a point. Therefore by Lemma 5.11, for every $C \in \mathrm{C}(G)$, there exists $\alpha_{C,p} \in \mathrm{C}(\mathbb{D})^\circ$ such that $\psi_{C,p} = \mathrm{id}_{\mathcal{F}_{C,p}}\alpha_{C,p}$. In particular, the claim holds. Therefore, for every $a \in \mathcal{A}_G$, $\psi'_{a,p} = \psi_{a,p}\alpha_a$ for some $\alpha_a \in \mathrm{C}(\mathbb{D})^\circ$.

We have $\mathcal{X} \subseteq \mathcal{S}(p)$. Define $g : \mathcal{A}_{\mathcal{X}} \to \mathrm{C}(\mathbb{D})^\circ$ by $(a)g = (\alpha_a)^{-1}$ for every $a \in \mathcal{A}_{\mathcal{X}}$. By the definition of $C(a)$, for every $a \in \mathcal{A}_G$, $C(a^{-1}) = C(a)^{-1}$ (see Lemma 6.3(i)). Therefore $\alpha_{a^{-1}} = (\alpha_a)^{-1}$, that is g satisfies (1). Suppose $C \in \mathrm{supp}_{\mathcal{C}}(X)$ is a walk with the sequence of arcs $a_1, \ldots, a_n$. Let $\psi_i = \psi_{a_i,p}$, let $\psi'_i = \psi'_{a_i,p}$, and let $\alpha_i = \alpha_{a_i}$ for every $i \in \{1, \ldots, n\}$. By the definition of the ψ'_i we have $\Pi_{i=1}^n\psi'_i = \mathrm{id}_{\mathcal{F}_{C,p}}$ (see Lemma 6.3(ii)), therefore

$$\mathrm{id}_{\mathcal{F}_{C,p}} = \Pi_{i=1}^n\psi'_i = \Pi_{i=1}^n(\psi_i\alpha_i) = (\Pi_{i=1}^n\psi_i)(\Pi_{i=1}^n\alpha_i) = \psi_{C,p}(\Pi_{i=1}^n\alpha_i)$$

This shows that $\psi_{C,p} = \mathrm{id}_{\mathcal{F}_{C,p}}[\Pi_{i=1}^n(a_i)g]$. Since by (Crc-cir) $\psi_C = \mathrm{id}_{\mathcal{F}_C}\alpha_C$ for some $\alpha_C \in \mathrm{C}(\mathbb{D})^\circ$, this implies that $\psi_C = \mathrm{id}_{\mathcal{F}_C}[\Pi_{i=1}^n(a_i)g]$, that is g satisfies (2). ∎

Remark 6.5. In Proposition 6.2(ii) condition (Crc-cir) is not necessary, since by Proposition 6.1(ii) the fact that, for all $C \in \mathrm{supp}_{\mathcal{C}}(X)$, $\psi_C = \mathrm{id}_{\mathcal{F}_C}\alpha_C$ follows from (PX-e).

7 Spanning set for $\mathrm{Ker}(\partial_{[\mathcal{C}]^*})$ in buildings and their Grassmannians

The results of this section are similar to “Auxiliary results” in [19], but we are interested in homology instead of homotopy; the results of this section are also related to the ordinary homology with integer coefficients of the simplicial complex of a building, which was calculated for spherical buildings by Solomon and Tits in [17]. The main results are Proposition 7.1 and Theorem 7.9.

7.1 Spanning set for $\mathrm{Ker}(\partial_{[\mathcal{C}]^*})$ in building chamber systems

We view buildings as chamber systems and we refer the reader to [19], [13], and [15] for definitions and notation.

Let $\mathcal{B}$ be a building with diagram $M = (m_{ij})$ over a type set I. Let $\mathcal{C}$ be the set of walks of $\mathcal{B}$ consisting of the circuits of length 3 and of the "apartments" of spherical residues of $\mathcal{B}$ of rank 2, that is of the circuits of $\mathcal{B}$ of all possible types $\mathrm{p}_{2m_{ij}}(i,j)$ with $\{i,j\} \subseteq I$ and m_{ij} an integer. Let $\partial_{[\mathcal{C}]^*}$ be defined as in Section 5.2. We let $K = \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$, and we let $K_{3,\mathrm{sph}}$ consist of all $X \in K$ with $\mathrm{supp}_{\mathcal{B}}(X) \subseteq R$, where R runs through the spherical residues of $\mathcal{B}$ of rank at most three. We will refer to this situation as **Hypothesis (HB)**. The purpose of this subsection is to prove the following.

Proposition 7.1. *Suppose that hypothesis (HB) holds. Then* $\langle K_{3,\mathrm{sph}} \rangle_{\mathrm{F}[\mathcal{C}]^*} = K$.

Suppose G is a graph and let $\mathcal{C}$ be a set of circular walks in G closed under taking inverses. An *orientation* of $\mathcal{C}$ is a subset $\mathrm{O}(\mathcal{C})$ of $\mathcal{C}$ such that, for every $C \in \mathcal{C}$, exactly one C or C^{-1} is in $\mathrm{O}(\mathcal{C})$.

Lemma 7.2. *Suppose (HB) holds and suppose that $\mathcal{B}$ is a thin spherical building of rank three. Suppose $C, D \in \mathcal{C}$ contain arcs a and a^{-1} respectively. Then there exists an orientation $\mathrm{O}(\mathcal{C})$ of $\mathcal{C}$ such that (i) $C, D \in \mathcal{C}$ and (ii) $\Sigma_{C \in \mathrm{O}(\mathcal{C})}[C]^* \in \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$.*

Proof. The graph of the chamber system $\mathcal{B}$ has a planar embedding with the faces being the residues of $\mathcal{B}$ of rank 2. Let $\mathrm{O}(\mathcal{C}) \subseteq \mathcal{C}$ consist of the elements of $\mathcal{C}$ oriented in the plane the same way as C. ∎

Let $G = (V, E)$ be a graph. Suppose $H = (V', E')$ is a subgraph of $\mathcal{G}$, and let $x \in V$. We say that H is *strongly gated* in G with respect to x with *gate* g if, for all $y \in V'$, $\mathrm{d}_G(x,y) = \mathrm{d}_G(x,g) + \mathrm{d}_H(g,y)$; the vertex g is denoted $\mathrm{gate}_H(x)$. Suppose $X \subseteq V$ and let $x \in V$. We define $\max_G(X,x) = \max\{\mathrm{d}_G(x,y) | y \in X\}$. We let $\mathrm{r}_X = \min\{\max_G(X,x) | x \in X\}$, that is r_X is the radius of the graph $G|X$.

Suppose $\mathcal{B}$ is a building. Every residue R of $\mathcal{B}$ is strongly gated with respect to every chamber x of $\mathcal{B}$ (see [15]); the chamber $\mathrm{gate}_R(x)$ is denoted $\mathrm{proj}_R(x)$ and is called the *projection* of x on R. For a walk W in $\mathcal{B}$, we denote $\mathrm{R}(W)$ the minimal by inclusion residue of $\mathcal{B}$ containing W.

Lemma 7.3. *Let $\mathcal{B}$ be the building of a generalized m-gon over a type set $I = \{i,j\}$, where m is an integer. Suppose C is a circuit of type $\mathrm{p}_{2m}(i,j)$ in $\mathcal{B}$, and suppose x is a chamber of $\mathcal{B}$. Then there exists a set S of circuits in $\mathcal{B}$, that consists of (1) triangles each containing one arc of C, and (2) circuits of type $\mathrm{p}_{2m}(i,j)$ containing x and a chamber of C opposite to x in $\mathcal{B}$, with the following properties.*

Let $C' = \mathrm{supp}_{\mathcal{B}}(C)$, let $X = \Sigma_{D \in S}[D]^*$, and let $\mathcal{X} = \cup_{D \in S}\mathrm{supp}_{\mathcal{B}}(D)$. Then

(i) $[C]^* - X \in \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$

(ii) $\max_{\mathcal{B}}(\mathcal{X},x) = \max_{\mathcal{B}}(C',x)$ *and* $\{y \in \mathcal{X} | \mathrm{d}_{\mathcal{B}}(x,y) = \max_{\mathcal{B}}(\mathcal{X},x)\} = \{y \in C' | \mathrm{d}_{\mathcal{B}}(x,y) = \max_{\mathcal{B}}(C',x)\}$.

Proof. For an arc (a,b) of C, there are exactly three possibilities:

$(+1)$ $\mathrm{d}_{\mathcal{B}}(x,a) < \mathrm{d}_{\mathcal{B}}(x,b)$

(0) $\mathrm{d}_{\mathcal{B}}(x,a) = \mathrm{d}_{\mathcal{B}}(x,b)$

(−1) $\mathrm{d}_{\mathcal{B}}(x,a) > \mathrm{d}_{\mathcal{B}}(x,b)$

and we say that (a,b) is of *type* $+1$, 0, *or* -1 *with respect to* x in these three cases. Suppose (a,b,c) is a segment of C and let $s = (s_1, s_2)$ be the types of the arcs (a,b) and (b,c) with respect to x. There are nine possibilities for s:

(1) $(1,1)$ or $(-1,-1)$.

(2) $(1,0)$ or $(0,-1)$; in this case either b and c, or a and b, are opposite to x in $\mathcal{B}$.

(3) $(1,-1)$; in this case b is opposite to x in $\mathcal{B}$.

(4) $(0,1)$ or $(-1,0)$.

(5) $(0,0)$; in this case a, b, and c are opposite to x in $\mathcal{B}$.

(6) $(-1,1)$.

First we describe the set S' of circuits of type $\mathrm{p}_{2m}(i,j)$ in S. Let O be the set of all chambers of C that are opposite to x in $\mathcal{B}$; since $m < \infty$, by (1)-(6) $O \neq \emptyset$. For each $b \in O$, let C_b be the unique circuit of type $\mathrm{p}_{2m}(i,j)$ with initial vertex x, containing b and x, and oriented as follows. Suppose (a,b,c) is the segment of C of length 2 containing b, and suppose that (b',b,b'') is the segment of C_b of length 2 containing b. Then the arcs (b',b) and (a,b) lie in one panel of $\mathcal{B}$ on b, and the arcs (b,b'') and (b,c) lie in the other panel of $\mathcal{B}$ on b. We let $S' = \{C_b | b \in O\}$.

Now we describe the set S'' of triangles in S. Let Q be the set of all arcs in C of type 0 with respect to x. Let $q \in Q$ and suppose that $q = (a,b)$. Let R be the panel of $\mathcal{B}$ containing a and b, and let $g = \mathrm{gate}_R(x)$. Let $T_q = (g,a,b,g)$. We let $S'' = \{T_q | q \in Q\}$.

Let $S = S' \cup S''$. The building $\mathcal{B}$ is a generalized m-gon. Using properties of generalized polygons and (1)-(6) above, one can check that S satisfies (i) and (ii) of the conclusion. ∎

Proof of Proposition 7.1. First, we introduce some notation. Suppose $x \in \mathcal{B}$, suppose $X \in \mathrm{F}[\mathcal{C}]^*$, and let $\mathcal{X} = \mathrm{supp}_{\mathcal{B}}(X)$. We define $\max_{\mathcal{B}}(X,x) = \max_{\mathcal{B}}(\mathcal{X},x)$, $\mathrm{r}_X = \mathrm{r}_{\mathcal{X}}$, and $(\mathrm{supp}_{\mathcal{B}}(X))_x = (\mathcal{X})_x = \{y \in \mathcal{X} | \mathrm{d}_{\mathcal{B}}(x,y) = \max_{\mathcal{B}}(X,x)\}$. We let $\mathrm{D}_X = \min\{|(\mathcal{X})_x| \, |x \in \mathcal{X} \text{ and } \max_{\mathcal{B}}(X,x) = \mathrm{r}_X\}$.

Let $r = \min\{\mathrm{r}_Y | Y \in K - \langle K_{3,\mathrm{sph}} \rangle_{\mathrm{F}[\mathcal{C}]^*}\}$, and let $X \in K - \langle K_{3,\mathrm{sph}} \rangle_{\mathrm{F}[\mathcal{C}]^*}$ be such that $\mathrm{r}_X = r$ and $\mathrm{D}_X = \min\{\mathrm{D}_Y | Y \in K - \langle K_{3,\mathrm{sph}} \rangle_{\mathrm{F}[\mathcal{C}]^*}$ and $\mathrm{r}_Y = r\}$.

Suppose first that $\mathrm{r}_X = 0$. Then $|\mathcal{X}| = 1$, therefore $X \in K_{3,\mathrm{sph}}$, a contradiction. Suppose next that $\mathrm{r}_X = 1$. Then $\mathcal{X} \subseteq \{x\} \cup \mathcal{B}(x)$ for some $x \in \mathcal{X}$. The subgraph of $\mathcal{B}$ induced on the set $\{x\} \cup \mathcal{B}(x)$ is the union of the panels of $\mathcal{B}$ on x, with any two distinct panels sharing exactly one chamber x. Therefore $X = \Sigma_{i \in I'} X_i$, where $I' \subseteq I$ is finite and, for each $i \in I'$, $X_i = \Sigma \lambda_C C$ with C ranging through a finite set of triangles of the $\{i\}$-panel of $\mathcal{B}$ on x. Since $X \in K$, we obtain that, for each $i \in I'$, $(X_i)\partial_{[\mathcal{C}]^*} = 0$. Therefore $X \in \langle K_{3,\mathrm{sph}} \rangle_{\mathrm{F}[\mathcal{C}]^*}$, a contradiction.

Suppose now that $\mathrm{r}_X \geq 2$. Let $x \in \mathcal{X}$ be such that $\max_{\mathcal{B}}(X,x) = \mathrm{r}_X$ and $|(\mathcal{X})_x| = \mathrm{D}_X$. We obtain a contradiction by showing that there exists $X' \in K - \langle K_{3,\mathrm{sph}} \rangle_{\mathrm{F}[\mathcal{C}]^*}$, such that either $\mathrm{r}_{X'} < \mathrm{r}_X$ or, else, $\mathrm{r}_{X'} = \mathrm{r}_X$ and $\mathrm{D}_{X'} < \mathrm{D}_X$. This will be done in three steps. On each step, we continue to call X the new element of $\mathrm{F}[\mathcal{C}]^*$ obtained from X. For $C \in \mathrm{supp}_{\mathcal{C}}(X)$, we denote λ_C the coefficient of C in the positive expansion of X.

Step 1. Suppose that $\mathrm{supp}_{\mathcal{C}}(X)$ contains circuits of type $\mathrm{p}_{2m_{ij}}(i,j)$. We "replace" these circuits with other circuits of type $\mathrm{p}_{2m_{ij}}(i,j)$ and with triangles.

Suppose that $C \in \mathrm{supp}_{\mathcal{C}}(X)$ is of type $\mathrm{p}_{2m_{ij}}(i,j)$ for some $\{i,j\} \subseteq I$. Then $\mathrm{R}(C)$ is of type $\{i,j\}$ and m_{ij} is an integer. Let $g = \mathrm{gate}_{\mathrm{R}(C)}(x)$, and let S be the set of circuits constructed from C and g as in Lemma 7.3. Then $[C]^* - \Sigma_{D\in S}[D]^* \in K_{3,\mathrm{sph}}$, and we replace X with $X + \lambda_C(-[C]^* + \Sigma_{D\in S}[D]^*)$. By Lemma 7.3(ii), this does not increase r_X, and does not add any new chambers to $(\mathcal{X})_x$. From now on we assume that (A1) holds.

(A1) For every circuit $C \in \mathrm{supp}_{\mathcal{C}}(X)$ of type $\mathrm{p}_{2m_{ij}}(i,j)$, C contains $\mathrm{gate}_{\mathrm{R}(C)}(x)$ and a chamber opposite to $\mathrm{gate}_{\mathrm{R}(C)}(x)$ in $\mathrm{R}(C)$; no arc of C is of type 0 with respect to x.

Step 2. Suppose $\mathrm{supp}_{\mathcal{C}}(X)$ contains triangles with all vertices in $(\mathcal{X})_x$. We "replace" all such triangles with triangles having exactly two vertices in $(\mathcal{X})_x$.

Suppose $T = (y_0, y_1, y_2, y_0)$ is a triangle such that $T \in \mathrm{supp}_{\mathcal{C}}(X)$ and $y_i \in (\mathcal{X})_x$ for all $i \in \{0,1,2\}$. All y_i lie in one panel Q of $\mathcal{B}$. Let $g = \mathrm{gate}_Q(x)$ and, for $i \in \{0,1,2\}$, let $T_i = (g, y_i, y_{i+1}, g)$, where the indices are added mod 3. Then $-[T]^* + [T_0]^* + [T_1]^* + [T_2]^* \in K_{3,\mathrm{sph}}$ and we replace X with $X + \lambda_T(-[T]^* + [T_0]^* + [T_1]^* + [T_2]^*)$. This does not increase r_X, and does not add any new chambers to $(\mathcal{X})_x$.

We claim that after Step 2 $\mathrm{supp}_{\mathcal{C}}(X)$ does not contain any triangles with vertices in $(\mathcal{X})_x$. Let $a = (y_0, y_1)$ be an arc of $\mathcal{B}$ such that $\{y_0, y_1\} \subseteq (\mathcal{X})_x$, let $g = \mathrm{gate}_{\mathrm{R}(a)}(x)$, and suppose $C \in \mathrm{supp}_{\mathcal{C}}(X)$ contains a. By (A1) C is a triangle and by Step 2 $[C] = [(g, y_0, y_1, g)]$. Since $\partial_{[C]^*}(X) = 0$, there must exist $D \in \mathrm{supp}_{\mathcal{C}}(X)$ containing a^{-1}. By (A1) and by Step 2 $[D] = [(g, y_1, y_0, g)]$. Therefore $[D] = [C^{-1}]$, contradicting the definition of $\mathrm{supp}_{\mathcal{C}}(X)$. Since the panels of $\mathcal{B}$ are strongly gated in $\mathcal{B}$, every triangle in $\mathrm{supp}_{\mathcal{C}}(X)$ having a vertex in $(\mathcal{X})_x$ must have at least two vertices in $(\mathcal{X})_x$, therefore we have proved the claim.

From now on we assume that condition (A2) holds

(A2) For all $y \in (\mathcal{X})_x$, every circuit in $\mathrm{supp}_{\mathcal{C}}(X)$ on y is of type $\mathrm{p}_{2m_{ij}}(i,j)$ for some $\{i,j\} \subseteq I$.

Let $y \in (\mathcal{X})_x$. We are going to "remove" y from $(\mathcal{X})_x$. To achieve this we define a directed graph $G = (V, \mathcal{A}, \alpha)$ with labeled arcs. The set of vertices V of G is the set of all $u \in \mathcal{X}$ such that (u, y) or (y, u) is a segment of at least one circuit $C \in \mathrm{supp}_{\mathcal{C}}(X)$. Before we define the arcs of G, we prove the following.

(*) For any walk $w = (u, y, v)$ in $\mathcal{B}$, there is at most one $C \in \mathrm{supp}_{\mathcal{C}}(X)$ containing w or w^{-1}.

Suppose $w = (u, y, v)$ is a segment of some $C \in \mathrm{supp}_{\mathcal{C}}(X)$ and let $Q = \mathrm{R}(w)$. By (A2) Q is of type $\{i,j\} \subseteq I$ with m_{ij} finite, and by (A1) C is the unique circuit in Q through y and $\mathrm{gate}_Q(x)$. Therefore, C is the unique circuit in $\mathrm{supp}_{\mathcal{C}}(X)$ containing w. By the same argument, any circuit $D \in \mathrm{supp}_{\mathcal{C}}(X)$ containing w^{-1} must be an associate of C^{-1}. Since $C \in \mathrm{supp}_{\mathcal{C}}(X)$, by its definition $\mathrm{supp}_{\mathcal{C}}(X)$ contains no associates of C^{-1}.

We let $(u, v) \in \mathcal{A}$ if and only if there exists $C \in \mathrm{supp}_{\mathcal{C}}(X)$ containing the segment (u, y, v), and we let $\alpha(u, v) = \lambda_C$. By (*), if $a \in \mathcal{A}$ then $a^{-1} \notin \mathcal{A}$.

Step 3. We "remove" arcs from G until G contains no walks of length 2.

Suppose (u_1, u_2, u_3) is a walk of length 2 in G. By (*) $u_1 \neq u_3$. Let $C_{1,2}, C_{2,3} \in \operatorname{supp}_{\mathcal{C}}(X)$ be circuits containing the segments (u_1, y, u_2) and (u_2, y, u_3) respectively. By (A2) $C_{1,2}$ and $C_{2,3}$ are of types $\mathrm{p}_{2m_{ij}}(i,j)$ and $\mathrm{p}_{2m_{jk}}(j,k)$ for some $i, j, k \in I$. Since the u_i are pairwise distinct, by (A1) i, j, and k are pairwise distinct. Let Q be the residue of $\mathcal{B}$ of type $\{i, j, k\}$ containing $\{u_1, u_2, u_3, y\}$, and let $g = \mathrm{gate}_Q(x)$. By (A1) $\mathrm{d}_{\mathcal{B}}(x, y) > \mathrm{d}_{\mathcal{B}}(x, u_i)$ for all $i \in \{1, 2, 3\}$, therefore the residue Q is spherical, and g and y are opposite to each other in Q. Let A be the unique apartment of Q containing g and y and let $\mathcal{C}_A = \mathcal{C} \cap \mathrm{C}(A)$. By convexity of A and by (A1), we have $C_{1,2}, C_{2,3} \in \mathcal{C}_A$. By Lemma 7.2 there exists an orientation $\mathrm{O}(\mathcal{C}_A)$ of A containing $C_{1,2}^{-1}$ and $C_{2,3}^{-1}$ and such that $\Sigma_{C \in \mathrm{O}(\mathcal{C}_A)}[C]^* \in K_{3,\mathrm{sph}}$. We replace X with $X + \Sigma_{C \in \mathrm{O}(\mathcal{C}_A)}[C]^*$.

For the moment, denote the new X by X', and let $\mathcal{X}' = \operatorname{supp}_{\mathcal{B}}(X')$. Then X' satisfies (A2). Let $D = \Sigma_{C \in \mathrm{O}(\mathcal{C}_A)}[C]^*$. Since A is an apartment of Q containing $\mathrm{gate}_Q(x)$ and y, and $\mathrm{gate}_Q(x)$ and y are opposite in A, (A1) holds for every $C \in \operatorname{supp}_{\mathcal{C}}(D)$. Since $\operatorname{supp}_{\mathcal{B}}(D) \subseteq A$, no chamber of $\operatorname{supp}_{\mathcal{B}}(D)$ except y is at distance r_X or more from x. This shows that $\mathrm{r}_{X'} \leq \mathrm{r}_X$ and $(\mathcal{X}')_x \subseteq (\mathcal{X})_x$.

We construct $G' = (V', \mathcal{A}', \alpha')$ from X' similarly to G. Then $\alpha'(u_1, u_2) = \alpha(u_1, u_2) - 1$, $\alpha'(u_2, u_3) = \alpha(u_2, u_3) - 1$, and $\alpha'(u_1, u_3) = \alpha(u_1, u_3) + 1$; for the other arcs $a \in \mathcal{A}'$ we have $\alpha'(a) = \alpha(a)$. Therefore

$$0 \leq \Sigma_{a \in \mathcal{A}'} \alpha'(a) = \Sigma_{a \in \mathcal{A}} \alpha(a) - 1 \tag{7.1}$$

Equation 7.1 shows that we can repeat the transformation of Step 3 until we reach X, such that the corresponding graph G contains no walks of length 2.

We claim that after Step 3 no circuit in $\operatorname{supp}_{\mathcal{C}}(X)$ contains y. Let $G = (V, \mathcal{A}, \alpha)$ be defined as before, and suppose $V \neq \emptyset$. Let $v \in V$. By the definition of the set V, (v, y) or (y, v) is a segment of at least one circuit in $\operatorname{supp}_{\mathcal{C}}(X)$. Since $X \in \operatorname{Ker}(\partial_{[\mathcal{C}]^*})$, there must be circuits C_1 and C_2 in $\operatorname{supp}_{\mathcal{C}}(X)$, such that (y, v) is a segment of C_1 and (v, y) is a segment of C_2. Suppose (u_1, y, v) and (v, y, u_2) are segments of C_1 and C_2 respectively. Then (u_1, v, u_2) is a walk of length 2 in G, a contradiction.

On Steps 1-3, we did not increase r_X, and did not add any new chambers to $(\mathcal{X})_x$. Therefore we have constructed $X' \in K - \langle K_{3,\mathrm{sph}} \rangle_{\mathrm{F}[\mathcal{C}]^*}$, such that either $\mathrm{r}_{X'} < \mathrm{r}_X$ (if $|\mathrm{D}_X|$ was 1), or $\mathrm{r}_{X'} = \mathrm{r}_X$ and $\mathrm{D}_{X'} = \mathrm{D}_X - 1$, contradicting the choice of X. ∎

7.2 Graph morphisms and walks

The main objective of this subsection is to prove Lemma 7.8 that applies to the following situation **(Geom)**.

Let $H = (V, E)$ be a connected graph, and let $\mathcal{P}$ and $\mathcal{L}$ be sets of induced subgraphs of H. Suppose the following conditions hold.

(Geom-geom) $|\mathcal{P}_L| \geq 2$ for all $L \in \mathcal{L}$, and $\mathcal{P}_L \neq \mathcal{P}_N$ for all pairs $L, N \in \mathcal{L}$ with $L \neq N$.

(Geom-vrt) Every vertex of H is a vertex of exactly one subgraph $p \in \mathcal{P}$.

(Geom-edg) Every edge of H is an edge of at least one subgraph $x \in \mathcal{P} \cup \mathcal{L}$.

Suppose (Geom) holds. We define $\Gamma = (\mathcal{P}, \mathcal{L})$ as the point-line geometry in which a point $p \in \mathcal{P}$ and a line $L \in \mathcal{L}$ are incident if and only if $p \in \mathcal{P}_L$. Let $G = (\mathcal{P}, \mathcal{E})$ be the point-collinearity graph of Γ. By (Geom-vrt) there is a map $\tau : V \to \mathcal{P}$ defined by $(v)\tau = p$ if and only if $v \in p$. By (Geom-edg) τ induces a morphism of graphs $H \to G$, which we also denote τ. Let W be a walk in H. The image w of W under τ is a stammering walk, that is a sequence of vertices in which any two consecutive vertices are either equal or adjacent. We denote $\mathrm{w}_G(W)$ the walk obtained from w by removing the repetitions. We denote $\mathcal{A}_H$ and $\mathcal{A}_G$ the arc sets of H and G. In the next lemma we use the following condition.

(Geom-lin) For every $L \in \mathcal{L}$ and for every pair of distinct points $p, q \in \mathcal{P}_L$, there exists an edge $\{x, y\}$ in L with $x \in p$ and $y \in q$.

Lemma 7.4. *Suppose hypothesis (Geom) holds.*

(i) The morphism τ is surjective on vertices; τ is surjective on edges if and only if (Geom-lin) holds.

(ii) If walks W_1 and W_2 in H have a common initial vertex, or a common terminal vertex, then $\mathrm{w}_G(W_1)$ and $\mathrm{w}_G(W_2)$ have a common initial vertex, or a common terminal vertex.

(iii) For any concatenation of walks $w_1 \circ w_2$ in H, $\mathrm{w}_G(w_1 \circ w_2) = \mathrm{w}_G(w_1) \circ \mathrm{w}_G(w_2)$.

(iv) For every walk C in H, $\mathrm{w}_G(C^{-1}) = [\mathrm{w}_G(C)]^{-1}$. If C is a circular walk in H, then $\mathrm{w}_G(C)$ is a circular walk in G and, for every associate D of C, $\mathrm{w}_G(D)$ is an associate of $\mathrm{w}_G(C)$.

Proof. Statement (i) is immediate from the definition of τ and from (Geom-lin). Statements (ii)-(iv) are true since τ is a morphism of graphs. ∎

Suppose (Geom) holds. Let $\mathcal{C}$ be a set of circular walks in H closed under orientation reversal and taking associates. Define $\mathrm{w}_G(\mathcal{C}) = \{\mathrm{w}_G(C) | C \in \mathcal{C}\}$. The set $\mathrm{w}_G(\mathcal{C})$ consists of circular walks and is closed under orientation reversal and taking associates.

Lemma 7.5. *Suppose hypothesis (Geom) holds. Let $\mathcal{C}$ be a set of circular walks in H closed under orientation reversal and taking associates. Then, for every $\mathcal{C}$-homotopy (W_1, W_2) in H, $(\mathrm{w}_G(W_1), \mathrm{w}_G(W_2))$ is a $\mathrm{w}_G(\mathcal{C})$-homotopy in G.*

Proof. Suppose (w, z) is a $\mathcal{C}$-homotopy in H, and suppose $w = w_1 \circ w_2 \circ w_3$ and $z = w_1 \circ w_2' \circ w_3$, where $w_2^{-1} \circ w_2' \in \mathcal{C}$. Then by Lemma 7.4 parts (iii) and (iv) $\mathrm{w}_G(w) = \mathrm{w}_G(w_1) \circ \mathrm{w}_G(w_2) \circ \mathrm{w}_G(w_3)$, $\mathrm{w}_G(z) = \mathrm{w}_G(w_1) \circ \mathrm{w}_G(w_2') \circ \mathrm{w}_G(w_3)$, and $\mathrm{w}_G(w_2)^{-1} \circ \mathrm{w}_G(w_2') = \mathrm{w}_G(w_2^{-1} \circ w_2') \in \mathrm{w}_G(\mathcal{C})$. ∎

Suppose (Geom) holds. In the next lemma we use condition (Geom-lin) together with the following condition.

(Geom-con) All subgraphs $p \in \mathcal{P}$ are connected.

Lemma 7.6. *Suppose hypothesis (Geom) and conditions (Geom-lin) and (Geom-con) hold. Let $\mathcal{C}$ be a set of circular walks in H closed under orientation reversal and taking associates.*

(i) For every walk w in G there exists a walk W in H, circular if w is circular, such that $\mathrm{w}_G(W) = w$.

(ii) If H is $\mathcal{C}$-simply connected, then G is $\mathrm{w}_G(\mathcal{C})$-simply connected.

Proof. (i) Suppose $w = (p_0, \ldots, p_n)$ is a walk in G. For every $i \in \{0, \ldots, n-1\}$ choose $a_i \in p_i$ and $b_{i+1} \in p_{i+1}$ adjacent to each other; this is possible by (Geom-lin). By (Geom-con), for every $i \in \{0, \ldots, n-1\}$, there exists a walk w_i from b_i to a_i in p_i; we let w_n be a walk from b_n to a_0 in p_n if w is circular, and the empty walk if w is not circular. Let $W = (a_0, b_1) \circ w_1 \circ \cdots \circ (a_{n-1}, b_n) \circ w_n$.

(ii) Suppose $C = (p_0, \ldots, p_n)$ is a circular walk in G. By part (i) there exists a circular walk D in H such that $\mathrm{w}_G(D) = C$. Since H is $\mathcal{C}$-simply connected, there exists a sequence of $\mathcal{C}$-homotopies $s = (D_0, \ldots, D_r)$ that starts with $D_0 = D$ and ends with a walk D_r of length 0. Let $s' = (\mathrm{w}_G(D_0), \ldots, \mathrm{w}_G(D_r))$. By Lemma 7.5 s' is a sequence of $\mathrm{w}_G(\mathcal{C})$-homotopies in G; it begins with C, and it ends with a walk $\mathrm{w}_G(D_r)$ of length 0. ∎

Suppose hypothesis (Geom) holds. Suppose $\mathcal{C}$ is a set of circular walks in H closed under orientation reversal and taking associates, and let $\mathcal{D} = \mathrm{w}_G(\mathcal{C})$. Let $\mathrm{F}[\mathcal{C}]^*$ and $\mathrm{F}[\mathcal{D}]^*$ be $\mathbb{Z}$-modules defined as in Section 5.2. The map w_G induces an element of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathcal{C}]^*, \mathrm{F}[\mathcal{D}]^*)$, which we also denote w_G, defined by the rule that, for every $X = \Sigma_{C \in \mathcal{C}} \lambda_C [C]^*$,

$$\mathrm{w}_G(X) = \Sigma_{C \in \mathcal{C}} \lambda_C [\mathrm{w}_G(C)]^*$$

Similarly, w_G induces an element of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}\mathcal{A}_H^*, \mathrm{F}\mathcal{A}_G^*)$, also denoted w_G. Let $\partial_{[\mathcal{C}]^*} \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathcal{C}]^*, \mathrm{F}[\mathcal{A}_H]^*)$ and $\partial_{[\mathcal{D}]^*} \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}[\mathcal{D}]^*, \mathrm{F}[\mathcal{A}_G]^*)$ be the boundary maps defined in Section 5.2.

Lemma 7.7. *Suppose hypothesis (Geom) holds. Let $\mathcal{C}$ be a set of circular walks in H, closed under orientation reversal and taking associates, and let $\mathcal{D} = \mathrm{w}_G(\mathcal{C})$.*

(i) For every $X \in \mathrm{F}[\mathcal{C}]^*$, we have $\mathrm{w}_G(([X]^*)\partial_{[\mathcal{C}]^*}) = (\mathrm{w}_G([X]^*))\partial_{[\mathcal{D}]^*}$.

(ii) If $X \in \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$, then $\mathrm{w}_G(X) \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$.

Proof. Statement (ii) follows from (i). Since w_G, $\partial_{[\mathcal{C}]^*}$, and $\partial_{[\mathcal{D}]^*}$ are $\mathbb{Z}$-linear, to prove (i) it suffices to show that $\mathrm{w}_G(([C]^*)\partial_{[\mathcal{C}]^*}) = (\mathrm{w}_G([C]^*))\partial_{[\mathcal{D}]^*}$ for every $C \in \mathcal{C}$. Suppose $C \in \mathcal{C}$ and let $D = \mathrm{w}_G(C)$. For $b \in \mathcal{A}_G$ define $\mathcal{A}_H(b) = \{a \in \mathcal{A}_H | \mathrm{w}_G(a) = b\}$. For every $b \in \mathcal{A}_G$, w_G induces a bijection from the set of segments of C in $\mathcal{A}_H(b)$, to the set of segments of $\mathrm{w}_G(C)$ equal to b. Therefore $\mu(D, b) = \Sigma\{\mu(C, a) | a \in \mathcal{A}_H(b)\}$. This implies $([D]^*)\partial_{[\mathcal{D}]^*} = \Sigma_{b \in \mathcal{A}_G} \mu(D, b) b^* = \Sigma_{b \in \mathcal{A}_G} \Sigma_{a \in \mathcal{A}_H(b)} \mu(C, a)(\mathrm{w}_G(a))^* = \mathrm{w}_G(([C]^*)\partial_{[\mathcal{C}]^*})$. ∎

Suppose (Geom) holds. Let $\mathcal{C}$ be a set of circular walks in H closed under orientation reversal and taking associates. For $z \in \mathcal{P} \cup \mathcal{L}$ we denote $\mathcal{C}_z$ the walks in $\mathcal{C}$ lying in the subgraph z. Consider the following conditions.

(Geom-pl) The geometry Γ is a partial linear space.

(Geom-flg) For every point-line flag (p, L) of Γ, the graph $p \cap L$ is connected.

(Geom-con$_{\mathcal{C}}$) Every subgraph $z \in \mathcal{P} \cup \mathcal{L}$ is $\mathcal{C}_z$-simply connected.

Lemma 7.8. *Suppose that hypothesis (Geom) holds. Let $\mathcal{C}$ be a set of circular walks in H closed under orientation reversal and taking associates, and let $\mathcal{D} = \mathrm{w}_G(\mathcal{C})$. Suppose conditions (Geom-pl), (Geom-flg), and (Geom-con$_{\mathcal{C}}$) hold. Suppose $X \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$. Let*

$\mathcal{X} = \mathrm{supp}_{\mathcal{D}}(X)$, and let $\mathcal{X}'$ and $\mathcal{X}''$ be the sets of vertices and edges of G appearing in the walks $C \in \mathcal{X}$. Then

$$X = X_0 - [\Sigma_{p\in\mathcal{X}'} D(p) + \Sigma_{e\in\mathcal{X}''} D(e)]$$

where (1) $X_0 \in \mathrm{w}_G(\mathrm{Ker}(\partial_{[\mathcal{C}]^*})$, (2) for every $p \in \mathcal{X}'$, $\mathrm{supp}_{\mathcal{P}}(D(p)) = \{p\}$, and (3) for every $e \in \mathcal{X}''$, $\mathrm{supp}_{\mathcal{P}}(D(e))$ is contained in the line $\langle e\rangle$ and $D(e) \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$.

Proof. We are going to construct $Y \in \mathrm{F}([\mathcal{C}]^*)$,

$$Y = Y_1 + \Sigma_{p\in\mathcal{X}'} C(p) + \Sigma_{e\in\mathcal{X}''} C(e) \tag{7.2}$$

such that $Y \in \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$, $\mathrm{w}_G(Y_1) = X$, and $X_0 := \mathrm{w}_G(Y)$ and $D(z) := \mathrm{w}_G(C(z))$, $z \in \mathcal{X}' \cup \mathcal{X}''$, satisfy the conclusion.

First, we define $Y_1 \in \mathrm{F}[\mathcal{C}]^*$. Let $\mathcal{Y}_1 \subseteq \mathcal{C}$ be such that $\mathcal{X} = \{\mathrm{w}_G(C)|C \in \mathcal{Y}_1\}$. Define

$$Y_1 = \Sigma_{C\in\mathcal{Y}_1} \lambda_{1,C}[C]^*$$

where $\lambda_{1,C} = \lambda_D$ for $D = \mathrm{w}_G(C)$. We have $\mathrm{w}_G(Y_1) = X$.

Next, for every $e \in \mathcal{X}''$, we define $C(e)$. Let $e \in \mathcal{X}''$ and suppose $e = \{p, q\}$, $p, q \in \mathcal{P}$. Let $a = (p, q)$. Let B and B' be the sets of arcs appearing in the walks $C \in \mathcal{Y}_1$, mapped by w_G to a and a^{-1} respectively. We have $B \cup B' \neq \emptyset$, since e is an edge of at least one walk $D \in \mathcal{X} \subseteq \mathrm{w}_G(\mathcal{C})$. By (Geom-pl) there is $L \in \mathcal{L}$ such that $B \cup B' \subseteq L$. To each $b \in B \cup B'$ we assign the weight $\rho_1(b) = \Sigma_{C\in\mathcal{Y}_1} \lambda_{1,C}\mu(C, b)$. Since, for every $C \in \mathcal{Y}_1$, $\Sigma_{b\in B}\mu(C, b) = \mu(\mathrm{w}_G(C), a)$, $\Sigma_{b\in B'}\mu(C, b^{-1}) = \mu(\mathrm{w}_G(C), a^{-1})$, and $X \in \mathrm{Ker}(\partial_{[\mathcal{D}]})$, we have

$$\Sigma_{b\in B}\rho_1(b) = \Sigma_{D\in\mathcal{X}}\lambda_D\mu(D, a) = \Sigma_{D\in\mathcal{X}}\lambda_D\mu(D, a^{-1}) = \Sigma_{b\in B'}\rho_1(b) > 0 \tag{7.3}$$

Suppose $b = (u, v) \in B$. Then $\rho_1(b) \neq 0$, therefore by equation 7.3 there exists $b' = (v', u') \in B'$. By (Geom-flg) there exist walks w_p and w_q in $L \cap p$ and $L \cap q$, connecting u' and u, and v and v' respectively. Let

$$C_1 = (b \circ w_q \circ b' \circ w_p)^{-1} \tag{7.4}$$

and reduce each of $\rho_1(b)$ and $\rho_1(b')$ by 1. We repeat this step until, after an integer number of steps $n \geq 1$, $\rho_1(b) = 0$ for every $b \in B \cup B'$; this produces circular walks $C_1, \ldots, C_n$ of the form 7.4. By (Geom-con$_{\mathcal{C}}$) and by Lemma 5.2, for each C_i there are walks $C_{ij} \in \mathcal{C}$, $j \in \{1, \ldots, n_i\}$, such that $(C_i)\partial_{[\mathcal{C}]^*} = (\Sigma_{j=1}^{n_i} C_{ij})\partial_{[\mathcal{C}]^*}$. Let $C(e) = \Sigma_{i=1}^{n}\Sigma_{j=1}^{n_i}[C_{ij}]^*$. Since by equation 7.4, for every $i \in \{1, \ldots, n\}$, $\mathrm{w}_G(C_i) \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$, by the $\mathbb{Z}$-linearity of w_G we have $\mathrm{w}_G(C(e)) \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$. We let $Y_2 = Y_1 + \Sigma_{e\in\mathcal{X}''} C(e)$.

Finally, we define $C(p)$, $p \in \mathcal{X}'$. Let $Y_2 = Y_1 + \Sigma_{e\in\mathcal{X}''} C(e)$, let $\mathcal{Y}_2 = \mathrm{supp}_{\mathcal{C}}(Y_2)$, let $Y_2 = \Sigma_{C\in\mathcal{Y}_2}\lambda_{2,C}[C]^*$ be the positive expansion for Y_2, and let $\mathcal{Y}_2'$ and $\mathcal{Y}_2''$ be the sets of vertices and arcs of H appearing in the walks $C \in \mathcal{Y}_2$. We consider the directed graph $H(Y_2) = (\mathcal{Y}_2', \mathcal{Y}_2'', \rho_2)$ with weighted arcs, where the weight of an arc $b \in \mathcal{Y}_2''$ is $\rho_2(b) = \Sigma_{C\in\mathcal{Y}_2}\lambda_{2,C}\mu(C, b)$. By the definition of Y_2, if $b \in \mathcal{Y}_2''$ connects two different subgraphs $p, q \in \mathcal{P}$, then

$$\rho_2(b) = \rho_2(b^{-1}) \tag{7.5}$$

Suppose $p \in \mathcal{X}'$, let $V_p = \mathcal{Y}_2' \cap p$, let $\mathcal{A}_p = \mathcal{Y}_2'' \cap p$, and let $H(p) = H(Y_2)|V_p$. Suppose $v \in V_p$ and let $\mathcal{A}_p^-(v)$ and $\mathcal{A}_p^+(v)$ be the sets of arcs in $\mathcal{A}_p$ with the terminal vertex v and with the initial vertex v respectively. We claim that

$$\Sigma_{b \in \mathcal{A}_p^-(v)} \rho_2(b) = \Sigma_{b \in \mathcal{A}_p^+(v)} \rho_2(b) \tag{7.6}$$

Let $\mathcal{B}_p^-(v)$ and $\mathcal{B}_p^+(v)$ be the sets of arcs in $\mathcal{Y}_2'' - \mathcal{A}_p$ with the terminal vertex v and with the initial vertex v. Equation 7.5 implies that

$$\Sigma_{b \in \mathcal{B}_p^-(v)} \rho_2(b) = \Sigma_{b \in \mathcal{B}_p^+(v)} \rho_2(b) \tag{7.7}$$

Suppose $C \in \mathcal{Y}_2$ has a segment $s = (u, v, w)$. There are the following mutually excluding possibilities:

(1) $\{u, w\} \subseteq p$; in this case s contributes $\lambda_{2,C}$ to each side of equation 7.6, and contributes 0 to each side of equation 7.7.

(2) $\{u, w\} \subseteq V - p$; then s contributes 0 to each side of equation 7.6, and contributes $\lambda_{2,C}$ to each side of equation 7.7.

(3) $u \in V - p$ and $w \in p$; then s contributes 0 to the $\mathcal{A}_p^-(v)$ side and $\lambda_{2,C}$ to the $\mathcal{A}_p^+(v)$ side of equation 7.6; and contributes $\lambda_{2,C}$ to the $\mathcal{B}_p^-(v)$ side and 0 to $\mathcal{B}_p^+(v)$ side of equation 7.7.

(4) $u \in p$ and $w \in V - p$; then s contributes $\lambda_{2,C}$ to the $\mathcal{A}_p^-(v)$ side and 0 to the $\mathcal{A}_p^+(v)$ side of equation 7.6, and contributes 0 to the $\mathcal{B}_p^-(v)$ side and $\lambda_{2,C}$ to the $\mathcal{B}_p^+(v)$ side of equation 7.7.

Comparing the contributions made by s to the sums in equations 7.6 and 7.7, we see that equation 7.6 follows from equation 7.7.

We call a directed graph a circuit if its vertices and arcs are comprised by a single circuit. Since equation 7.6 holds for every $v \in V_p$, the graph $H(p)$ is a union of not necessarily distinct circuits $C_1, \ldots, C_n$. By (Geom-con$_{\mathcal{C}}$) and by Lemma 5.2, for each C_i there are walks $C_{ij} \in \mathcal{C}$, $j \in \{1, \ldots, n_i\}$, such that $(C_i)\partial_{[\mathcal{C}]^*} = (\Sigma_{j=1}^{n_i} C_{ij})\partial_{[\mathcal{C}]^*}$. Let $C(p) = \Sigma_{i=1}^{n} \Sigma_{j=1}^{n_i} [C_{ij}]^*$.

Define Y by equation 7.2. Then $Y \in \mathrm{Ker}(\partial_{[\mathcal{C}]^*})$. ∎

7.3 Spanning set for $\mathrm{Ker}(\partial_{[w_{\mathcal{G}}(\mathcal{C})]^*})$ in building Grassmannians

Suppose $\mathcal{B}$ is a chamber system and let Y be a set of chambers of $\mathcal{B}$. We define $\mathrm{typ}(Y) = \cup \mathrm{typ}(e)$, where e runs through the edges of $\mathcal{B}|Y$ and $\mathrm{typ}(e)$ is the type of the edge e.

Let $\mathcal{B}$ be a building of type M, a Coxeter diagram over a type set I. Suppose $J \subseteq I$ and let $I' = I - J$. The *Grassmann geometry* $\Gamma = (\mathcal{P}, \mathcal{L})$ of $\mathcal{B}$ of type J is a point-line geometry whose points $\mathcal{P}$ are the residues of $\mathcal{B}$ of type I'. The lines $\mathcal{L}$ of Γ are the $\mathcal{P}$-shadows of the panels of $\mathcal{B}$ of all possible types $\{j\} \subseteq J$, where we regard any two equal shadows as the same line; a point $p \in \mathcal{P}$ and a line $L \in \mathcal{L}$ are incident if and only if $p \in L$.

Given $L \in \mathcal{L}$, more than one panel of $\mathcal{B}$ can have the point shadow L. More precisely, suppose P is a panel of $\mathcal{B}$ whose point shadow is $L \in \mathcal{L}$. Then P is of type $\{j\} \subseteq J$. Let $T = \{j\} \cup [I' - D_{0,1}(j)]$, where $D_{0,1}(j)$ denotes the set consisting of j and the nodes $i \in I$ of the diagram M connected to the node j by at least one bond, and let R be the residue of $\mathcal{B}$ of type T containing P. The panels of $\mathcal{B}$ whose point shadow is L are precisely the panels of R of type $\{j\}$, therefore we can regard R as a line of Γ. The residue R is the unique maximal by inclusion residue of $\mathcal{B}$ with point shadow L (see [18] Chapter 12, [12] Chapter 5, [8]). The residue R is also the unique maximal by inclusion residue of $\mathcal{B}$ containing P and having the form $P \times P'$, where $\times$ denotes the direct product of chamber systems and P' is a residue of $\mathcal{B}$ intersecting P such that $\mathrm{typ}(P') \subseteq I'$.

For $p \in \mathcal{P}$, the residue p of $\mathcal{B}$ will be denoted R_p when we want to emphasize that we are looking at a residue of $\mathcal{B}$. For $L \in \mathcal{L}$, we denote R_L the maximal by inclusion residue of $\mathcal{B}$ whose point shadow is L.

Theorem 7.9. *Suppose hypothesis (HB) of Subsection 7.1 holds, and let $\mathcal{D} = \mathrm{w}_{\mathcal{G}}(\mathcal{C})$. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a Grassmann geometry of $\mathcal{B}$ with point-collinearity graph $\mathcal{G}$. Suppose $X \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$. Let $\mathcal{X} = \mathrm{supp}_{\mathcal{D}}(X)$, and let $\mathcal{X}'$ and $\mathcal{X}''$ be the sets of vertices and edges of $\mathcal{G}$ appearing in the walks $C \in \mathcal{X}$. Then*

$$X = X_0 + \Sigma_{p \in \mathcal{X}'} D(p) + \Sigma_{e \in \mathcal{X}''} D(e)$$

where (1) $X_0 \in \langle \mathrm{w}_{\mathcal{G}}(K_{3,\mathrm{sph}}) \rangle_{\mathrm{F}[\mathcal{D}]^*}$, (2) for every $p \in \mathcal{X}'$, $\mathrm{supp}_{\mathcal{P}}(D(p)) = \{p\}$, and (3) for every $e \in \mathcal{X}''$, $\mathrm{supp}_{\mathcal{P}}(D(e))$ is contained in the line $\langle e \rangle$ and $D(e) \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$.

Proof. Since Γ is a Grassmann geometry of $\mathcal{B}$, hypothesis (Geom) and conditions (Geom-pl) and (Geom-flg) hold for the graphs $\mathcal{B}$ and $\mathcal{G}$ (see [8]). Since buildings are 2-simply connected chamber systems and every residue of a buildings is a building ([19]; see also [15]), condition (Geom-con$_{\mathcal{C}}$) holds. Therefore the conclusion follows from Proposition 7.1, Lemma 7.8, and the $\mathbb{Z}$-linearity of $\mathrm{w}_{\mathcal{G}}$. ∎

8 Embeddings of building geometries

In this section we prove Theorems 1.1 and 1.2. All geometries considered in Theorems 1.1 and 1.2 are Grassmann geometries of spherical buildings. Theorem 1.1 was first proved by Veldkamp [21]; a different proof was given by Tits [18]; the infinite rank version of Theorem 1.1 is proved in Shult [15].

8.1 Projective embeddings and presheaves

Let V be a left vector space over a division ring $\mathbb{D}$. The projective space $\mathbb{P}(V) = (\mathrm{P}_1(V), \mathrm{P}_2(V))$ of V is the point-line geometry of 1- and 2-dimensional subspaces of V in the roles of points and lines, with the incidence being symmetrized containment. The *projective dimension* of $\mathbb{P}(V)$ is $\dim(V) - 1$. Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a point-line geometry. A *projective embedding* or just an *embedding* $\xi : \Gamma \to \mathbb{P}(V)$ of Γ over $\mathbb{D}$ is a pair of injective maps $\mathcal{P} \to \mathrm{P}_1(V)$ and $\mathcal{L} \to \mathrm{P}_2(V)$ such that (1) for every $L \in \mathcal{L}$, the image of L regarded as a set of points is a full line of $\mathbb{P}(V)$, and (2) the images of the points of Γ span $\mathbb{P}(V)$. For an embedding ξ, we denote by the same symbol ξ the corresponding maps $\mathcal{P} \to \mathrm{P}_1(V)$ and $\mathcal{L} \to \mathrm{P}_2(V)$. A *morphism* of embeddings $\xi_1 \to \xi_2$ over a division ring $\mathbb{D}$ is a $\mathbb{D}$-semilinear transformation of the underlying vector spaces that, for every $p \in \mathcal{P}$, maps $(p)\xi_1$ to $(p)\xi_2$. An embedding of Γ over $\mathbb{D}$ is *absolutely universal over* $\mathbb{D}$ if every embedding of Γ over $\mathbb{D}$ is its homomorphic image. An embedding ξ of Γ over $\mathbb{D}$ is *universal relatively to* an embedding ξ_1 of Γ over $\mathbb{D}$ if every embedding of Γ over $\mathbb{D}$, with a homomorphic image ξ_1, is a homomorphic image of ξ.

Every projective embedding ξ of Γ over $\mathbb{D}$ gives rise to a point-line presheaf $\mathcal{F}_\xi = (\{\mathcal{F}_{\xi,p}\}_p, \{\mathcal{F}_{\xi,L}\}_L, \{\phi_{\xi,pL}\}_{pL})$ on Γ over $\mathbb{D}$ where, for every $z \in \mathcal{P} \cup \mathcal{L}$, $\mathcal{F}_{\xi,z} = (z)\xi$ and, for every point-line flag (p, L) of Γ, the connecting map $\phi_{\xi,pL}$ is the inclusion $(p)\xi \hookrightarrow (L)\xi$. We say that $\mathcal{F}_\xi$ is an *embedding presheaf* on Γ arising from ξ. Every morphism of embeddings induces an isomorphism of the corresponding embedding presheaves. Therefore, if Γ is a connected geometry and $\tau : \xi_1 \to \xi_2$ is a morphism of projective embeddings of Γ, then by Lemma 2.3(ii) and by Proposition 2.4 τ is unique up to multiplication by an element of $\mathbb{D}$.

8.2 H-chains

In this subsection we adapt results from [14] to our needs. We are unable to simply quote [14] since the exact statements we need, although all implicit, cannot be found there. What we call a projective embedding of a point-line geometry is a faithful projective embedding in the terminology of [14].

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. A *geometric hyperplane* or just a *hyperplane* of Γ is a proper subspace H of Γ such that, for every $L \in \mathcal{L}$, $L \cap H \neq \emptyset$. Suppose that $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ is a point-line presheaf on Γ over $\mathbb{D}$ and let H be a hyperplane of Γ. Let $H^* = \mathcal{P} - H$ and let $\mathcal{L}^* = \{L \in \mathcal{L} | L \nsubseteq H\}$. An *$H$-chain in $\mathcal{F}$* is a set of vectors $\{v_p \in \mathcal{F}_p | p \in H^*\}$ such that, for every line $L \in \mathcal{L}^*$ and for every pair of points $p, q \in L - H$, we have

$$(v_p)\phi_{pL} - (v_q)\phi_{qL} \in (\mathcal{F}_r)\phi_{rL} \tag{8.1}$$

where $\{r\} = L \cap H$.

Remark 8.1. If $C = \{v_p | p \in \mathcal{P}\}$ is an H-chain in a presheaf $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ over a division ring $\mathbb{D}$ then, since the connecting maps ϕ_{pL} are $\mathbb{D}$-linear, for every $\alpha \in \mathbb{D} - \{0\}$ the set $C' = \{\alpha v_p | p \in \mathcal{P}\}$ is also an H-chain.

We construct the following graph $G_{\mathcal{F},H} = (V_{\mathcal{F},H}, E_{\mathcal{F},H})$ associated with $\mathcal{F}$ and H. Let $V_{\mathcal{F},H} = \cup\{\mathcal{F}_p | p \in H^*\}$. Suppose $p, q \in H^*$ are distinct points and suppose $u \in \mathcal{F}_p$ and $v \in \mathcal{F}_q$. Then $\{u, v\} \in E_{\mathcal{F},H}$ if and only if there exists a line $L \in \mathcal{L}$ incident with both p and q, and

$$(u)\phi_{pL} - (v)\phi_{qL} \in (\mathcal{F}_r)\phi_{rL}, \tag{8.2}$$

where $\{r\} = L \cap H$. We define a map $\tau_{\mathcal{F},H} : V_{\mathcal{F},H} \to H^*$ by

$$(u)\tau_{\mathcal{F},H} = p$$

for every $p \in H^*$ and for every $u \in \mathcal{F}_p$. Let $G = (\mathcal{P}, \mathcal{E})$ be the point-collinearity graph of Γ. The map $\tau_{\mathcal{F},H}$ induces a morphism of graphs $G_{\mathcal{F},H} \to (G|H^*)$ which we also denote $\tau_{\mathcal{F},H}$. Suppose $w = (p_1, \ldots, p_n)$ is a walk in $G|H^*$ and let $W = (u_1, \ldots, u_n)$ be a walk in $G_{\mathcal{F},H}$, such that $(u_i)\tau_{\mathcal{F},H} = p_i$ for every $i \in \{1, \ldots, n\}$. Then we say that W is a *lift* of w under $\tau_{\mathcal{F},H}$. If Γ is not a partial linear space, then w can have more than one lift at each point in the preimage under $\tau_{\mathcal{F},H}$ of its initial vertex.

Suppose that $\Gamma|H^*$ is connected. Then, following [14], we say that the map $\tau_{\mathcal{F},H}$ is a *trivial covering* if $G_{\mathcal{F},H}$ is a union of connected components isomorphic to $G|H^*$, and $\tau_{\mathcal{F},H}$ maps each connected component isomorphically onto $G|H^*$. Using equations 8.1 and 8.2 and Remark 8.1 one can prove the following.

Proposition 8.2 (cf. Corollary 1 of Theorem 1 of Ronan [14])**.** *Let Γ be a point-line geometry, let $\mathcal{F}$ be a point-line presheaf on Γ, and let H be a geometric hyperplane of Γ. Suppose that $\Gamma|H^*$ is connected. Then an H-chain exists in $\mathcal{F}$ if and only if $\tau_{\mathcal{F},H}$ is a trivial covering.* ∎

Remark 8.3. Under the hypothesis of Proposition 8.2 the morphism $\tau_{\mathcal{F},H}$ is a trivial covering if and only if every lift of every circular walk is a circular walk (see [15]).

Corollary 8.4 (cf. Corollary 3 of Theorem 1 of Ronan [14])**.** *Let Γ be a point-line geometry with point-collinearity graph G, let $\mathcal{F}$ be a point-line presheaf on Γ, and let H be a geometric hyperplane of Γ. Suppose that there is a set of full subgeometries $\mathcal{S}$ of Γ with point-collinearity graphs $G(S)$ such that (1) the graph $G|(\mathcal{P} - H)$ is $\mathcal{C}$-simply connected, where $\mathcal{C}$ is the set consisting of the circular walks of the graphs $G(S)$, $S \in \mathcal{S}$, and (2) for every $S \in \mathcal{S}$, a $(\mathcal{P}(S) \cap H)$-chain exists in $\mathcal{F}|S$. Then an H-chain exists in $\mathcal{F}$.*

Proof. Let $H^* = \mathcal{P} - H$. By Proposition 8.2 and Remark 8.3 we need to show that, for every circular walk in $G|H^*$, every lift under $\tau_{\mathcal{F},H}$ is a circular walk in $G_{\mathcal{F},H}$. Since $G|H^*$ is $\mathcal{C}$-simply connected, it suffices to show that every lift of every walk in $\mathcal{C}$ is a circular walk.

Let $C \in \mathcal{C}$ and let C' be a lift of C under $\tau_{\mathcal{F},H}$. By the definition of $\mathcal{C}$ there exists $S \in \mathcal{S}$ such that C is a walk in $G(S)$. Let $\mathcal{F}' = \mathcal{F}|S$, let $H' = H \cap \mathcal{P}(S)$, and let $V' = \cup\{\mathcal{F}_p | p \in \mathcal{P}(S) - H'\}$. Then H' is a hyperplane of S, $G_{\mathcal{F}',H'}$ is a subgraph of $G_{\mathcal{F},H}$, and $\tau_{\mathcal{F}',H'} = \tau_{\mathcal{F},H}|G_{\mathcal{F}',H'}$. Therefore C' is a lift of C under $\tau_{\mathcal{F}',H'}$. By hypothesis a H'-chain exists in $\mathcal{F}'$ therefore, by Proposition 8.2 applied to $\mathcal{F}'$ and H', C' is a circular walk in $G_{\mathcal{F},H}$. ∎

Suppose that $\Gamma = (\mathcal{P}, \mathcal{L})$ is a point-line geometry and let V be a left vector space over a division ring. Suppose ξ is a map $\mathcal{P} \to \{\{0\}\} \cup \mathrm{P}_1(V)$, and assume that the image of ξ spans V. We say that a hyperplane H of Γ *arises from* a hyperplane of V under the map ξ, or just that H arises from a hyperplane of V when ξ is understood, if there exists a subspace V' of V of codimension 1 such that $(H)\xi = (\mathcal{P})\xi \cap (\{\{0\}\} \cup \mathrm{P}_1(V'))$. If the map ξ induces an embedding, then we say that H arises from the embedding.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry and let $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ be a point-line presheaf on Γ over a division ring $\mathbb{D}$. We define the following vector space $\mathrm{H}_0(\mathcal{F})$ associated with $\mathcal{F}$. Let $V' = \bigoplus\{\mathcal{F}_z | z \in \mathcal{P} \cup \mathcal{L}\}$. Let V'' be the subspace of V' spanned by all vectors of the form $v - (v)\phi_{pL}$, where (p, L)

runs through the point-line flags of Γ and, for each flag (p, L), v runs through the vectors of $\mathcal{F}_p$. Define

$$\mathrm{H}_0(\mathcal{F}) = V'/V'' \tag{8.3}$$

We denote $\xi_{\mathcal{F}}$ the map $\mathcal{P} \to \{\{0\}\} \cup \mathrm{P}_1(V)$ that takes each $z \in \mathcal{P}$ to the image of $\mathcal{F}_z$ in $\mathrm{H}_0(\mathcal{F})$ under the quotient map of equation 8.3. One can show that, for a hyperplane H, existence of an H-chain in $\mathcal{F}$ is equivalent to existence of $f \in \mathrm{Hom}_{\mathbb{D}}(\mathrm{H}_0(\mathcal{F}), \mathbb{D})$ that vanishes on $(H)\xi_{\mathcal{F}}$ and does not vanish on $(\mathcal{P} - H)\xi_{\mathcal{F}}$. Therefore the following holds.

Theorem 8.5 (Theorem 2 of Ronan [14]). *Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a point-line geometry, let $\mathcal{F}$ be a presheaf on Γ, and let H be a hyperplane of Γ. Then H arises from a vector space hyperplane of $\mathrm{H}_0(\mathcal{F})$ under the map $\xi_{\mathcal{F}}$ if and only if an H-chain exists in $\mathcal{F}$.*

Theorem 8.6 (Corollary 3 of Theorem 2 of Ronan [14]). *Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a point-line geometry and let $\mathcal{F}$ be a presheaf on Γ. Then the map $\xi_{\mathcal{F}}$ induces a projective embedding of Γ into $\mathbb{P}(\mathrm{H}_0(\mathcal{F}))$ if and only if for every pair of distinct points of Γ there is a geometric hyperplane H of Γ, containing one point but not the other, such that an H-chain exists in $\mathcal{F}$.*

Proposition 8.7 (Proposition 3 of Ronan [14]). *Suppose Γ is a point-line geometry, let ϵ be a projective embedding of Γ, and let $\mathcal{F} = \mathcal{F}_\epsilon$. Then the map $\xi_{\mathcal{F}}$ induces an embedding $\overline{\epsilon} : \Gamma \to \mathbb{P}(\mathrm{H}_0(\mathcal{F}))$, universal relatively to ϵ.*

Remark 8.8. Since a morphism of embeddings induces an isomorphism of the corresponding embedding presheaves, it follows from Proposition 8.7 that a point-line geometry, embeddable over a division ring $\mathbb{D}$, has an an absolutely universal embedding over $\mathbb{D}$ if and only if all its embedding presheaves over $\mathbb{D}$ are isomorphic to each other.

8.3 Polar spaces

In this subsection we follow Chapter 7 of [15] (see also [18] and [5]). For a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ and for $p \in \mathcal{P}$, we denote $p^\perp$ the set of all points of Γ collinear with p (the set $p^\perp$ includes p itself).

A *polar space* is a point-line geometry $(\mathcal{P}, \mathcal{L})$ with the property (PolSp).

(PolSp) *For every $p \in \mathcal{P}$ and for every $L \in \mathcal{L}$ not on p, the intersection $p^\perp \cap L$ is either a single point or all of L.*

A polar space $\Gamma = (\mathcal{P}, \mathcal{L})$ is *nondegenerate* if, for all $p \in \mathcal{P}$, $p^\perp \neq \mathcal{P}$. Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a nondegenerate polar space. Then, for all $p \in \mathcal{P}$, $p^\perp$ is a hyperplane of Γ and, for all pairs of distinct points p and q, $p^\perp \neq q^\perp$. Suppose that Γ is a nondegenerate polar space with thick lines. Then all singular subspaces of Γ are projective spaces. If there is a maximal singular subspace of Γ of finite projective dimension n, then all maximal singular subspaces of Γ have projective dimension n, and the *polar rank* or just the *rank* of Γ is defined to be $n + 1$. A nondegenerate polar space of rank 1 is any set of cardinality at least 2. By Theorem 4 of Buekenhout and Shult [5] and by Theorem 7.4 of Tits [18] the nondegenerate polar spaces of finite rank $n \geq 2$ are precisely the buildings of type $(\mathsf{B}/\mathsf{C})_n$. The chambers of the building $\mathcal{B}$ are the maximal flags $S_1 \leq \cdots \leq S_n$ of the polar space Γ, where S_i

denotes a singular subspace of Γ of projective dimension $i-1$; two chambers are i-adjacent in $\mathcal{B}$ if and only if the flags differ by exactly one subspace, of projective dimension $i-1$.

Theorem 8.9 (Cooperstein and Shult [6]; Shult [15]). *Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a nondegenerate polar space of rank at least 3 with thick lines, and let G be the point-collinearity graph of Γ. Then, for every geometric hyperplane H of Γ, the graph $G|(\mathcal{P} - H)$ is simply connected.*

Theorem 8.9 can also be deduced from Corollary of Theorem 5 of Pasini [11] (stated as Propositions 8.29 and 12.50 in [12]), combined with Theorem 12.64 of Pasini [12].

8.4 Proofs of Theorems 1.1 and 1.2

We need the following facts regarding projective spaces all of which can be found in [15]. Suppose Γ and Γ' are projective spaces of projective dimension $n \geq 2$, and suppose $\phi : \Gamma \to \Gamma'$ is an isomorphism of point-line geometries. If $\xi : \Gamma \to \mathbb{P}(V)$ and $\xi' : \Gamma' \to \mathbb{P}(V')$ are embeddings of Γ and Γ' over division rings $\mathbb{D}$ and $\mathbb{D}'$, then $\mathbb{D} = \mathbb{D}'$ and there is a $\mathbb{D}$-semilinear map $V \to V'$ inducing ϕ (this is the Fundamental Theorem of Projective Geometry). Therefore, up to isomorphism, ξ is the unique projective embedding of Γ and $\mathcal{F}_\xi$ is the unique embedding presheaf on Γ. Embeddable projective spaces are called classical. Every projective space of projective dimension at least 3 is classical (Veblen and Young [20]). Every hyperplane of a classical projective space Γ arises from every embedding of Γ. Using Theorem 8.5 and Proposition 8.7 we obtain the following (one can also prove this directly).

Lemma 8.10. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a projective space and suppose $\mathcal{F}$ is an embedding presheaf on Γ. Then, for every hyperplane H of Γ, there is an H-chain in $\mathcal{F}$.* ■

We will need the following facts regarding Grassmann geometries of buildings.

Remark 8.11. Let $\mathcal{B}$ be a building with diagram M over a type set I, let $J \subseteq I$, and let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the J-Grassmann geometry of $\mathcal{B}$ (see Subsection 7.3 for definition).

1. Suppose $\mathcal{B}$ is spherical. If R is a residues of $\mathcal{B}$ of type opposite to $I - J$, then the set $H_R = \{p \in \mathcal{P} | \mathrm{R}_p \text{ is not opposite to } R \text{ in } \mathcal{B}\}$ is a hyperplane of Γ ([1]). Therefore, for every pair of distinct points in $\mathcal{P}$, there is a hyperplane of Γ of the form H_R containing exactly one of the points. If $\Gamma = (\mathcal{P}, \mathcal{L})$ is a nondegenerate polar space, then the hyperplanes H_R are precisely the hyperplanes of Γ of the form $p^\perp$, $p \in \mathcal{P}$.

2. Every triangle of the point-collinearity graph of Γ is contained in a subspace of Γ which is a projective plane. For every projective plane π of Γ, there is a residue R of $\mathcal{B}$ such that $\pi = \mathcal{P}_R$, the diagram $M|\mathrm{typ}(R)$ has exactly one connected component K meeting J, and $M|K$ is A_2 ([9]).

Lemma 8.12. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be (1) a nondegenerate polar space of rank at least three, all of whose planes are embeddable, or (2) one of the geometries of Theorem 1.2. Suppose that Γ has thick lines.*

Let $\mathcal{S}$ be the set of projective planes of Γ and suppose $\mathcal{F} = (\{\mathcal{F}_p\}_p, \{\mathcal{F}_L\}_L, \{\phi_{pL}\}_{pL})$ is a presheaf on Γ over $\mathbb{D}$ such that, for every $S \in \mathcal{S}$, $\mathcal{F}|S$ is an embedding presheaf on $\Gamma|S$. Then the following statements hold.

(i) For every hyperplane H of Γ, there is an H-chain in $\mathcal{F}$.

(ii) $\xi_{\mathcal{F}}$ induces a projective embedding of Γ into the space $\mathbb{P}(\mathrm{H}_0(\mathcal{F}))$.

Proof. Let G be the point-collinearity graph of Γ.

(i) Suppose H is a hyperplane of Γ and let $H^* = \mathcal{P} - H$. The graph $G|H^*$ is simply connected. This was proved by Cooperstein and Shult [6, 15] for polar spaces (we stated this as Theorem 8.9), $\mathsf{D}_{5,5}$, and $\mathsf{E}_{6,6}$; by Shult [16] for $\mathsf{E}_{7,7}$; by Kasikova [7] for $\mathsf{F}_{4,1}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, and $\mathsf{E}_{8,8}$. In all cases the proof relies on point-line properties of the geometry. By Remark 8.11(2) every triangle of G is contained in a plane of Γ. If Γ is a polar space, then the planes of Γ are embeddable by hypothesis. If Γ is one of the geometries of Theorem 1.2, then using Remark 8.11(2) and looking at the diagrams of the buildings we see that every plane π of Γ is contained (1) in a symplecton of Γ in the case of $\mathsf{F}_{4,1}$ or (2) in a projective space of projective dimension at least 3 for the rest of the geometries. Therefore π is embeddable (in the case of $\mathsf{F}_{4,1}$ the symplecta are embeddable by hypothesis). By Lemma 8.10, for every plane of Γ, there exists an $(H \cap S)$-chain in $\mathcal{F}|S$. Therefore by Corollary 8.4 there exists an H-chain in $\mathcal{F}$.

(ii) By Remark 8.11(1), for every pair of distinct points in Γ, there is a hyperplane H containing one point but not the other, and by part (i) there exists an H-chain in $\mathcal{F}$. Therefore (ii) holds by Theorem 8.6. ■

Lemma 8.13. *Let $\mathcal{B}$ be a building with diagram M over a type set I. Suppose $\{i, j, k\} \subseteq I$, let $\mathcal{P}$ and $\mathcal{S}$ be the sets of residues of $\mathcal{B}$ of types $I - \{i\}$ and $I - \{j\}$, and let R be a residue of $\mathcal{B}$ of type $I - \{k\}$.*

(i) If there is a unique path in M from i to j, and k is on that path, then $\mathcal{P}_R \subseteq \mathcal{P}_S$ for every $S \in \mathcal{S}_R$.

(ii) If there is a unique path in M from i to j, and k is on that path, then $\mathcal{P}_R \subseteq \cap\{\mathcal{P}_S | S \in \mathcal{S}_R\}$.

(iii) If there is a unique path in M from i to k, and j is on that path, then $\cup\{\mathcal{P}_S | S \in \mathcal{S}_R\} \subseteq \mathcal{P}_R$.

Proof. (i) By hypothesis of (i) i and j lie in different connected components of the diagram $M|\mathrm{typ}(R)$. Therefore $R = R' \times R''$, where R' and R'' are residues of $\mathcal{B}$ of types I' and I'', $i \in I'$, and $j \in I''$. Suppose $p \in \mathcal{P}_R$ and suppose $S \in \mathcal{S}_R$. Then $p \cap R$ contains a residue Q'' of R of type I'', and $S \cap R$ contains a residue Q' of R of type I'. Therefore $p \cap S \neq \emptyset$, that is $p \in \mathcal{P}_S$.

Statement (ii) is immediate from (i). To prove (iii) let $S \in \mathcal{S}_R$. Applying (i) to S and R with their roles interchanged we obtain $\mathcal{P}_S \subseteq \mathcal{P}_R$. ■

Proof of Theorems 1.1 and 1.2. Let Γ be one of the geometries in Theorems 1.1 and 1.2. Let $\mathcal{B}$ be the building associated with Γ. We denote M the diagram of $\mathcal{B}$ over the type set $I = \{1, \ldots, n\}$, and in case of Theorem 1.1 we assume that M is $(\mathsf{B}/\mathsf{C})_n$

For each geometry Γ we define a geometry $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$, where $\mathcal{S}$ is a set of subspaces of Γ such that every plane of Γ is contained in a member of $\mathcal{S}$. Let $\mathcal{G}'$ denote the point-collinearity graph of Γ'. We show that the hypothesis of Corollary 5.6 holds for $\mathcal{G}'$. Therefore there exists a presheaf $\mathcal{F}$ on Γ over a division ring $\mathbb{D}$ such that, for every $S \in \mathcal{S}$, $\mathcal{F}|S$ is an embedding presheaf on $\Gamma|S$. Since by Remark 8.11(2) every plane of Γ is contained in a member of $\mathcal{S}$, by Lemma 8.12(ii) Γ has a projective embedding into the space $\mathbb{P}(\mathrm{H}_0(\mathcal{F}))$.

First, we describe the geometries $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$.

If Γ is a polar space, then $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ is the dual polar space; it is a building Grassmannian of type $(\mathsf{B}/\mathsf{C})_{n,n}$. The points $\mathcal{S}$ of Γ' are the maximal singular subspaces of Γ; as singular subspaces of Γ they have projective dimension $n-1 \geq 3$ and, therefore, are embeddable. The lines $\mathcal{L}_{\mathcal{S}}$ of Γ' are the hyperplanes of the maximal singular subspaces of Γ (viewed as sets of the elements of $\mathcal{S}$ containing them); they have projective dimension $n-2 \geq 2$.

For $\mathsf{F}_{4,1}$ the geometry $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ is $\mathsf{F}_{4,4}$. The points of Γ' are the symplecta of Γ; by hypothesis they are embeddable. The lines of Γ' are the planes of Γ.

For $\mathsf{D}_{5,5}$ the geometry $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ is $\mathsf{D}_{5,1}$. The points of Γ' are the symplecta of Γ; they are polar spaces of type D_4 and are embeddable by Theorem 1.1. The lines of Γ' are the maximal singular subspaces of Γ of projective dimension 3.

For $\mathsf{D}_{n,1}$ the geometry $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ is $\mathsf{D}_{n,n}$. The points of Γ' are one class of maximal singular subspaces of Γ; they have projective dimension $n-1 \geq 4$ and therefore are embeddable. The lines of Γ' are the subspaces of codimension 2 of the maximal singular subspaces of Γ; they are the singular subspaces of Γ of projective dimension $n-3 \geq 2$.

For $\mathsf{E}_{n,n}$, $n \in \{6,7,8\}$, and for $\mathsf{E}_{7,1}$ the geometry $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ is $\mathsf{E}_{n,2}$. In all cases the points of Γ' are the maximal singular subspaces of Γ of projective dimension $n-1 \geq 5$, therefore they are embeddable. In the case of $\mathsf{E}_{n,n}$ the lines of Γ' are the singular subspaces of Γ of projective dimension $n-4 \geq 2$; in the case of $\mathsf{E}_{7,1}$ the lines of Γ' are the planes of Γ.

For $\mathsf{E}_{6,2}$ the geometry $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ is $\mathsf{E}_{6,1}$, The points of Γ' are the shadows of the residues of $\mathcal{B}$ of type $I - \{1\}$; they are one class of convex subspaces of $\mathsf{E}_{6,2}$ isomorphic to $\mathsf{D}_{5,5}$, and are embeddable by part (ii) of Theorem 1.2. The lines of Γ' are one class of the maximal singular subspaces of Γ; they have projective dimension 4.

We denote $\mathcal{A}'$ the set of arcs of $\mathcal{G}'$. We let $i, j, k \in I$ be the nodes of M corresponding to $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{S}$. That is, the points $\mathcal{P}$ are the residues of $\mathcal{B}$ of type $I - \{i\}$, the lines $\mathcal{L}$ are the residues of type $I - \{j\}$, and the elements of $\mathcal{S}$ are the residues of type $I - \{k\}$.

1. Hypothesis (PSh). Suppose $(S, T) \in \mathcal{A}'$. It was remarked in each case that S and T are embeddable; we denote $\mathbb{D}$ and $\mathbb{D}'$ the corresponding division rings, and we denote $\mathcal{F}_S$ and $\mathcal{F}_T$ the corresponding embedding presheaves. It was also noted in each case that the intersection $S \cap T$ is a singular subspace of Γ of projective dimension at least 2. Therefore $\mathbb{D} = \mathbb{D}'$ and there exists an isomorphism of embedding presheaves $\psi_{S,T} : \mathcal{F}_S|(S \cap T) \to \mathcal{F}_T|(S \cap T)$. It follows from the connectedness of $\mathcal{G}'$ that all $S \in \mathcal{S}$ are embeddable over the same division ring $\mathbb{D}$. We let $\mathcal{F} = \{\mathcal{F}_S | S \in \mathcal{S}\}$ and let $\Psi = \{\psi_{S,T} | (S, T) \in \mathcal{A}'\}$, where the presheaf isomorphisms $\psi_{S,T}$ are chosen so that (PSh-inv) holds.

2. Condition (Con$_{\mathcal{L}}$). Let $L \in \mathcal{L}$. The graph $\mathcal{G}'|\mathcal{S}(L)$ is the point-collinearity graph of a Grassmann geometry of the residue of $\mathcal{B}$ of type $I - \{j\}$ corresponding to L, therefore it is connected.

3. Hypothesis (Crc). We use the notation of hypothesis (HB) of Subsection 7.1, and we let $\mathcal{D} = \mathrm{w}_{\mathcal{G}'}(\mathcal{C})$. We show that $\mathcal{D}$ satisfies (Crc) of Section 5.3.

3.1 Conditions (Crc-con) and (Crc-pnt). Suppose $p \in \mathcal{P}$. The graphs $\mathcal{G}'$ and $\mathcal{G}'|\mathcal{S}(p)$ are point-collinearity graphs of a Grassmann geometry of $\mathcal{B}$, and of a Grassmann geometry of the residue of $\mathcal{B}$ of type $I - \{i\}$ corresponding to the point p. The buildings $\mathcal{B}$ and R_p are respectively $\mathcal{C}$- and $\mathcal{C}_p$-simply connected. Therefore by Theorem 12.64 of [12] or, alternatively, by Lemma 7.6(ii), the graph $\mathcal{G}'$ is $\mathcal{D}$-simply connected and the graph $\mathcal{G}'|\mathcal{S}(p)$ is $\mathcal{D}_p$-simply connected.

To prove conditions (Crc-cir) and (Crc-ker) we use the following. In all cases, in the diagram M, the nodes i and k are connected by a unique path w, and the node j lies on w. The length of w is at least 4, and the length of the segment w' of w, starting with j and ending with k, is at least 3. For example, if Γ is a polar space and $\mathcal{B}$ is of type $(\mathsf{B}/\mathsf{C})_n$, then $i = 1$, $j = 2$, $k = n$, $w = (1, 2, \ldots, n)$, and $w' = (2, 3, \ldots, n)$. We write the group $\mathrm{C}(\mathbb{D})^\circ$ multiplicatively.

3.2 Condition (Crc-cir). We show that, for every $D \in \mathcal{D}$, the set $\mathrm{supp}_{\mathcal{S}}(D)$ satisfies condition (PX-l), therefore by Proposition 6.1(i) $\mathcal{D}$ satisfies (Crc-cir).

Let $D \in \mathcal{D}$. Let $C \in \mathcal{C}$ be such that $D = \mathrm{w}_{\mathcal{G}'}(C)$, and let Q be a residue of $\mathcal{B}$ of rank 2 containing C. Let l be a vertex of the walk w' not contained in $\mathrm{typ}(Q)$, and let R be the residue of $\mathcal{B}$ of type $I - \{l\}$ containing Q. Then, by Lemma 8.13(i) $\mathcal{P}_R$ contains a line of Γ, and by Lemma 8.13(ii) $\mathcal{P}_R \subseteq \cap\{S | S \in \mathcal{S}_R\} \subseteq \mathcal{P}_D$.

3.3 Condition (Crc-ker). Suppose $X \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$. By Theorem 7.9 $X = X_0 + \Sigma_{S \in \mathcal{X}'} D(S) + \Sigma_{e \in \mathcal{X}''} D(e)$, where $\mathcal{X}'$ and $\mathcal{X}''$ are sets of vertices and edges of $\mathcal{G}'$, and the following conditions hold: (1) $X_0 \in \langle \mathrm{w}_{\mathcal{G}'}(K_{3,\mathrm{sph}}) \rangle_{\mathrm{F}[\mathcal{D}]^*}$, (2) for every $S \in \mathcal{X}'$, $\mathrm{supp}_{\mathcal{S}}(D(S)) = \{S\}$, and (3) for every $e \in \mathcal{X}''$, $\mathrm{supp}_{\mathcal{S}}(D(e))$ is contained in the line $\langle e \rangle_{\Gamma'}$ and $D(e) \in \mathrm{Ker}(\partial_{[\mathcal{D}]^*})$. In each case, from the definition of the geometry Γ' we see that, for every $e \in \mathcal{X}''$, $\mathcal{P}_{D(e)} \neq \emptyset$. Therefore, since (Crc-pnt) holds, by Proposition 6.2(i) and by the $\mathbb{Z}$-linearity of $f_{\Psi,[\mathcal{D}]^*}$, $(\Sigma_{p \in \mathcal{X}'} D(p) + \Sigma_{e \in \mathcal{X}''} D(e)) f_{\Psi,[\mathcal{D}]^*} = 1$. It remains to show that $(X_0) f_{\Psi,[\mathcal{D}]^*} = 1$.

Suppose $Z \in \mathrm{w}_{\mathcal{G}'}(K_{3,\mathrm{sph}})$. Let $Y \in K_{3,\mathrm{sph}}$ be such that $\mathrm{w}_{\mathcal{G}'}(Y) = Z$ and let Q be a residue of $\mathcal{B}$ of rank 3 containing Y. Let l be a vertex of the path w not contained in $\mathrm{typ}(Q)$, and let R be the residue of $\mathcal{B}$ of type $I - \{l\}$ containing Q. Then $\mathcal{P}_R$ contains a point of Γ, and by Lemma 8.13(ii) $\mathcal{P}_R \subseteq \cap\{S | S \in \mathcal{S}_R\} \subseteq \mathcal{P}_Z$. Therefore $\mathcal{P}_Z \neq \emptyset$ and, since (Crc-pnt) holds, by Proposition 6.2(i) $(Z) f_{\Psi,[\mathcal{D}]^*} = 1$. It follows by the $\mathbb{Z}$-linearity of $f_{\Psi,[\mathcal{D}]^*}$ that $(X_0) f_{\Psi,[\mathcal{D}]^*} = 1$. ■

Remark 8.14. (1) All geometries considered in Theorems 1.1 and 1.2 are building Grassmannians corresponding to an end node of the diagram, for which it is known that the subgraph of the point-collinearity graph induced on the complement of an arbitrary hyperplane is simply connected.

(2) In the proof of Theorem 1.2 there are other choices for geometries Γ'. For example, if Γ is $\mathsf{E}_{n,n}$ one can let $\mathcal{S}$ be the set of symplecta of Γ and let $\mathcal{L}_{\mathcal{S}}$ be the set of the maximal singular subspaces of Γ of projective dimension $n - 2 \geq 4$; then Γ' is the geometry $\mathsf{E}_{n,1}$.

Now, we give a different proof of Theorem 1.1 which illustrates the use of condition (PX-e) of Section 6.

Second proof of Theorem 1.1. We let $\mathcal{B}$ denote the building with diagram M of type $(\mathsf{B}/\mathsf{C})_n$, associated with Γ.

Let $\mathcal{S}$ be the set of planes of Γ and let $\Gamma' = (\mathcal{S}, \mathcal{L}_{\mathcal{S}})$, where the set $\mathcal{L}_{\mathcal{S}}$ consists of the sets $\{\pi \in \mathcal{S} | L \subseteq \pi \subseteq X\}$, such that $L \in \mathcal{L}$ and X is a singular subspace of Γ of projective dimension 3; Γ' is the Grassmann geometry of $\mathcal{B}$ of type $(\mathsf{B}/\mathsf{C})_{n,3}$. We denote $\mathcal{G}'$ the point-collinearity graph of Γ' and we denote $\mathcal{A}'$ the set of arcs of $\mathcal{G}'$. We show that the hypothesis of Corollary 5.6 holds, therefore there exists a presheaf on Γ such that, for every plane S of Γ, $\mathcal{F}|S$ is the unique embedding presheaf on $\Gamma|S$. Then by Lemma 8.12(ii) Γ has a projective embedding into $\mathbb{P}(\mathrm{H}_0(\mathcal{F}))$.

Hypothesis (PSh). Let $(S, T) \in \mathcal{A}'$. Then $S \cup T \subseteq X$, where X is a singular subspace of Γ of projective dimension 3. The space X is embeddable over a division ring $\mathbb{D}$. Therefore S and T are embeddable over $\mathbb{D}$ and, if we denote $\mathcal{F}_S$ and $\mathcal{F}_T$ the corresponding embedding presheaves, then there exists an isomorphisms of presheaves $\psi_{S,T} : \mathcal{F}_S|(S \cap T) \to \mathcal{F}_T|(S \cap T)$. It follows from the connectedness of Γ' that all $S \in \mathcal{S}$ are embeddable over the same division ring $\mathbb{D}$. Let $\mathcal{F} = \{\mathcal{F}_S | S \in \mathcal{S}\}$ and let $\Psi = \{\psi_{S,T} | (S, T) \in \mathcal{A}'\}$, where the presheaf isomorphisms $\psi_{S,T}$ are chosen so that they satisfy (PSh-inv).

Condition (Con$_{\mathcal{L}}$). Let $L \in \mathcal{L}$. The graph $\mathcal{G}'|\mathcal{S}(L)$ is the point-collinearity graph of a Grassmann geometry of a residue of $\mathcal{B}$ of type $I - \{j\}$, therefore it is connected (this geometry is a polar space of rank $n - 2 \geq 2$).

Hypothesis (Crc). Let $\mathcal{C}$ be as in hypothesis (HB) and let $\mathcal{D} = \mathrm{w}_{\mathcal{G}'}(\mathcal{C})$. We show that $\mathcal{D}$ satisfies (Crc) of Section 5.3. The proof of conditions (Crc-con) and (Crc-pnt) is as in the first proof of Theorem 1.1 and we omit it. To prove (Crc-cir) and (Crc-ker), first, we observe that, for every $S \in \mathcal{S}$, $\Gamma|S$ is a projective space and, for every $(S, T) \in \mathcal{A}'$, $\Gamma|(S \cap T)$ is a line. Therefore using Corollary 2.6 we obtain the following.

(**) *For every $S \in \mathcal{S}$, $\mathcal{F}_S$ is the unique embedding presheaf on $\Gamma|S$. For every $(S, T) \in \mathcal{A}'$, the presheaf isomorphism $\psi_{S,T}$ is unique up to multiplication by an element of* $\mathrm{C}(\mathbb{D})^\circ$.

We write the group $\mathrm{C}(\mathbb{D})^\circ$ multiplicatively.

Condition (Crc-cir). We show that every $D \in \mathcal{D}$ satisfies (PX-l) or (PX-e), therefore by Proposition 6.1(i) or (ii) $\mathcal{D}$ satisfies (Crc-cir).

Let $D \in \mathcal{D}$, let $C \in \mathcal{C}$ be such that $\mathrm{w}_{\mathcal{G}'}(C) = D$, and let Q be a residue of $\mathcal{B}$ of rank 2 containing C. Suppose, first that there exists a residue R of $\mathcal{B}$ of type $I - \{l\}$ containing Q with $l \in \{2, 3\}$. Then $\mathcal{P}_R$ is a line or plane of Γ and by Lemma 8.13(ii) $\mathcal{P}_R \subseteq \cap\{S | S \in \mathcal{S}_R\} \subseteq \mathcal{P}_D$. Therefore (PX-l) holds for D.

Suppose now that Q is of type $\{2, 3\}$. Then C is contained in a residue R of $\mathcal{B}$ of type $I - \{4\}$. The set $\mathcal{P}_R$ is a singular subspace of Γ of projective dimension 3 and by Lemma 8.13(iii) $\mathcal{P}^D \subseteq \cup\{S | S \in \mathcal{S}_R\} \subseteq \mathcal{P}_R$. Therefore $\Gamma|\mathcal{P}^D$ is embeddable over D. Since (**) holds, (PX-e) holds for D.

Condition (Crc-ker). As in the first proof of Theorem 1.1, we need to show that, for every $Z \in \mathrm{w}_{\mathcal{G}'}(K_{3,\mathrm{sph}})$, $(Z)f_{\Psi,[\mathcal{D}]^*} = 0$. Suppose $Z \in \mathrm{w}_{\mathcal{G}'}(K_{3,\mathrm{sph}})$. Let $Y \in K_{3,\mathrm{sph}}$ be such that $\mathrm{w}_{\mathcal{G}'}(Y) = Z$ and let Q be a residue of $\mathcal{B}$ of rank 3 containing Y.

Suppose first that there exists a residue R of $\mathcal{B}$ of type $I - \{l\}$ containing Q with $l \in \{1, 2, 3\}$. Then $\mathcal{P}_R$ is a point, line, or plane of Γ and by Lemma

8.13(i) $\mathcal{P}_R \subseteq \cap\{S|S \in \mathcal{S}_R\} \subseteq \mathcal{P}_X$. Therefore $\mathcal{P}_Z \neq \emptyset$ and by Proposition 6.2(i) $(Z)f_{\Psi,[\mathcal{D}]^*} = 0$.

Suppose now that Q is of type $\{1,2,3\}$. Let R be the residue of Γ of type $I - \{4\}$ containing Q. The point shadow $\mathcal{P}_R$ of R is a singular subspace of Γ of projective dimension 3 and by Lemma 8.13(iii) $\mathcal{P}^X \subseteq \cup\{S|S \in \mathcal{S}_R\} \subseteq \mathcal{P}_R$. Therefore $\Gamma|\mathcal{P}^X$ is embeddable over D. Since (**) holds, (PX-e) holds for $\mathrm{supp}_{\mathcal{S}}(X)$. Therefore by Proposition 6.2(ii) $(Z)f_{\Psi,[\mathcal{D}]^*} = 0$. ■

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Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403 USA
email: annakas@bgsu.edu