Common fixed point theorems for a pair of multivalued mappings under weak contractive conditions in ordered metric spaces

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Abstract

We present common fixed point theorems for a pair of weakly isotone increasing multivalued mappings satisfying general weak contractive conditions, as well as almost contractive conditions in ordered complete metric spaces. Examples are presented to show the usage of these results.

1 Introduction and preliminaries

Fixed point theory for multivalued mappings was originally initiated by von Neumann in the study of Game theory. Fixed point theorems for multivalued mappings are quite useful in Control theory and have been frequently used in solving problems in Economics and Game theory.

The study of fixed points for multivalued contraction mappings has been an active topic, as well. The development of geometric fixed point theory for multifunctions was initiated by the work of Nadler [22] in 1969. He used the concept of Hausdorff metric to establish the multivalued contraction principle containing the Banach contraction principle as a special case, as follows.

Theorem 1.1. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ be a mapping from $\mathcal{X}$ into $CB(\mathcal{X})$ such that for all $x, y \in \mathcal{X}$,*

$$H(\mathcal{T}x, \mathcal{T}y) \leq \lambda d(x, y)$$

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where, $0 \leq \lambda < 1$. *Then* $\mathcal{T}$ *has a fixed point.*

Since then, this discipline has been developed further, and many profound concepts and results have been established with considerable generality; see, for example, the works of Hong [13], Hong et al. [15], Itoh and Takahashi [16], Kaneko [18], Kaneko and Sessa [19], Mizoguchi and Takahashi [21], Rhoades [27], Rouhani and Moradi [28], and references cited therein.

Let $(\mathcal{X}, d)$ be a metric space. We denote the classes of nonempty, resp. nonempty and bounded subsets of $\mathcal{X}$ by $N(\mathcal{X})$, resp. $B(\mathcal{X})$. For $\mathcal{A}$, $\mathcal{B} \in B(\mathcal{X})$, expressions $D(\mathcal{A}, \mathcal{B})$ and $\delta(\mathcal{A}, \mathcal{B})$ are defined as follows:

$$D(\mathcal{A}, \mathcal{B}) = \inf\{d(a,b) : a \in \mathcal{A},\ b \in \mathcal{B}\},$$
$$\delta(\mathcal{A}, \mathcal{B}) = \sup\{d(a,b) : a \in \mathcal{A},\ b \in \mathcal{B}\}.$$

If $\mathcal{A} = \{a\}$, then we write $D(\mathcal{A}, \mathcal{B}) = D(a, \mathcal{B})$ and $\delta(\mathcal{A}, \mathcal{B}) = \delta(a, \mathcal{B})$. If, additionally, $\mathcal{B} = \{b\}$, then $D(\mathcal{A}, \mathcal{B}) = \delta(\mathcal{A}, \mathcal{B}) = d(a, b)$. Obviously, $D(\mathcal{A}, \mathcal{B}) \leq \delta(\mathcal{A}, \mathcal{B})$.

For all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in B(\mathcal{X})$, the definition of $\delta(\mathcal{A}, \mathcal{B})$ yields the following:

$$\delta(\mathcal{A}, \mathcal{B}) = \delta(\mathcal{B}, \mathcal{A}),$$
$$\delta(\mathcal{A}, \mathcal{B}) \leq \delta(\mathcal{A}, \mathcal{C}) + \delta(\mathcal{C}, \mathcal{B}),$$
$$\delta(\mathcal{A}, \mathcal{B}) = 0 \text{ iff } \mathcal{A} = \mathcal{B} = \{a\}.$$

One can refer to above notation in [11, 12].

Definition 1.2. *A point* $x^* \in \mathcal{X}$ *is called a fixed point of a multivalued operator* $\mathcal{T} : \mathcal{X} \to B(\mathcal{X})$ *if* $x^* \in \mathcal{T}x^*$.

Definition 1.3. *[20]. A function* $F : [0, +\infty) \to [0, +\infty)$ *is called an altering distance function if the following properties are satisfied:*

1. *F is nondecreasing and continuous,*

2. $F(t) = 0$ *if and only if* $t = 0$.

Weak contractive conditions with functions of this and related types have been used to establish fixed point results in a number of works, some of which are noted in [5, 8, 10, 23, 26, 29, 30].

On the other hand, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [25, Theorem 2.1] who presented its applications to matrix equations. Subsequently, Nieto and Rodríguez-López [24] extended the result of [25] for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed point theorems in ordered metric spaces. Beg and Butt [2] worked on set-valued mappings and proved common fixed point results for mappings satisfying implicit relation in partially ordered metric space. Recently, Choudhury and Metiya [6] have proved fixed point theorems for multivalued mappings in the framework of a partially ordered metric space. Hong [14] proved new hybrid fixed point theorems involving multivalued operators which satisfy weakly generalized contractive conditions in an ordered complete metric space and presented an application to hyperbolic differential inclusions.

We will use the following terminology

Definition 1.4. *Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called an ordered metric space if:*

(i) $(\mathcal{X}, d)$ *is a metric space,*

(ii) $(\mathcal{X}, \preceq)$ *is a partially ordered set.*

Elements $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 1.5. *[2]. Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a partially ordered set $(\mathcal{X}, \preceq)$. The relation $\preceq_1$ between $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:*

$$\mathcal{A} \preceq_1 \mathcal{B} \iff \text{for each } a \in \mathcal{A} \text{ there exists } b \in \mathcal{B} \text{ such that } a \preceq b.$$

The results of this paper are divided in two parts. First, in Section 2, we establish the existence of common fixed points for a pair of weakly isotone increasing multivalued mappings under a general weakly contractive condition in partially ordered metric spaces. The consequences of the main theorem are also mentioned. Section 3 is devoted to common fixed point results for a pair of weakly isotone increasing multivalued mappings under a variant of so-called almost contractive conditions of Berinde [3]. Our results generalize the results of Choudhury and Metiya [6] by considering comparatively more general contractive and weakly contractive conditions for a pair of weakly isotone increasing multivalued mappings. They also extend results of Choudhury et al. [5], Berinde [4] and Babu et al. [1] from single valued mappings in metric spaces to multivalued mappings in ordered metric spaces (see also the recent paper of Ćirić et al. [7]). Examples are presented to show the usage of the results and, in particular, that the order can be crucial.

2 Common fixed point results under weak contractive conditions

In this section, we prove common fixed point theorems for a pair of weakly isotone increasing multivalued mappings under general weak contractive condition. In order to formulate the results, we extend to multivalued mappings the notion of weakly isotone increasing mappings given by Dhage, O'Regan and Agarwal [9].

Definition 2.1. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set. Two maps $\mathcal{S}, \mathcal{T} : \mathcal{X} \to N(\mathcal{X})$ are said to be weakly isotone increasing if for any $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq_1 \mathcal{T}y$ for all $y \in \mathcal{S}x$ and $\mathcal{T}x \preceq_1 \mathcal{S}y$ for all $y \in \mathcal{T}x$.*

Note that, in particular, single-valued mappings $\mathcal{T},\mathcal{S}:\mathcal{X}\to\mathcal{X}$ are weakly isotone increasing [9] if $\mathcal{S}x\preceq\mathcal{T}\mathcal{S}x$ and $\mathcal{T}x\preceq\mathcal{S}\mathcal{T}x$ hold for each $x\in\mathcal{X}$.

In what follows, $\mathcal{F}$ will denote the set of altering distance functions (see Definition 1.3). Ψ will be the set of functions $\psi:[0,+\infty)\to[0,+\infty)$ which are nondecreasing, right-continuous and $\psi(t)>0$ for $t>0$.

The main result of this section is the following

Theorem 2.2. *Let $(\mathcal{X},d,\preceq)$ be an ordered complete metric space. Let $\mathcal{T},\mathcal{S}:\mathcal{X}\to B(\mathcal{X})$ be multivalued mappings such that*

$$F(\delta(\mathcal{T}x,\mathcal{S}y))\leq F(M(x,y))-\psi(F(\max\{d(x,y),\delta(x,\mathcal{T}x),\delta(y,\mathcal{S}y)\}))\tag{2.1}$$

for all comparable $x,y\in\mathcal{X}$, where $F\in\mathcal{F}$, $\psi\in\Psi$ and

$$M(x,y)=\max\left\{d(x,y),D(x,\mathcal{T}x),D(y,\mathcal{S}y),\frac{1}{2}[D(x,\mathcal{S}y)+D(y,\mathcal{T}x)]\right\}.\tag{2.2}$$

Also suppose that $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing and there exists an $x_0\in\mathcal{X}$ such that $\{x_0\}\prec_1\mathcal{S}x_0$. If the condition

$$\begin{cases} \text{if } \{x_n\}\subset\mathcal{X} \text{ is a non-decreasing sequence with } x_n\to z \text{ in } \mathcal{X},\\ \text{then } x_n\preceq z \text{ for all } n\end{cases}\tag{2.3}$$

holds, then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.

Proof. First of all we show that, if $\mathcal{S}$ or $\mathcal{T}$ has a fixed point, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Indeed, let, e.g., z be a fixed point of $\mathcal{S}$. Assume that $\delta(z,\mathcal{T}z)>0$. If we use the inequality (2.1) for $x=y=z$, we have, taking into account properties of functions $F\in\mathcal{F}$ and $\psi\in\Psi$,

$$\begin{aligned}F(\delta(\mathcal{T}z,z))&\leq F(\delta(\mathcal{T}z,\mathcal{S}z))\\&\leq F(M(z,z))-\psi(F(\max\{d(z,z),\delta(z,\mathcal{T}z),\delta(z,\mathcal{S}z)\}))\\&\leq F(\delta(\mathcal{T}z,z))-\psi(F(\delta(\mathcal{T}z,z))<F(\delta(\mathcal{T}z,z)),\end{aligned}$$

which is a contradiction. Thus $\delta(z,\mathcal{T}z)=0$ and so z is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Starting with the given x_0, define a sequence $\{x_n\}$ as follows:

$$x_{2n+1}\in\mathcal{S}x_{2n},\ x_{2n+2}\in\mathcal{T}x_{2n+1}\text{ for } n\in\{0,1,\dots\}.\tag{2.4}$$

If $x_{n_0}\in\mathcal{S}x_{n_0}$ or $x_{n_0}\in\mathcal{T}x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n\neq x_{n+1}$ for all n.

Since $\{x_0\}\preceq_1\mathcal{S}x_0$, $x_1\in\mathcal{S}x_0$ can be chosen so that $x_0\preceq x_1$. Since $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing, it is $\mathcal{S}x_0\preceq_1\mathcal{T}x_1$; in particular, $x_2\in\mathcal{T}x_1$ can be chosen so that $x_1\preceq x_2$. Now, $\mathcal{T}x_1\preceq_1\mathcal{S}x_2$ (since $x_2\in\mathcal{T}x_1$); in particular, $x_3\in\mathcal{S}x_2$ can be chosen so that $x_2\preceq x_3$. Continuing this process, we conclude that $\{x_n\}$ can be an increasing sequence in $\mathcal{X}$:

$$x_1\preceq x_2\preceq\cdots\preceq x_n\preceq x_{n+1}\preceq\cdots.$$

Now we claim that

$$F(d(x_{n+1}, x_n)) < F(d(x_n, x_{n-1})). \tag{2.5}$$

Setting $x = x_{2n+1}$ and $y = x_{2n}$ in (2.2), we have

$$\begin{aligned} M(x_{2n+1}, x_{2n}) &= \max\Big\{ d(x_{2n+1}, x_{2n}), D(\mathcal{T}x_{2n+1}, x_{2n+1}), D(\mathcal{S}x_{2n}, x_{2n}), \\ &\qquad \frac{D(x_{2n+1}, Sx_{2n}) + D(Tx_{2n+1}, x_{2n})}{2} \Big\} \\ &= \max\Big\{ d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, x_{2n})}{2} \Big\}. \end{aligned}$$

Since $\frac{d(x_n, x_{n+2})}{2} \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$, it follows that

$$M(x_{2n+1}, x_{2n}) \leq \max\{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1})\}.$$

Therefore, from (2.1)

$$\begin{aligned} F(d(x_{2n+2}, x_{2n+1})) &\leq F(\delta(\mathcal{T}x_{2n+1}, \mathcal{S}x_{2n})) \\ &\leq F(M(x_{2n+1}, x_{2n})) \\ &\quad - \psi(F(\max\{d(x_{2n+1}, x_{2n}), \delta(x_{2n+1}, \mathcal{T}x_{2n+1}), \delta(x_{2n}, \mathcal{S}x_{2n+1}))\})) \\ &= F(\max\{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1})\}) \\ &\quad - \psi(F(\max\{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1})\})). \end{aligned} \tag{2.6}$$

Suppose that $d(x_{2n}, x_{2n+1}) \leq d(x_{2n+1}, x_{2n+2})$, for some positive integer n. Then from (2.6), we have

$$\begin{aligned} F(d(x_{2n+2}, x_{2n+1})) &\leq F(d(x_{n+1}, x_{n+2})) - \psi(F(d(x_{n+1}, x_{n+2}))) \\ &< F(d(x_{2n+1}, x_{2n+2})), \end{aligned}$$

that is, $F(d(x_{n+1}, x_{n+2})) \leq 0$, which implies that $d(x_{2n+1}, x_{2n+2}) = 0$, and $x_{2n+1} = x_{2n+2}$, contradicting our assumption that $x_n \neq x_{n+1}$, for each n. Therefore,

$$F(d(x_{2n+2}, x_{2n+1})) < F(d(x_{2n+1}, x_{2n})). \tag{2.7}$$

Analogously, we have

$$F(d(x_{2n+1}, x_{2n})) < F(d(x_{2n}, x_{2n-1})). \tag{2.8}$$

Thus from (2.7) and (2.8), we get that (2.5) holds for all $n \in \mathbb{N}$. Monotonicity of F implies that also $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of positive real numbers, hence $\lim_{n\to\infty} d(x_{n+1}, x_n) = r \geq 0$. Suppose that $r > 0$. It follows from (2.6) that

$$F(d(x_{2n+2}, x_{2n+1})) \leq F(d(x_{2n+1}, x_{2n})) - \psi(F(d(x_{2n+1}, x_{2n}))).$$

Passing to the limit when $n \to \infty$ we get that

$$F(r) \leq F(r) - \psi(F(r)),$$

which is possible only if $r = 0$, because of the assumed properties of functions $F \in \mathcal{F}$ and $\psi \in \Psi$. Hence, we have proved that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.9}$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We proceed by negation and suppose that it is not. That is, there exists $\varepsilon > 0$ such that $d(x_n, x_m) \geq \varepsilon$ for infinitely many values of m and n with $m < n$. This assures that there exist two sequences $\{m(k)\}, \{n(k)\}$ of natural numbers, with $m(k) < n(k)$, such that for each $k \in \mathbb{N}$,

$$d(x_{2m(k)}, x_{2n(k)+1}) > \varepsilon. \tag{2.10}$$

It is not restrictive to suppose that $n(k)$ is the least positive integer exceeding $m(k)$ and satisfying (2.10). We have

$$\begin{aligned}\varepsilon &< d(x_{2m(k)}, x_{2n(k)+1})\\ &\leq d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2n(k)+1})\\ &\leq \varepsilon + d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2n(k)+1})\end{aligned}$$

and passing to the limit as $k \to \infty$, we have $d(x_{2m(k)}, x_{2n(k)+1}) \to \varepsilon$. We note that

$$\begin{aligned}&d(x_{2m(k)}, x_{2n(k)+1}) - d(x_{2m(k)}, x_{2m(k)+1}) - d(x_{2n(k)+2}, x_{2n(k)+1})\\ &\quad\leq d(x_{2m(k)+1}, x_{2n(k)+2})\\ &\quad\leq d(x_{2m(k)}, x_{2n(k)+1}) + d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2n(k)+2}, x_{2n(k)+1}),\end{aligned}$$

and thus $d(x_{2m(k)+1}, x_{2n(k)+2}) \to \varepsilon$ as $k \to \infty$. We have that

$$\begin{aligned}&M(x_{2n(k)+1}, x_{2m(k)}))\\ &\quad= \max\{d(x_{2n(k)+1}, x_{2m(k)}), d(x_{2n(k)+1}, x_{2n(k)+2}), d(x_{2m(k)}, x_{2m(k)+1}),\\ &\qquad \tfrac{1}{2}[d(x_{2n(k)+1}, x_{2m(k)+1}) + d(x_{2m(k)}, x_{2n(k)+2})]\}\\ &\quad\leq \max\{d(x_{2n(k)+1}, x_{2m(k)}), d(x_{2n(k)+1}, x_{2n(k)+2}), d(x_{2m(k)}, x_{2m(k)+1}),\\ &\qquad \tfrac{1}{2}[d(x_{2n(k)+1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)+2})]\}\end{aligned}$$

and so passing to the limit as $k \to \infty$ we have $\lim_{k\to\infty} M(x_{2n(k)+1}, x_{2m(k)}) \leq \varepsilon$. Therefore we have

$$\begin{aligned}&F(d(x_{2m(k)+1}, x_{2n(k)+2}))\\ &\quad\leq F(\delta(\mathcal{S}x_{2m(k)}, \mathcal{T}x_{2n(k)+1}))\\ &\quad\leq F(M(x_{2n(k)+1}, x_{2m(k)})) - \psi(F(\max\{d(x_{2n(k)+1}, x_{2m(k)}),\\ &\qquad \delta(x_{2n(k)+1}, Tx_{2n(k)+1}), \delta(x_{2m(k)}, Sx_{2m(k)})\}))\\ &\quad\leq F(M(x_{2n(k)+1}, x_{2m(k)})) - \psi(F(\max\{d(x_{2n(k)+1}, x_{2m(k)}),\\ &\qquad d(x_{2n(k)+1}, x_{2n(k)+2}), d(x_{2m(k)}, x_{2m(k)+1})\}))\end{aligned}$$

and passing to the limit as $k \to \infty$ in the above equation, F being continuous and ψ right-continuous, we get

$$F(\varepsilon) \leq F(\varepsilon) - \psi(F(\varepsilon)) < F(\varepsilon),$$

a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$ which is complete. Then, there exists $z \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now suppose that $\delta(z, \mathcal{S}z) > 0$. From (2.3), we have $x_{2n} \preceq z$ for all $n \in \mathbb{N}$. Hence, we can apply the considered contractive condition. Then, setting $x = x_{2n+1}$ and $y = z$ in (2.1), we obtain

$$\begin{aligned}
&F(\delta(x_{2n+2}, \mathcal{S}z)) \le F(\delta(\mathcal{T}x_{2n+1}, \mathcal{S}z)) \\
&\quad \le F(M(x_{2n+1}, z)) - \psi(F(\max\{d(x_{2n+1}, z), \delta(x_{2n+1}, \mathcal{T}x_{2n+1}), \delta(z, \mathcal{S}z)\}) \\
&\quad \le F(M(x_{2n+1}, z)) - \psi(F(\max\{d(x_{2n+1}, z), d(x_{2n+1}, x_{2n+2}), \delta(z, \mathcal{S}z)\})
\end{aligned}$$

where

$$\begin{aligned}
M(x_{2n+1}, z) &= \max\{d(x_{2n+1}, z), D(x_{2n+1}, \mathcal{T}x_{2n+1}), D(z, \mathcal{S}z), \\
&\qquad \tfrac{1}{2}[D(x_{2n+1}, \mathcal{S}z) + D(z, \mathcal{T}x_{2n+1})]\} \\
&= \max\{d(x_{2n+1}, z), d(x_{2n+1}, x_{2n+2}), D(z, \mathcal{S}z), \\
&\qquad \tfrac{1}{2}[d(x_{2n+1}, \mathcal{S}z) + d(z, x_{2n+2})]\}.
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality and using the continuity of F and right-continuity of ψ, we have

$$\begin{aligned}
F(\delta(z, \mathcal{S}z)) &\le F(D(z, \mathcal{S}z)) - \psi(F(\delta(z, \mathcal{S}z))) \\
&\le F(\delta(z, \mathcal{S}z)) - \psi(F(\delta(z, \mathcal{S}z))) \\
&< F(\delta(z, \mathcal{S}z)),
\end{aligned}$$

a contradiction. Therefore $\delta(z, \mathcal{S}z) = 0$ and thus $z \in \mathcal{S}z$. Hence, z is a fixed point of $\mathcal{S}$. Analogously, starting from $x = z$ and $y = x_{2n+1}$, one can prove that $z \in \mathcal{T}z$. It follows that $z \in \mathcal{S}z \cap \mathcal{T}z$, that is, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. ∎

In Theorem 2.2, if $\mathcal{T}, \mathcal{S}$ are single valued mappings and condition (2.3) is replaced by requiring that one of $\mathcal{T}$ and $\mathcal{S}$ is continuous, then we have the following result.

Theorem 2.3. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T}, \mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ be mappings such that*

$$F(d(\mathcal{T}x, \mathcal{S}y)) \le F(M(x, y)) - \psi(F(\max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{S}y)\}))$$

for all comparable $x, y \in \mathcal{X}$, where $F \in \mathcal{F}$, $\psi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{S}y), \frac{d(x, \mathcal{S}y) + d(y, \mathcal{T}x)}{2}\right\}.$$

Also suppose that $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing. If one of $\mathcal{S}$ and $\mathcal{T}$ is continuous, then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.

Proof. Consider $\mathcal{T}$ as a multivalued mapping for which $\mathcal{T}x$ is a singleton set for every $x \in \mathcal{X}$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 2.2. Following the line of its proof, we have that $\{x_n\}$ is a Cauchy sequence and

$$\lim_{n\to\infty} x_n = z.$$

Then, if $\mathcal{T}$ is continuous, we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}x_n = \mathcal{T}z$$

and this proves that z is a fixed point of $\mathcal{T}$ and so z is a fixed point of $\mathcal{S}$. Similarly, if $\mathcal{S}$ is continuous, we have the result. Thus it is immediate to conclude that $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. ∎

Putting $\mathcal{S} = \mathcal{T}$ in Theorem 2.2 we obtain the following

Corollary 2.4. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T} : \mathcal{X} \longrightarrow B(\mathcal{X})$ be a multivalued mapping such that*

$$F(\delta(\mathcal{T}x, \mathcal{T}y)) \leq F(M(x,y)) - \psi(F(\max\{d(x,y), \delta(x,\mathcal{T}x), \delta(y,\mathcal{T}y)\})) \tag{2.11}$$

for all comparable $x, y \in \mathcal{X}$, where $F \in \mathcal{F}$, $\psi \in \Psi$ and

$$M(x,y) = \max\left\{d(x,y), D(x,\mathcal{T}x), D(y,\mathcal{T}y), \frac{D(x,\mathcal{T}y) + D(y,\mathcal{T}x)}{2}\right\}.$$

Also suppose that $\mathcal{T}x \preceq_1 \mathcal{T}(\mathcal{T}x)$ for all $x \in \mathcal{X}$ and there is $x_0 \in \mathcal{X}$ such that $\{x_0\} \prec_1 \mathcal{T}x_0$. If the condition

$$\begin{cases} \textit{if } \{x_n\} \subset \mathcal{X} \textit{ is a non-decreasing sequence with } x_n \to z \textit{ in } \mathcal{X}, \\ \textit{then } x_n \preceq z \textit{ for all } n \end{cases}$$

holds, then $\mathcal{T}$ has a fixed point.

If $\mathcal{T}$ is a single-valued mapping in Corollary 2.4, then we have the following

Corollary 2.5. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping such that*

$$F(d(\mathcal{T}x, \mathcal{T}y)) \leq F(M(x,y)) - \psi(F(\max\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y)\}))$$

for all comparable $x, y \in \mathcal{X}$, where $F \in \mathcal{F}$, $\psi \in \Psi$ and

$$M(x,y) = \max\left\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), \frac{d(x,\mathcal{T}y) + d(y,\mathcal{T}x)}{2}\right\}.$$

Also suppose that $\mathcal{T}x \preceq \mathcal{T}(\mathcal{T}x)$ for all $x \in \mathcal{X}$. If the condition

$$\begin{cases} \textit{if } \{x_n\} \subset \mathcal{X} \textit{ is a non-decreasing sequence with } x_n \to z \textit{ in } \mathcal{X}, \\ \textit{then } x_n \preceq z \textit{ for all } n \end{cases}$$

holds, then $\mathcal{T}$ has a fixed point.

Remark 2.6. In [25, Corollary 2.5], it was proved that if

$$\text{every pair of elements has a lower bound and an upper bound,} \tag{2.12}$$

then for every $x \in \mathcal{X}$,

$$\lim_{n\to\infty} \mathcal{T}^n(x) = y,$$

where y is the fixed point of $\mathcal{T}$ such that

$$y = \lim_{n\to\infty} \mathcal{T}^n(x_0)$$

and hence $\mathcal{T}$ has a unique fixed point. If condition (2.12) fails, it is possible to find examples of mappings $\mathcal{T}$ with more than one fixed point. There exist some examples to illustrate this fact in [24].

We illustrate the results of this section by an example showing also that the use of order can be crucial.

Example 2.7. Let $\mathcal{X} = \{A, B, C\}$, where $A = (0,0)$, $B = (1,1)$, $C = (2,0) \in \mathbb{R}^2$. Metric d is defined as $d((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ so that $d(A,B) = 1$, $d(A,C) = 2$ and $d(B,C) = 1$. Order $\preceq$ is introduced by $(x_1,y_1) \preceq (x_2,y_2)$ iff $x_1 \le x_2$ and $y_1 \le y_2$, so that $A \preceq B$ and $A \preceq C$, while B and C are incomparable.

Consider the mapping $\mathcal{T} : \mathcal{X} \to B(\mathcal{X})$ given by

$$\mathcal{T} = \begin{pmatrix} A & B & C \\ \{A\} & \{A\} & \{A,B\} \end{pmatrix},$$

and functions $F \in \mathcal{F}$, $\psi \in \Psi$ given by $F(t) = \frac{1}{2}t$, $\psi(t) = \frac{1}{3}t$. To prove that condition (2.11) of Corollary 2.4 holds, it is enough to check that it is satisfied for $x = A$, $y = B$ and for $x = A$, $y = C$ (in the case when $x = y$, (2.11) is trivially satisfied).

If $x = A$, $y = B$, then $\mathcal{T}x = \mathcal{T}y = \{A\}$ and $\delta(\mathcal{T}x, \mathcal{T}y) = 0$, so (2.11) holds. If $x = A$, $y = C$, then

$$\delta(\mathcal{T}x, \mathcal{T}y) = \delta(\{A\}, \{A,B\}) = d(A,B) = 1,$$

and

$$\begin{aligned} M(x,y) &= \max\{d(A,C), D(A,\{A\}), D(C,\{A,B\}), \\ &\qquad \tfrac{1}{2}(D(A,\{A,B\}) + D(C,\{A\}))\} \\ &= \max\{2, 0, 1, \tfrac{1}{2}(0+2)\} = 2, \\ m(x,y) &= \max\{d(x,y), \delta(x,\mathcal{T}x), \delta(y,\mathcal{T}y)\} = \max\{2,0,2\} = 2. \end{aligned}$$

Hence, $F(\delta(\mathcal{T}x,\mathcal{T}y)) = \frac{1}{2} < \frac{2}{3} = F(M(x,y)) - \psi(F(m(x,y)))$. All other conditions of Corollary 2.4 are fulfilled and $\mathcal{T}$ has a fixed point A.

Note that for (incomparable) points $x = B$, $y = C$ condition (2.11) is not satisfied, and so the respective result in metric space without order cannot be applied to reach the conclusion. Indeed, in this case, $\mathcal{T}x = \{A\}$, $\mathcal{T}y = \{A, B\}$,

$$\begin{aligned}\delta(\mathcal{T}x, \mathcal{T}y) &= d(A, B) = 1,\\ M(x,y) &= \max\{1, 1, 1, \tfrac{1}{2}(0+2)\} = 1,\\ m(x,y) &= \max\{1,1,2\} = 2\end{aligned}$$

and $F(\delta(\mathcal{T}x, \mathcal{T}y)) = \frac{1}{2} > \frac{1}{6} = F(M(x,y)) - \psi(F(m(x,y)).$

3 Common fixed point results under almost contractive conditions

In this section, we prove common fixed point theorems for a pair of weakly isotone increasing multivalued mappings satisfying a variant of so-called almost contractive condition.

Theorem 3.1. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Assume that there is a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and that $\mathcal{T}, \mathcal{S} : \mathcal{X} \longrightarrow B(\mathcal{X})$ are multivalued mappings such that*

$$\begin{aligned}\delta(\mathcal{T}x, \mathcal{S}y) \leq{}& M(x,y)\\ &+ L\min\{\varphi(D(x,\mathcal{T}x)), \varphi(D(y,\mathcal{S}y)), \varphi(D(x,\mathcal{S}y)), \varphi(D(y,\mathcal{T}x))\},\end{aligned} \tag{3.1}$$

for all comparable $x, y \in \mathcal{X}$, where $L \geq 0$, and

$$\begin{aligned}M(x,y) = \max\Big\{&\varphi(d(x,y)), \varphi(D(x,\mathcal{T}x)), \varphi(D(y,\mathcal{S}y)),\\ &\varphi\left(\frac{D(x,\mathcal{S}y) + D(y,\mathcal{T}x)}{2}\right)\Big\}.\end{aligned} \tag{3.2}$$

Also suppose that $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing and there exists an $x_0 \in \mathcal{X}$ such that $\{x_0\} \prec_1 \mathcal{S}x_0$. If the condition

$$\begin{cases} \text{if } \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to z \text{ in } \mathcal{X},\\ \text{then } x_n \preceq z \text{ for all } n\end{cases} \tag{3.3}$$

holds, then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.

Proof. First of all we show that, if $\mathcal{S}$ or $\mathcal{T}$ has a fixed point, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Indeed, let z be a fixed point of $\mathcal{S}$ and assume $\delta(z, \mathcal{T}z) >$

0. If we use the inequality (3.1) for $x = y = z$, and properties of φ, we have

$$\begin{aligned}
&\delta(\mathcal{T}z, \mathcal{S}z) \\
&\quad \leq \max\Big\{\varphi(d(z,z)), \varphi(D(z,\mathcal{T}z)), \varphi(D(z,\mathcal{S}z)), \\
&\qquad\quad \varphi\left(\frac{D(z,\mathcal{S}z) + D(z,\mathcal{T}z)}{2}\right)\Big\} \\
&\qquad + L\min\{\varphi(D(z,\mathcal{T}z)), \varphi(D(z,\mathcal{S}z)), \varphi(D(z,\mathcal{S}z)), \varphi(D(z,\mathcal{T}z))\} \\
&\quad = \max\left\{\varphi(D(z,\mathcal{T}z)), \varphi\left(\frac{1}{2}D(z,\mathcal{T}z)\right)\right\} \\
&\quad < D(z,\mathcal{T}z),
\end{aligned}$$

which is a contradiction. Thus $\delta(z, \mathcal{T}z) = 0$ and so z is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Starting with the given x_0, define a sequence $\{x_n\}$ as follows:

$$x_{2n+1} \in \mathcal{S}x_{2n}, \ x_{2n+2} \in \mathcal{T}x_{2n+1} \ \text{ for } \ n \geq 0. \tag{3.4}$$

Note $x_1 \in \mathcal{S}x_0$ such that $x_0 \preceq x_1$ and since $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing, $\mathcal{S}x_0 \preceq_1 \mathcal{T}y$ for all $y \in \mathcal{S}x_0$. Hence, we have $\mathcal{S}x_0 \preceq_1 \mathcal{T}x_1$. In particular $\mathcal{S}x_0 \preceq_1 x_2$, and so $x_1 \preceq x_2$. Continuing this process we construct a monotone increasing sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots . \tag{3.5}$$

If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Hence we shall assume that $x_n \neq x_{n+1}$, for all $n \geq 0$.

Now we claim that for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}). \tag{3.6}$$

From (3.5) we have that $x_n \prec x_{n+1}$ for all $n \in \mathbb{N}$. Then from (3.1) with $x = x_n$, $y = x_{n+1}$ and $n = 2k - 1$, $k \in \mathbb{N}$, we get

$$\begin{aligned}
&d(x_{n+1}, \ x_{n+2}) \\
&\quad \leq \delta(\mathcal{T}x_n, \mathcal{S}x_{n+1}) \\
&\quad \leq M(x_n, x_{n+1}) + L\min\{\varphi(D(x_n, \mathcal{T}x_n)), \\
&\qquad \varphi(D(x_{n+1}, \mathcal{S}x_{n+1})), \varphi(D(x_n, \mathcal{S}x_{n+1})), \varphi(D(x_{n+1}, \mathcal{T}x_n))\} \\
&\quad \leq M(x_n, x_{n+1}) + L\min\{\varphi(d(x_n, x_{n+1})), \varphi(d(x_{n+1}, x_{n+2})), \\
&\qquad \varphi(d(x_n, x_{n+2})), \varphi(d(x_{n+1}, x_{n+1}))\}.
\end{aligned} \tag{3.7}$$

By (3.2), we have

$$\begin{aligned}
&M(x_n, x_{n+1})\\
&\quad = \max\{\varphi(d(x_n, x_{n+1})), \varphi(D(x_n, \mathcal{T}x_n)), \varphi(D(x_{n+1}, \mathcal{S}x_{n+1})),\\
&\qquad \varphi\left(\frac{D(x_n, \mathcal{S}x_{n+1}) + D(x_{n+1}, \mathcal{T}x_n)}{2}\right)\right\}\\
&\quad = \max\{\varphi(d(x_n, x_{n+1})), \varphi(d(x_n, x_{n+1})), \varphi(d(x_{n+1}, x_{n+2})),\\
&\qquad \varphi\left(\frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right)\right\}\\
&\quad = \max\left\{\varphi(d(x_n, x_{n+1})), \varphi(d(x_{n+1}, x_{n+2})), \varphi\left(\frac{1}{2}d(x_n, x_{n+2})\right)\right\}.
\end{aligned}$$

• If $M(x_n, x_{n+1}) = \varphi(d(x_{n+1}, x_{n+2}))$, by (3.7) and using the fact that $\varphi(t) < t$ for all $t > 0$, we have

$$d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_{n+1}, x_{n+2})) < d(x_{n+1}, x_{n+2}),$$

a contradiction.
• If $M(x_n, x_{n+1}) = \varphi\left(\frac{1}{2}d(x_n, x_{n+2})\right)$, we get

$$d(x_{n+1}, x_{n+2}) \leq \varphi\left(\frac{1}{2}d(x_n, x_{n+2})\right) < \frac{1}{2}d(x_n, x_{n+2}).$$

On the other hand, by the triangular inequality, we have

$$\frac{1}{2}d(x_n, x_{n+2}) \leq \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_{n+2}).$$

Thus, we have

$$d(x_{n+1}, x_{n+2}) < \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_{n+2}),$$

which implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}).$$

• If $M(x_n, x_{n+1}) = \varphi(d(x_n, x_{n+1}))$, we get

$$d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}).$$

Then, in all cases, we have $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n = 2k-1$, $k \in \mathbb{N}$. Similarly, we can prove that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n = 2k$, $k \in \mathbb{N}$. Therefore, we conclude that (3.6) holds.

Now, from (3.6) it follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing. Therefore, there is some $\lambda \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lambda. \tag{3.8}$$

We are able to prove that $\lambda = 0$. In fact, by the triangular inequality, we get

$$\frac{1}{2}d(x_n, x_{n+2}) \leq \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_{n+2}).$$

By (3.6), we have

$$\frac{1}{2}d(x_n, x_{n+2}) \le d(x_n, x_{n+1}). \tag{3.9}$$

From (3.9), taking the upper limit as $n \to \infty$, we get

$$\limsup_{n\to\infty} \frac{1}{2}d(x_{2n}, x_{2n+2}) \le \lim_{n\to\infty} d(x_{2n}, x_{2n+1}).$$

If we set

$$\limsup_{n\to\infty} \frac{1}{2}d(x_{2n}, x_{2n+2}) = b, \tag{3.10}$$

then clearly $0 \le b \le \lambda$. As φ is continuous and taking the upper limit on both sides of (3.7), we get

$$\limsup_{n\to+\infty} d(x_{2n+1}, x_{2n+2}) \le \max\Big\{ \varphi(\limsup_{n\to+\infty} d(x_{2n+1}, x_{2n+2})), \\ \varphi(\limsup_{n\to+\infty}(d(x_{2n+1}, x_{2n})), \varphi\left(\frac{1}{2}(\limsup_{n\to+\infty} d(x_{2n}, x_{2n+2}))\right) \Big\}.$$

Hence by (3.8) and (3.10), we deduce

$$\lambda \le \max\{\varphi(\lambda), \varphi(b)\}.$$

If we suppose that $\lambda > 0$, then we have

$$\lambda \le \max\{\varphi(\lambda), \varphi(b)\} < \max\{\lambda, b\} = \lambda,$$

a contradiction. Thus $\lambda = 0$ and consequently

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.11}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. To this end, it is sufficient to verify that $\{x_{2n}\}$ is a Cauchy sequence. Suppose, on the contrary, that it is not. Then, there exists an $\varepsilon > 0$ such that for each even integer $2k$ there are even integers $2n(k)$, $2m(k)$ with $2m(k) > 2n(k) > 2k$ such that

$$r_k = d(x_{2n(k)}, x_{2m(k)}) \ge \varepsilon \text{ for } k \in \{1, 2, \dots\}. \tag{3.12}$$

For every even integer $2k$, let $2m(k)$ be the smallest number exceeding $2n(k)$ satisfying condition (3.12) for which

$$d(x_{2n(k)}, x_{2m(k)-2}) < \varepsilon. \tag{3.13}$$

From (3.12), (3.13) and the triangular inequality, we have

$$\varepsilon \le r_k \le d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}) \\ \le \varepsilon + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}).$$

Hence by (3.11), it follows that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{3.14}$$

Now, from the triangular inequality, we have

$$|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leq d(x_{2m(k)-1}, x_{2m(k)}).$$

Letting $k \to \infty$ and using (3.11) and (3.14), we get

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}
&d(x_{2n(k)}, x_{2m(k)}) \qquad (3.16)\\
&\quad\leq d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)})\\
&\quad\leq d(x_{2n(k)}, x_{2n(k)+1}) + \delta(\mathcal{S}x_{2n(k)}, \mathcal{T}x_{2m(k)-1})\\
&\quad\leq d(x_{2n(k)}, x_{2n(k)+1}) + M(x_{2m(k)-1}, x_{2n(k)})\\
&\qquad + L\min\{\varphi(d(x_{2m(k)-1}, \mathcal{T}x_{2m(k)-1})), \varphi(D(x_{2n(k)}, \mathcal{S}x_{2n(k)})),\\
&\qquad \varphi(D(x_{2m(k)-1}, \mathcal{S}x_{2n(k)})), \varphi(D(x_{2n(k)}, \mathcal{T}x_{2m(k)-1}))\}\\
&\quad= d(x_{2n(k)}, x_{2n(k)+1}) + M(x_{2m(k)-1}, x_{2n(k)})\\
&\qquad + L\min\{\varphi(d(x_{2m(k)-1}, x_{2m(k)})), \varphi(d(x_{2n(k)}, x_{2n(k)+1})),\\
&\qquad \varphi(d(x_{2m(k)-1}, x_{2n(k)+1})), \varphi(d(x_{2n(k)}, x_{2m(k)}))\},
\end{aligned}$$

where

$$\begin{aligned}
M(x_{2m(k)-1}, x_{2n(k)}) = \max\Big\{&\varphi(d(x_{2m(k)-1}, x_{2n(k)})),\\
&\varphi(d(x_{2m(k)-1}, x_{2m(k)})), \varphi(d(x_{2n(k)}, x_{2n(k)+1})),\\
&\varphi\left(\frac{d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)-1}, x_{2n(k)+1})}{2}\right)\Big\}.
\end{aligned}$$

From

$$\begin{aligned}
d(x_{2m(k)-1}, x_{2n(k)+1}) \leq\ & d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2n(k)})\\
&+ d(x_{2n(k)}, x_{2n(k)+1}),
\end{aligned}$$

taking the upper limit as $k \to \infty$, using (3.11) and (3.14), we get

$$\limsup_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)+1}) \leq \varepsilon.$$

On the other hand, we have

$$\begin{aligned}
\varepsilon \leq\ & d(x_{2m(k)}, x_{2n(k)})\\
\leq\ & d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)})
\end{aligned}$$

and taking the lower limit as $k \to \infty$, we get

$$\varepsilon \leq \liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) \leq \liminf_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)+1}).$$

It follows that

$$\varepsilon \le \liminf_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)+1}),$$

and so

$$\lim_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)+1}) = \varepsilon. \tag{3.17}$$

Now, using (3.11), (3.14), (3.15), (3.17) and the continuity of φ, we get

$$\lim_{k\to\infty} M(x_{2m(k)-1}, x_{2n(k)}) = \max\{\varphi(\varepsilon), 0, 0, \varphi(\varepsilon)\} = \varphi(\varepsilon). \tag{3.18}$$

Letting $k \to \infty$ in (3.16), we obtain

$$\varepsilon \le \varphi(\varepsilon) < \varepsilon,$$

a contradiction. Thus, assumption (3.12) is wrong. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of $\mathcal{X}$, there exists a $z \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = z.$$

As the limit point z is independent of the choice of x_n, we also get

$$\lim_{n\to\infty} D(\mathcal{S}x_{2n}, z) = \lim_{n\to\infty} D(\mathcal{T}x_{2n+1}, z) = 0.$$

Now we show that z is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

By the assumption (3.3), $x_n \preceq z$, for all n. Then by the property of sequence $\{x_n\}$, for $x = x_{2n+1}$ and $y = z$, we have

$$\begin{aligned} &\delta(\mathcal{T}x_{2n+1}, \mathcal{S}z) \\ &\le \max\Big\{\varphi(d(x_{2n+1}, z)), \varphi(D(x_{2n+1}, \mathcal{T}x_{2n+1})), \varphi(D(z, \mathcal{S}z)), \\ &\quad \varphi\left(\frac{D(z, \mathcal{T}x_{2n+1}) + D(x_{2n+1}, \mathcal{S}z)}{2}\right)\Big\} + L\min\{\varphi(D(x_{2n+1}, \\ &\quad \mathcal{T}x_{2n+1})), \varphi(D(z, \mathcal{S}z)), \varphi(D(x_{2n+1}, \mathcal{S}z)), \varphi(D(z, \mathcal{T}x_{2n+1}))\}. \end{aligned} \tag{3.19}$$

Letting $n \to \infty$ and using properties of φ, we have

$$\delta(z, \mathcal{S}z) \le \max\{\varphi(D(z, \mathcal{S}z)), \varphi(D(z, \mathcal{S}z)/2)\} < D(z, \mathcal{S}z),$$

which is a contradiction, unless $\delta(z, \mathcal{S}z) = 0$ and so $z \in \mathcal{S}z$. Analogously, starting from $x = z$ and $y = x_{2n}$, one can prove that $z \in \mathcal{T}z$. It follows that $z \in \mathcal{S}z \cap \mathcal{T}z$, that is, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. ∎

Remark 3.2. (i) The condition

$$\begin{aligned} \delta(\mathcal{T}x, \mathcal{S}y) &\le \varphi(M_1(x, y)) \\ &\quad + L\min\{\varphi(D(x, \mathcal{T}x)), \varphi(D(y, \mathcal{S}y)), \varphi(D(x, \mathcal{S}y)), \varphi(D(y, \mathcal{T}x))\}, \end{aligned} \tag{3.20}$$

where

$$M_1(x, y) = \max\left\{d(x, y), D(x, \mathcal{T}x), D(y, \mathcal{S}y), \frac{D(y, \mathcal{T}x) + D(x, \mathcal{S}y)}{2}\right\},$$

implies condition (3.1).

(ii) Condition (3.20) is equivalent to condition (3.1) if we suppose that φ is a non-decreasing function.

(iii) From Theorem 3.1 we can derive a corollary involving condition (3.20);

(iv) Under the hypothesis that φ is a non-decreasing function, we can state many others corollaries using the equivalences established by Jachymski in [17] for single valued mappings.

Example 3.3. *Let $\mathcal{X} = [0, +\infty)$ be equipped with the standard metric d and order $\preceq$ given by*

$$x \preceq y \iff x \geq y.$$

Consider the following mappings $\mathcal{T}, \mathcal{S} : \mathcal{X} \to B(\mathcal{X})$:

$$\mathcal{T}x = \left[\frac{1}{4}x, \frac{1}{3}x\right], \qquad \mathcal{S}x = \left[\frac{1}{5}x, \frac{3}{10}x\right], \qquad x \in [0, +\infty).$$

Check first that $\mathcal{T}$ and $\mathcal{S}$ are weakly isotone increasing. Suppose that $y \in \mathcal{S}x = [\frac{1}{5}x, \frac{3}{10}x]$ and $z \in \mathcal{S}x = [\frac{1}{5}x, \frac{3}{10}x]$. Then $u \in \mathcal{T}y$ implies that $u \leq \frac{1}{3} \cdot \frac{3}{10}x = \frac{1}{10}x < \frac{1}{5}x \leq z$ and so $z \preceq u$. This means that for any $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq_1 \mathcal{T}y$ for all $y \in \mathcal{S}x$. Similarly, one can prove that for each $x \in \mathcal{X}$ we have $\mathcal{T}x \preceq_1 \mathcal{S}y$ for all $y \in \mathcal{T}x$.

Let $\varphi(t) = \frac{1}{2}t$ for $t \in [0, +\infty)$ and $L = 1$. Now we check that condition (3.1) *holds for all $x, y \in \mathcal{X}$. Consider the following two possibilities.*

1. $x \preceq y$, i.e., $x \geq y$. Denote $y = xz$, $0 \leq z \leq 1$. Then

$$\begin{aligned}
\delta(\mathcal{T}x, \mathcal{S}y) &= \delta([\tfrac{1}{4}x, \tfrac{1}{3}x], [\tfrac{1}{5}y, \tfrac{3}{10}y]) = \tfrac{1}{3}x - \tfrac{1}{5}y = x(\tfrac{1}{3} - \tfrac{1}{5}z) \\
&\leq \frac{1}{3}x, \\
M(x, y) &= \frac{1}{2}\max\{x - y, \tfrac{2}{3}x, \tfrac{7}{10}y, \tfrac{1}{2}[(x - \tfrac{3}{10}y) + |y - \tfrac{1}{3}x|]\} \\
&= \frac{1}{2}x\max\{1 - z, \tfrac{2}{3}, \tfrac{7}{10}z, \tfrac{1}{2}[(1 - \tfrac{3}{10}z) + |z - \tfrac{1}{3}|]\} \\
&\geq \frac{1}{2}x \cdot \frac{2}{3} = \frac{1}{3}x, \\
m(x, y) &= \min\{\varphi(D(x, \mathcal{T}x)), \varphi(D(y, \mathcal{T}y)), \varphi(D(x, \mathcal{S}y)), \varphi(D(y, \mathcal{S}x))\} \\
&= \frac{1}{2}x\min\{\tfrac{2}{3}, \tfrac{7}{10}z, 1 - \tfrac{3}{10}z, |z - \tfrac{1}{3}|\} \geq 0.
\end{aligned}$$

Hence, we obtain that

$$\delta(\mathcal{T}x, \mathcal{S}y) \leq M(x, y) + Lm(x, y) \tag{3.21}$$

is satisfied.

2. $x \succ y$, i.e., $x < y$ and $x = yz$ for some $z \in (0,1)$. Then

$$\begin{aligned}
\delta(\mathcal{T}x, \mathcal{S}y) &= y\delta([\tfrac{1}{4}z, \tfrac{1}{3}z], [\tfrac{1}{5}, \tfrac{3}{10}]) = y \times \begin{cases} \frac{3}{10} - \frac{1}{4}z, & 0 < z < \frac{4}{5} \\ \frac{1}{10}, & \frac{4}{5} \le z < \frac{9}{10} \\ \frac{1}{3}z - \frac{1}{5}, & \frac{9}{10} \le z < 1, \end{cases} \\
&\le \frac{3}{10}y, \\
M(x,y) &= \frac{1}{2}y \max\{1 - z, \tfrac{2}{3}z, \tfrac{7}{10}, \tfrac{1}{2}[\psi(z) + (1 - \tfrac{1}{3}z)]\}, \\
&\ge \frac{1}{2}y \cdot \frac{7}{10} = \frac{7}{20}y > \frac{3}{10}y \\
m(x,y) &= \frac{2}{3}y \min\{\tfrac{2}{3}z, \tfrac{7}{10}, \psi(z), 1 - \tfrac{1}{3}z\} \ge 0.
\end{aligned}$$

Again we obtain that condition (3.21) *is satisfied.*

Thus, all the conditions of Theorem 3.1 are fulfilled, and $\mathcal{T}$ and $\mathcal{S}$ have a fixed point ($z = 0$).

Similar corollaries can be obtained as in the previous section. Putting $\mathcal{S} = \mathcal{T}$ in Theorem 3.1, we obtain immediately the following result.

Corollary 3.4. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Assume that there is a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and that $\mathcal{T} : \mathcal{X} \longrightarrow B(\mathcal{X})$ is a multivalued mapping such that*

$$\begin{aligned}
&\delta(\mathcal{T}x, \mathcal{T}y) \le M(x,y) \\
&\qquad + L \min\{\varphi(D(x, \mathcal{T}x)), \varphi(D(y, \mathcal{T}y)), \varphi(D(x, \mathcal{T}y)), \varphi(D(y, \mathcal{T}x))\},
\end{aligned}$$

for all comparable $x, y \in \mathcal{X}$, where $L \ge 0$, and

$$\begin{aligned}
M(x,y) = \max\Big\{ &\varphi(d(x,y)), \varphi(D(x, \mathcal{T}x)), \varphi(D(y, \mathcal{T}y)), \\
&\varphi\left(\frac{D(x, \mathcal{T}y) + D(y, \mathcal{T}x)}{2}\right)\Big\}.
\end{aligned}$$

Also suppose that $\mathcal{T}x \preceq_1 \mathcal{T}(\mathcal{T}x)$ for all $x \in \mathcal{X}$ and that there is $x_0 \in \mathcal{X}$ such that $\{x_0\} \prec_1 \mathcal{T}x_0$. If the condition

$$\begin{cases} \text{if } \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to z \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq z \text{ for all } n \end{cases}$$

holds, then $\mathcal{T}$ has a fixed point.

To conclude this section, we provide a sufficient condition to ensure the uniqueness of the fixed point in the above Theorem 3.1.

First, we recall the usual definition of the diameter of a set $\mathcal{A}$ in a metric space $(\mathcal{X}, d)$:

$$\operatorname{diam}(\mathcal{A}) := \sup\{d(x,y) : x, y \in \mathcal{A}\} \text{ i.e. } \delta(\mathcal{A}, \mathcal{A}) = \operatorname{diam}\mathcal{A},$$

for any subset $\mathcal{A}$ of $\mathcal{X}$. Then, we get the following theorem.

Theorem 3.5. *Adding to the hypotheses of Theorem 3.1 the following condition:*

$$\lim_{n\to\infty} \operatorname{diam}\left((\mathcal{T}\circ\mathcal{S})^n(\mathcal{X})\right) = 0,$$

where $\circ$ *denotes the composition of mappings, we obtain the uniqueness of the common fixed point of* $\mathcal{S}$ *and* $\mathcal{T}$*.*

Proof. Let z and z' be two common fixed points of $\mathcal{S}$ and $\mathcal{T}$, that is,

$$z \in \mathcal{T}z \cap \mathcal{S}z \quad \text{and} \quad z' \in \mathcal{T}z' \cap \mathcal{S}z'.$$

It is immediate to show that for all $n \in \mathbb{N}$, we have:

$$(\mathcal{T}\circ\mathcal{S})^n x = x, \text{ for all } x \in \{z, z'\}.$$

Then

$$d(z,z') = \delta((\mathcal{T}\circ\mathcal{S})^n z, (\mathcal{T}\circ\mathcal{S})^n z') \le \operatorname{diam}((\mathcal{T}\circ\mathcal{S})^n(\mathcal{X})) \to 0 \text{ as } n \to \infty.$$

Hence $z = z'$ and the proof is completed. ∎

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