Real elliptic surfaces with double section and real tetragonal curves

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Abstract

We give an upper bound for the number of nested ovals of a real tetragonal curve R embedded in an Hirzubruch surface in terms of the genus of the curve.

1 Introduction

A real elliptic surface will be a morphism $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ defined over $\mathbb{R}$, when Y is a real algebraic surface such that over all but finitely many points in the basic curve, the fibre is a nonsingular curve of genus one. See [18, Chapter 7]
We suppose that Π_1 admits at least one singular fibre, the surface is called regular $(H^1(X, \mathcal{O}_X) = 0)$ and that no fibre of Π contains a (-1)-curve; the fibration is called relatively minimal.

Now, one can associate to $\Pi_1 : Y \longrightarrow \mathbb{P}^1$, its Jacobian fibration $\Pi : Jac(Y) \longrightarrow \mathbb{P}^1$, defined over $\mathbb{R}$. By construction Π' admits a real section $s : \mathbb{P}^1 \longrightarrow Jac(Y)$. see[18, Chapter 7].

After contracting all components of the fibres of Π that do not intersect the curve $s(\mathbb{P}^1)$, we obtain the Weierstrass model of the Jacobian elliptic surface J^w with a proper map $\widetilde{\Pi} : J^w \longrightarrow B$ where J^w has at worst simple singular points, and a section $s' : \mathbb{P}^1 \longrightarrow J^w$ not passing through the singular points of J^w. One

Received by the editors June 2010.
Communicated by M. Van den Bergh.
2000 *Mathematics Subject Classification :* 14J27, 14P25.
Key words and phrases : real conic bundle, real elliptic surface, Betti numbers, real tetragonal curve.

can obtain J from J^w by resolving its singularities. The quotient of J^w by the fiberwise multiplication by (-1)-automorphism is a geometrically ruled surface $\mathbb{F}_{2n}$ over $\mathbb{P}^1$, the section s mapping to a section of $pro : \mathbb{F}_{2n} \longrightarrow \mathbb{P}^1$.
The projection $J^w \longrightarrow \mathbb{F}_{2n}$ is a double covering defined by the (fiberwise) linear system $|2s'|$ on J^w; its branch curve is the disjoint union of the exceptional section s' and some trigonal curve C on $\mathbb{F}_{2n}$.
Over $\mathbb{P}^1$, the topology of real elliptic surfaces with real section and real trigonal curves is due to F. Bihan and F. Mangolte, see [4].
Over a base of an arbitrary genus, the study of equivariant deformation of real elliptic surfaces and real trigonal curves is due to to A. Degtyarev, I. Itenberg and V. Kharlamov. see [7]
A reduced smooth curve is called tetragonal if it admits a $(4 : 1)$-cover to $\mathbb{P}^1$. Then R admits a natural embedding into a certain Hirzebruch surface F.
The aim of the paper is to obtain an upper bound for the number of nested ovals of a real tetragonal curve R embedded in an Hirzebruch surface in terms of the genus of the curve as it is cited in theorem 7.4.

Acknowlgements: We are grateful to A. Degtyarev and F. Mangolte for their helpful remarks and suggestions.
We are grateful to the referee for valuable comments on a previous version of the paper.
We thank also J. C. Douai for many stimulating discussions and encouragement during the initial phase of this work.
The second author would like to thank R. Van Luijk for the hospitality he enjoyed at Leiden.
Finally, the second author would like to thank M. Reid and S. Siksek for a visit in Warwick.
This paper is partially supported by research team 05/UR/15-02 Bizerte and the university of Monastir.

2 Main results

A real structure on a complex variety X is an anti-holomorphic involution $\sigma_X : X \longrightarrow X$.
A real variety is a complex variety X equipped with a real structure σ_X. The fixed points set $Fix(\sigma_X)$ is called the real part of X and is denoted $X(\mathbb{R})$. An holomorphic map $f : X \longrightarrow Y$ between two real varieties (X, σ_X) and (Y, σ_Y) is called real if it commutes with real structures.
The following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \sigma_X \downarrow & & \downarrow \sigma_Y \\ X & \xrightarrow{f} & Y \end{array}$$

see [18, Chapter 1].
The paper is organized as follows:

In section 3, we explicit the fiberwise double covering, defined over $\mathbb{R}$, of a given real projective rational surface branched along some real curve C.

Now, let $pro_1 : \mathbb{F}_e \longrightarrow \mathbb{P}^1$ be a real geometrically ruled surface[1] with a real curve $R \subset \mathbb{F}_e$ which induces a 4:1 map $pro_1 : R \longrightarrow \mathbb{P}^1$ that we call real tetragonal curve.
In section 5, we explicit the equivariant Recillas geometric construction of the trigonal curve associated to the tetragonal curve R. Next, we prove:
PropositionTwo elliptic surfaces with double section are deformation equivalents over $\mathbb{C}$, if and only if $g(R) = g(R')$, where R and R' are tetragonal curves associated to the given surfaces.

Then, we give a simple algebraic characterization for real elliptic surfaces with double section (over $\mathbb{C}$) which have real section.
Proposition Let Y be an elliptic surface over $\mathbb{R}$ given by the equation:

$$y^2 = a_0(t)x^4 + a_1(t)x^3 + a_2(t)x^2 + a_3(t)x + a_4(t)$$

and the triplet $(\Sigma, R, \mathbb{P}^1)$ is the geometrically ruled surface $pro : \Sigma \longrightarrow \mathbb{P}^1$ with a tetragonal curve R associated to Y.
Then Y admit a section over $\mathbb{R}$ if and only if there exist a fiberwise automorphism defined over $\mathbb{R}$ which sends R to real tetragonal curve R' given by the equation $a'_0x^4 + a'_1x^3 + a'_2x^2 + a'_3x + a'_4 = 0$ where a'_0 is a square of a real rational map defined over $\mathbb{P}^1\mathbb{C}$.

In section 6, we recall some basic results concerning topology of real part $X(\mathbb{R})$ of a given real smooth algebraic surface X.
Finally, in section 7, we study topology of the real part $R(\mathbb{R})$ for a real tetragonal curve embedded in an Hirzubruch surface and we show how one can use real elliptic surface with double real or complex conjugate section to compute topology of the real part for real tetragonal curves.

We prove:

Proposition A real elliptic surface with double section (over $\mathbb{C}$) is a $(M-d)$-variety if and only if is the associated tetragonal curve is a $(M-d)-curve$.

From [4], we deduce :

Proposition Let C is a real trigonal curve in a geometrically ruled surface $\mathbb{F}_{2n}$. If C has real scheme $< a, b >$, then $a \leq 5\frac{g+2}{6}$ and $b \leq 5\frac{g+2}{6}$, where g is the genus of C.

Next, we give a relation between the genus of the tetragonal curve and an associated trigonal curve.

[1]Miles Reid suggests that the 'F' might stand for 'Fritz' Hirzebruch's firstname

Proposition Let R be a tetragonal curve, and C be an associated trigonal curve. Then,

$$g(R) = g(C) - 1.$$

Finally, we give bounds for the number of nested ovals (denoted by a) and the others ovals (denoted by b) of a given real tetragonal curve embedded in an Hirzubruch surface.

Theorem If R is a real tetragonal curve embedded in an Hirzubruch surface. Then,

$$(i)\ \ a \leq \frac{5}{6}(g(R) + 3),$$

and

$$(ii)\ \ b \leq \frac{5}{6}(g(R) + 3).$$

In this present paper, we use the recent works of A. Degtyarev, I. Itenberg and V. Kharlamov about equivariant deformation of real elliptic surfaces $\Pi : Y \longrightarrow \mathbb{P}^1$, not necessary with real section, see [7] and of some results of the second author (joint with F. Mangolte), see [1].

3 Double covers of rational surfaces

Let Y a real smooth projective rational surface, $B \subset Y$ a real reduced smooth effective divisor on Y or zero. Suppose we have a real line bundle $\mathcal{L}$ on Y such that $\mathcal{O}_Y(B) = \mathcal{L}^2$ and a real section $s \in \Gamma(Y, \mathcal{O}_Y(B))$ vanishing exactly along B (if $B = 0$ we take for s the constant function 1).

We denote by L the total space of $\mathcal{L}$ and we let $p : L \longrightarrow Y$ be the bundle projection. If $t \in \Gamma(L, p^*\mathcal{L})$ is the tautological section, then the zero divisor of $p^*s - t^2$ defines a real analytic subspace X in L.

If $B \neq 0$, since B is reduced and smooth, then also X is smooth $\Pi = p/X$ exhibits X as a real 2-fold ramified covering of Y with branch-locus B. We call $f : X \longrightarrow Y$ the real 2-cyclic covering (or real ramified double covering) of Y branched along B, determined by $\mathcal{L}$.

If $B = 0$, we take $\mathcal{L} \neq \mathcal{O}_Y$; in this case $f : X \longrightarrow Y$ is called the real 2-cyclic unramified covering of Y determined by $\mathcal{L}$.

Conversely, given $\Pi : X \longrightarrow Y$ a real finite morphism of degree two between smooth real projective surfaces, we can recover B and $\mathcal{L}$ as follows:

Let $\tau : X \longrightarrow X$ be the sheet interchange equivariant involution, i. e. , $\tau^2 = id$, $\Pi \circ \tau = \Pi$. Then B is the image under τ of the fixed set of τ and $\Pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L}^{-1}$, where the direct sum decomposition corresponds to taking the $+1$ and -1 eigenspaces of τ acting on $\Pi_*\mathcal{O}_X$.

The morphism $f : X \longrightarrow Y$ induces the natural real homomorphism

$$f^* : Pic(Y) \longrightarrow Pic(X).$$

Now, Y and X are projective, by using the natural map $f_* : Pic(X) \longrightarrow Pic(Y)$ between the group of divisors on X and on Y, one obtains a real homomorphism $f_* : Pic(X) \longrightarrow Pic(Y)$.

Remark 3.1. The surface X has at most singularities over singular points of B. In particular, if B is reduced and smooth then also X is smooth.

We recall a general theorem relation the 2-torsion in $Br(X)$ of a double cover X of rational surface to the 2-torsion of Jacobian $J(C)$ of the branch curve.

Theorem 3.2. *Let $f : X \longrightarrow Y$ be a double cover of a smooth complex projective rational surface Y branched along a smooth curve C. Let $G = \mathbb{Z}/2\mathbb{Z}$ be the subgroup of Aut(X) generated by the covering involution σ. Assume that σ acts trivially on $Pic(X)$ is torsion-free.*

Then there is a natural inclusion

$$J(C)_2 \hookrightarrow Br(X)_2$$

with $Br(X)_2/J(C)_2 \cong (\mathbb{Z}/2\mathbb{Z})^n$ and $n = 2(1 + b_2(Y) - b_0(C)) - \rho$ where $\rho = rank \ \ [NS(X)]$ and b_i is the ith Betti number.

Proof: See [3]. ∎

Assume now, Y is a geometrically ruled surface. We denote it by $\mathbb{F}_n$ with fibration $pro : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ which is the compactification of the line bundle $\mathcal{O}(n) \in Pic(\mathbb{P}^1)$ with a rational curve C_∞ at infinity, and with self-intersection $C_\infty^2 = -n$.

It is well-known that this surface, is obtained by blowing up the cone in $\mathbb{P}^{n+1}$ over the rational curve of degree n in the vertex, for $n \geq 1$. In the case $n = 0$; the surface $\mathbb{F}_0$ is isomorphic to the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$.

Now, if f is a fiber of the fibration $pro : \mathbb{F}_n \longrightarrow \mathbb{P}^1$, then

$$Pic(\mathbb{F}_n) \cong \mathbb{Z}f \oplus \mathbb{Z}C_\infty; \ \ f^2 = 0; \ \ fC_\infty = 1 \ \ C_\infty^2 = -n.$$

The image in $\mathbb{F}_n = \mathcal{O}(n) \coprod C_\infty$ of a global section of $\mathcal{O}(n)$ is a section H of $pro : \mathbb{F}_n \longrightarrow \mathbb{P}^1$. It is also an hyperplane section of the cone in $\mathbb{P}^{n+1}$ which doesn't meet the vertex.

The Chern class of canonical line bundle K on $\mathbb{F}_n$ is given by

$$c_1(K) = -2C_\infty - (n+2)f.$$

In particular, $K^2 = 8$.

4 Jacobian elliptic surface

In This section, we discuss certain complex aspects of Jacobian elliptic surfaces. Most of the material presented here is not very original. It has a large overlap with Bert Van Geemen's paper [3].

Let $\Pi : X \longrightarrow \mathbb{P}^1$ be a regular elliptic surface with a section $s : \mathbb{P}^1 \longrightarrow X$. Then

X is a double cover of a ruled surface $pro : \mathbb{F}_{2n} \longrightarrow \mathbb{P}^1$ for some positif integer n. The branch locus of the double cover $X \longrightarrow \mathbb{F}_{2n}$ consists of a section at infinity C_∞ and a curve C such that the restriction $pro : C \longrightarrow \mathbb{P}^1$ is a 3:1 covering of $\mathbb{P}^1$.

Hurwitz formula gives us

$$\mathcal{X}(X) = 2\mathcal{X}(\mathbb{F}_{2n}) - \mathcal{X}(C \coprod C_\infty) = 8 - (2 - 2g + 2) = 4 + 2g$$

with $g = g(C)$.

Now, we suppose the surface $\Pi : X \longrightarrow \mathbb{P}^1$ is relatively minimal. By Noether formula, we have:

$$\mathcal{X}(X) = 12\mathcal{X}(O_X) = 12n.$$

In particular,

$$g = \frac{12n - 4}{2} = 6n - 2, \quad and \quad g \equiv 4[6].$$

When $Pic(X) \cong \mathbb{Z}^2$, we get

$$Br(X) \cong H^2(X, \mathbb{Q}/\mathbb{Z})/(Pic(X) \otimes \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{2g}$$

and C is smooth and irreductible. Since $H^1(X, \mathbb{C}) = 0$, the second Betti number of complex part is $B_2 = 2(g + 1)$.

Proposition 4.1. *Let* $\Pi : X \longrightarrow \mathbb{P}^1$ *a regular elliptic surface with a section* $s : \mathbb{P}^1 \longrightarrow X$*, and a fiberwise double cover* $X \longrightarrow \mathbb{F}_{2n}$*, with branch curve* $C \coprod C_\infty$*.*

We assume that $Pic(X) \cong \mathbb{Z}^2$. Then the injection $J(C)_2 \hookrightarrow Br_2(X)$ is an isomorphism.

Proof: This follows from 3.2 applied to the double cover $X \longrightarrow \mathbb{F}_{2n}$. Note that $b_0(C) = 2$ and $Pic(X)$ is generated by the class of a section and a fiber, so the covering group $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $Pic(X) \cong \mathbb{Z}^2$. ■

5 Double cover of jacobian fibration

Let X be a smooth projective surface with an elliptic fibration $\Pi : X \longrightarrow \mathbb{P}^1$, with section. The Brauer group of X is isomorphic to the Tate-Shafarevich group $TS(X) = H^1_{ét}(\mathbb{P}^1, X^\#)$ where $X^\#$ is the sheaf of groups on $\mathbb{P}^1$ of local sections of $p_X : X \longrightarrow \mathbb{P}^1$. [10]

A non trivial element of order n in $TS(X)$ is a genus one fibration $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ (without a section) and n is the minimal degree of a multisection of $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ and in this case the surface X correspond to Jacobian of Y.

The description of the real version of the Tate-Shafarevich group is found in [7].

Now, we consider a real genus one fibration $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ with a double section and we recover the associated Jacobian fibration $\Pi : X \longrightarrow \mathbb{P}^1$. For this we use the classical construction of the Jacobian of a double cover of $\mathbb{P}^1$ branched over four points, due to Hermite.

This construction gives us a real conic bundle over X with corresponds to $\alpha \in Br(X)$ defining Y.

5.1 Jacobian fibration

Let $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ be a genus one fibration defined by an equation

$$Y : w^2 = a_0 v^4 + 4a_1 v^3 + 6a_2 v^2 + 4a_3 v + a_4$$

Then, the Jacobian fibration of Π_1 as above is the elliptic surface $\Pi : X \longrightarrow \mathbb{P}^1$ given by

$$X : y^2 = 4x^3 - px - q,$$

with

$$p = a_0 a_4 - 4a_1 a_3 + 3a_2^2,$$
$$q = a_0 a_2 a_4 - a_0 a_3^2 - a_1^2 a_4 + 2a_1 a_2 a_3 - a_2^3.$$

5.2 Trigonal curve

Definition 5.1. Let $pro : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ be a geometrically ruled surface with a section at infinity C_∞. A trigonal curve on $\mathbb{F}_n$ (called also trisection) is a reduced curve $C \subset \mathbb{F}_n$ disjoint from C_∞ and such that the restriction $pro : C \longrightarrow \mathbb{P}^1$ is a 3:1 covering of $\mathbb{P}^1$.

A trigonal curve $C \subset \mathbb{F}_n$ is called generic if the discriminant $\Delta = 4p^3 + 27q^2$ has only simple roots and g_2, g_3, Δ have distinct roots.
A trigonal curve is a curve of bidegree (3,0) on $\mathbb{F}_n$ and C does not intersect C_∞. In other words, the intersection number of C with a fiber $\mathbb{P}^1$ is 3 and with the section C_∞ is 0, so $deg\ C = 3$.
Any trigonal curve can be given by an affine equation

$$x^3 + g_2 x + g_3 = 0$$

where g_2 and g_3 are certain sections of $\mathcal{O}(4n)$and $\mathcal{O}(6n)$ such that $g_2 = \frac{-p}{4}$ and $g_3 = \frac{-q}{4}$.

Definition 5.2. The j-invariant of generic trigonal curve $C \subset \mathbb{F}_n$ is the function $j : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ given by:

$$j = \frac{4g_2^3}{\Delta}; \quad \Delta = 4g_2^3 + 27g_3^2.$$

Geometrically, the value of j at a generic point $b \in \mathbb{P}^1$ is the usual j-invariant of the quadruple of points cut by the union $C \bigcup C_\infty$ in the projective line $pro^{-1}(b)$.
The sections g_2,g_3 defining C must satisfy the following conditions:
(1) The discriminant $\Delta = 4g_2^3 + 27g_3^2$ is not identically zero.
(2) At each point $b \in \mathbb{P}^1$, one has $min(3ord_b(g_2), 2ord_b(g_3)) < 12$.
The explicit expression for p and q imply that the equation of C is the determinant of a symmetric matrix M :

$$4x^3 - px - q = det(M),$$

$$M = \begin{pmatrix} a_0 & a_1 & a_2 + 2x \\ a_1 & a_2 - x & a_3 \\ a_2 + 2x & a_3 & a_4 \end{pmatrix}.$$

The matrix M defines a real conic bundle on $\mathbb{F}_{2n}$ which degenerates over the curve C. In particular, it defines an étale $2:1$ cover of C (the conics split in two lines over points in C), thus we get a point α of order two in $Pic(C)$.
When $\Pi : X \longrightarrow \mathbb{P}^1$ is real, the pull-back along $\nu : X \longrightarrow \mathbb{F}_{2n}$ gives an unramified real conic bundle on $\nu^{-1}(\mathbb{F}_{2n} - C)$.
Due to the branching of order two along C, this conic bundle extends to all of X (locally near a point of X on the ramification locus, the conic bundle is defined by a real homogeneous polynomial of the form $X^2 + Y^2 + t^2 Z^2$, where $t = 0$ defines the ramification curve. Changing coordinates $Z = tZ$, one can show that the conic bundle does not ramify in codimension one, so does not ramify at all.

5.3 Geometric observation

We start with an example inspired by an example of R. Silhol [18, page 181].
We consider the surface X defined in $\mathbb{P}^3$ by $x^4 - y^4 = z^4 - w^4$. This is the equation of a smooth quartic in $\mathbb{P}^3$, and hence of a $K3$ surface with a real elliptic fibration over $\mathbb{P}^1$ whose fibers are the curves defined by:

$$\begin{cases} x^2 + y^2 = t(z^2 - w^2) \\ t(x^2 - y^2) = z^2 + w^2 \end{cases}$$

A generic fiber X_t of this elliptic fibration is the intersection of two quadrics $\varphi_t(x, y, z, w) = \varphi'_t(x, y, z, w) = 0$. Let Y_t be the curve parametrizing families of lines on the quadrics of the pencil spanned by φ_t and φ'_t. Then Y_t is canonically identified with the degree 2 components $Pic^2(X_t)$ of the Picard group of X_t: one associates to a family of lines on a quadric the divisor class of the intersection of any line of this family with X_t. The curve Y comes with a map $\xi : X_t \longrightarrow Y_t = Pic^2(X_t)$ sending a point P to the divisor class of $2P$.
Geometrically, this is the family of lines containing the tangent to X_t at P. On the other hand, Y_t is isomorphic to the curve

$$\mu^2 = det(\lambda\varphi_t - \varphi'_t)$$

5.4 Tetragonal curves

The quotient of Y by the fiberwise (-1)-automorphism is a geometrically ruled surface Σ with some tetragonal curve

$$R : a_0 v^4 + 4a_1 v^3 + 6a_2 v^2 + 4a_3 v + a_4 = 0$$

which corresponds to g^1_4 pencil over $\mathbb{P}^1$.

Now, there is a natural bijection between étale double covers of trigonal curves and tetragonal curves. In fact the tetragonal construction associates to a $4:1$

cover of $\mathbb{P}^1$, that is an extension of degree four of $k(\mathbb{P}^1)$ given by the Lagrange resolvent and its natural quadratic extension. see [6] and [17]

Remark 5.3. The Recillas trigonal construction gives us a relation between (C, α) and R with α is some point of order two in $J(C)$.

5.5 Equivariant Recillas construction

Let $\Pi_1 : R \longrightarrow \mathbb{P}^1$ a simple equivariant branched covering of degree 4. Let the equivariant symmetric fibred product, quotient of $R \times_{\mathbb{P}^1} R$ by the involution which exchanges factors.
It is the union of the diagonal (isomorphic to R) and a real smooth curve $\widetilde{R}(\Pi_1)$ which is a branched covering of $\mathbb{P}^1$. Now, the natural $\mathbb{P}^1$-involution i over $\widetilde{R}(\Pi_1)$: if $a \in \mathbb{P}^1$ is not a branched point of Π_1, a point of $\widetilde{R}_a(\Pi_1)$ is identified to a subset A containing two elements of the fiber $\Pi_1^{-1}(a)$ and then $i(a) = \Pi_1^{-1}(a) - A$. This involution is without fixed points, then the curve $R(\Pi_1) = \widetilde{R}(\Pi_1)/i$ is smooth and the natural map $\Pi(\Pi_1) : \widetilde{R}(\Pi_1) \longrightarrow R(\Pi_1)$ is an étale real covering of degree 2.
The projection from $R(\Pi_1)$ over $\mathbb{P}^1$ defines a simple equivariant branched covering $g(\Pi_1) : R(\Pi_1) \longrightarrow \mathbb{P}^1$ of degree 3.

Remark 5.4. Over $\mathbb{C}$, two regular relatively minimal elliptic surfaces with no multiple fibers are deformation equivalent if and only if their holomorphic Euler characteristic are equal, see [13].

Proposition 5.5. *Two elliptic surfaces X and Y with double section (over $\mathbb{C}$) are deformation equivalents over $\mathbb{C}$, if and only if $g(R) = g(R')$ where R , R' are tetragonal curves associated to the given surface.*

Proof. The surfaces X and Y have sections over $\mathbb{C}$, so Y is deformation equivalent to X over $\mathbb{C}$ if and only if $\mathcal{X}(Y) = \mathcal{X}(X)$.
We have:

$$\mathcal{X}(Y) = 2\mathcal{X}(\Sigma) - \mathcal{X}(R') = 8 - \mathcal{X}(R')$$
$$\mathcal{X}(X) = 2\mathcal{X}(\Sigma) - \mathcal{X}(R) = 8 - \mathcal{X}(R)$$

Then $\mathcal{X}(X) = \mathcal{X}(Y)$ equivalent $g(R) = g(R')$. ∎

5.6 Resolution of singularities

Let $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ be a real elliptic surface over $\mathbb{R}$ with a double section given by the affine equation

$$y^2 = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4.$$

Double points at infinity are singular and give us a section defined over $\mathbb{R}$, which can be desingularised by taking the fiberwise isomorphism:

$$\begin{array}{ccc} Y & \longrightarrow & Y^* \\ (x, y) & \longmapsto & (\frac{1}{x}, \frac{y}{x^2}) \end{array}$$

for $x \neq 0$, which is well defined by fiberwise (-1)-automorphism.
The quotient of Y by fiberwise (-1)-automorphism is the geometrically ruled surface Σ with a tetragonal curve

$$R : a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

such that $x = \infty$ corresponds to the section at infinity, and $x = 0$ the zero section.
Over Σ, we set the fiberwise automorphism of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which sends the section at infinity to the zero section. Let R^* be a tetragonal curve given by

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

The double cover of Σ ramified over R^* is an elliptic surface Y^* given by the affine equation $y^2 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ such that the section at infinity is sending to the section $(0, \pm\sqrt{a_0(t)})$ which is defined over $\mathbb{R}$ if and only if $a_0(t)$ is a square of a real rational map defined over $\mathbb{P}^1\mathbb{C}$.
The affine equation of Y^* given is $y^2 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where the section at infinity on Y is replaced by two sections $(0, \pm\sqrt{a_0(t)})$ which is defined over $\mathbb{R}$ if only if $a_0(t)$ is a square of a real rational map defined over $\mathbb{P}^1\mathbb{C}$.

Proposition 5.6. *Let Y be an elliptic surface over $\mathbb{R}$ given by the equation:*

$$y^2 = a_0(t)x^4 + a_1(t)x^3 + a_2(t)x^2 + a_3(t)x + a_4(t)$$

and the triplet $(\Sigma, R, \mathbb{P}^1)$ is the geometrically ruled surface pro : $\Sigma \longrightarrow \mathbb{P}^1$ with a tetragonal curve R associated to Y.
Then Y admit a section over $\mathbb{R}$ if and only if there exist a fiberwise automorphism defined over $\mathbb{R}$ which sends R to real tetragonal curve R' given by the equation $a_0'x^4 + a_1'x^3 + a_2'x^2 + a_3'x + a_4' = 0$ where a_0' is a square of a real rational map defined over $\mathbb{P}^1\mathbb{C}$.

Proof. Let Y is an elliptic surface given by the equation:

$$y^2 = a_0(t)x^4 + a_1(t)x^3 + a_2(t)x^2 + a_3(t)x + a_4(t).$$

If $a_0(t)$ is a square of real rational map defined over $\mathbb{P}^1\mathbb{C}$ then Y admit a section over $\mathbb{R}$.
Conversely, if Y admits a section, the quotient by fiberwise (-1)-automorphism on Y gives a geometrically ruled surface Σ with an exceptional section and a tetragonal curve R given by the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$. With a fiberwise automorphism of ruled surfaces, one constructs a tetragonal curve

$$R' : a_0'x^4 + a_1'x^3 + a_2'x^2 + a_3'x + a_4' = 0,$$

and sending the exceptional section of Σ to the infinity section of Σ, in this case a_0' will be a square of a real rational map defined over $\mathbb{P}^1\mathbb{C}$.
The double cover of Σ ramified over the curve R' gives an elliptic surface Y', with a section , given by the affine equation $y^2 = a_0'x^4 + a_3'x^3 + a_2'x^2 + a_3'x + a_4'$ such that a_0' is a square of a real rational map defined over $\mathbb{P}^1\mathbb{C}$. ∎

Now, we suppose that the Jacobian fibration $\Pi : X \longrightarrow \mathbb{P}^1$ has only singular fibers of type I_1. [18, Chapter 7]. The fibration is said generic. Note that the fiber of type I_1 is a singular cubic and the singular point has distinct real tangents(real type I_1^-) or has two complex conjugate tangents (real type I_1^+) . See [1]
By silhol's classification of structure of the fibers of an elliptic surface, the singular fibers of the fibration $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ must be of type I_1^+, or I_1^-or (I_1 with an isolated real point). We also say $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ generic elliptic fibration (not necessary with real section). See[19]

Remark 5.7. From now, we suppose that the fibrations Π_1 is generic and we confuse Π_1 with its non-singular projective completion.

Let $\mathbb{F}_e$ be a real geometrically ruled surface with fibration $pro_1 : \mathbb{F}_e \longrightarrow \mathbb{P}^1$ and $R \subset \mathbb{F}_e$ a real tetragonal curve. The fibration $pro_1 : \mathbb{F}_e \longrightarrow \mathbb{P}^1$ induces a 4:1 map
$pro_1 : R \longrightarrow \mathbb{P}^1$.

Now, the double cover of $\mathbb{F}_e$ with branch-locus R, gives us a real elliptic surface $\Pi_1 : Y \longrightarrow \mathbb{P}^1$(with double section). Let $\Pi : X \longrightarrow \mathbb{P}^1$ the Jacobian fibration associated to Π_1 and $pro : C \subset \mathbb{F}_{2n} \longrightarrow \mathbb{P}^1$ the associated trigonal curve.

Lemma 5.8. *Let $a \in \mathbb{P}^1\mathbb{R}$ generic point.*
If $pro_{1/R}^{-1}(a)(\mathbb{R}) = \emptyset$, then $pro_{/C}^{-1}(a)(\mathbb{R})$ contains exactly 3 points.

Proof: The double cover of $\mathbb{P}^1$ ramified at $pro_{1/R}^{-1}(a)$ is a curve E of genus one. Now, $E(\mathbb{R}) = \emptyset$ (see[18, Chapter 7]). Then, $J(E)(\mathbb{R})$ has two components. So, $pro_{/C}^{-1}(a)(\mathbb{R})$ contains exactly 3 points. See [7]

Example: We consider the trivial product $\mathbb{C} \times \mathbb{P}^1 = \Sigma$ with an affine tetragonal curve defined by

$$R : -x^4 + 6(t-1)x^2 + 36t = 0$$

The double cover of Σ in a fiber preserving manner ramified over R is an elliptic surface given by the equation

$$y^2 = -x^4 + 6(t-1)x^2 + 36t$$

and note that the fiber over $t = 0$ is of type I_1(see [14]). This implies that the singular fibre is defined by

$$y^2 = -x^4 - 6x^2$$

which has one isolated real point.

6 Bounds for the number of connected components

In this section, we give some classical restrictions on the topology of a real algebraic surface (X, σ) in term of the numerical invariants of the complex surface X.

Let (X, σ) be a real algebraic manifold. We will use the following notations:
$B_i(X) = dimH_i(X, \mathbb{Z}/2)$ the i^{th} mod 2 Betti number of X,

$h_i(X(\mathbb{R})) = dimH_i(X(\mathbb{R}), \mathbb{Z}/2)$ the i-th mod 2 Betti number of $X(\mathbb{R})$,

$h_*(X(\mathbb{R})) = \sum h_i(X(\mathbb{R}))$,

$B_*(X) = \sum B_i(X)$.

The first important prohibition is given by the Milnor-Smith-Thom inequality.

Proposition 6.1. *(Milnor-Smith-Thom inequality) Let (X, σ) be a real algebraic manifold, then*

$$h_*(X(\mathbb{R})) \leq B_*(X);$$
$$B_*(X) - h_*(X(\mathbb{R})) \equiv 0 \ mod \ 2.$$

Proof: See [18, Chapter 2]. ∎

Let $2d = B_*(X) - h_*(X(\mathbb{R}))$. A real algebraic surface is an M-surface if $d = 0$ and an (M-d)-surface if $d \neq 0$. Other restrictions of Euler characteristic of the real part of a real algebraic surface, in the case of M-surface or (M-1)-surface, given in term of congruence. See [4] or [9]

The Smith-Thom inequality gives bounds for the number of connected components and the first Betti number $h_1(X(\mathbb{R})) = dim_{\mathbb{Z}/2}H^1(X(\mathbb{R}), \mathbb{Z}/2)$ of a real elliptic surface.
Remember that a surface X is regular if the Hodge number $h^{0,1} = 0$.
For a real algebraic surface X, we have

$$2\#X(\mathbb{R}) + h_1(X(\mathbb{R})) \leq 2 + 2B_1 + B_2$$

where B_i is the i-th Betti number of X.
In the case of X a real elliptic surface, relatively minimal, we have $h^{0,1}(X) = 0$, $\mathcal{X}(X) = 12\mathcal{X}(\mathcal{O}_X)$, (Noether formula).
On the other hand, the Hodge decomposition theorem give $h^{1,1}(X) = B_2 - 2h^{0,2}(X)$, and X is regular, then $B_2 = 12\mathcal{X}(\mathcal{O}_X) - 2$ and $h^{0,2}(X) = \mathcal{X}(\mathcal{O}_X) - 1$, then

$$2\#X(\mathbb{R}) + h_1(X(\mathbb{R})) \leq 12\mathcal{X}(\mathcal{O}_X).$$

In particular, $\#X(\mathbb{R}) \leq 6\mathcal{X}(\mathcal{O}_X)$.

Theorem 6.2. *(Kharlamov)Let $\Pi : X \longrightarrow \mathbb{P}^1$ be a real regular elliptic surface with no multiple fibers, then*

$$h_1(X(\mathbb{R})) \leq h^{1,1}(X)$$

Remark 6.3. Recall that $\Pi : X \longrightarrow \mathbb{P}^1$ is relatively minimal, with no multiple fiber. Using the Hodge decomposition, we get

$$B_1(X) = B_3(X) = 0 \, , \, h^{1,1}(X) = 10\mathcal{X}(\mathcal{O}_X)$$

We obtain,

Theorem 6.4. *[15] Let $\Pi : X \longrightarrow \mathbb{P}^1$ be a real regular relatively minimal elliptic surface with no multiple fibers, then*

$$h_1(X(\mathbb{R})) \leq 10\mathcal{X}(\mathcal{O}_X)$$

Proof. See [1]. ∎

Theorem 6.5. *Let $\Pi : X \longrightarrow \mathbb{P}^1$ be a real regular elliptic surface with a real section, then*

$$\#X(\mathbb{R}) \leq 5\mathcal{X}(\mathcal{O}_X)$$

For $k \geq 1$, there exist a real elliptic surface X with a real section such that $\mathcal{X}(\mathcal{O}_X) = k$ and $\#X(\mathbb{R}) = 5\mathcal{X}(\mathcal{O}_X)$.
Proof: See [1]. ∎

7 Ovals of a real tetragonal curve

Consider a real geometrically ruled surface $\mathbb{F}_{2n}$ and C a real non singular trigonal curve in $\mathbb{F}_{2n}$.
We consider the non singular curve $C \cup C_\infty$ which is of bidegree $(3,0) + (1,-2n) = (4,-2n)$. Since the bidegree is even, the polynomial H defining $C \cup C_\infty$ has well-defined sign on $\mathbb{F}_{2n}(\mathbb{R})$.
Hence the ovals of C can be divided into two groups according to the sign of H in the interior of each oval.
It follows that, with the exception of $C(\mathbb{R})$ consisting of three pseudo-lines, the real scheme of C,that is the pair $(\mathbb{F}_{2n}(\mathbb{R}), C(\mathbb{R}))$ up the homeomorphism, is uniquely determined by the numbers a and b of ovals in each of these two groups and will be denoted by $< a, b >$. See [4].
One can combine proposition 4. 5 and proposition 4. 3 of [4] to get:

Proposition 7.1. *Let C is a real trigonal curve in a geometrically ruled surface $\mathbb{F}_{2n}$. If C has real scheme $< a, b >$, then $a \leq 5\frac{g+2}{6}$ and $b \leq 5\frac{g+2}{6}$, where g is the genus of C.*

Consider a real geometrically ruled surface $\mathbb{F}_e$ and R a real non singular tetragonal curve in $\mathbb{F}_e$. Each oval of $R(\mathbb{R})$ has a well-defined interior (homeomorphic to a disc). We say that two ovals are nested if one lies in the interior of the other. We denote a by the number of nested ovals and b otherwise. See [21], [12], [22], [16].

The following proposition states that a real elliptic surface with double section (over $\mathbb{C}$) and its branch curve have same discrepancy.

Proposition 7.2. *A real elliptic surface with double section (over $\mathbb{C}$) is a $(M-d)$-surface if and only if is the associated tetragonal curve is a $(M-d)$-curve.*

Proof. Let $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ a real genus one fibration with a double section which can be real or complex conjugate and we recover the associated Jacobian fibration $\Pi : X \longrightarrow \mathbb{P}^1$. Over $\mathbb{C}$, Y has a section, then Y is simply connected, see[11, Proposition 2. 2. 1], gives Y is regular, the other Betti numbers are controlled using the Riemann-Hurwitz formula

$$\mathcal{X}(Y) = 2\mathcal{X}(\Sigma) - \mathcal{X}(R) = 8 - \mathcal{X}(R)$$

and Poincaré duality.

The Betti numbers of the real part $Y(\mathbb{R})$ are found using simple comparaison with the description of nested ovals of $R(\mathbb{R})$ and others components of $R(\mathbb{R})$. ■

Let now, $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ be a real algebraic elliptic surface with double section (over $\mathbb{C}$), and $\Pi : X \longrightarrow \mathbb{P}^1$ the Jacobian fibration associated to Y. Over $\mathbb{C}$, the map Π_1 admits a section. So, it hasn't multiple fibers. By a simple comparaison of singular fibres of Π_1 and Π [18, chapitre7], we get

$$h_1(Y(\mathbb{R})) \leq h_1(X(\mathbb{R})),$$

and

$$\#Y(\mathbb{R}) \leq \#X(\mathbb{R}).$$

Next, we will give a relation between the genus of tetragonal curve and the associated trigonal curve.

Proposition 7.3. *Let R be a tetragonal curve, and C be an associated trigonal curve. Then,*

$$g(R) = g(C) - 1.$$

Proof. Let $\Pi : X \longrightarrow \mathbb{P}^1$ be the complex regular elliptic surface associated to the curve C (See section 4) and $\Pi_1 : Y \longrightarrow \mathbb{P}^1$ be the complex regular elliptic surface associated to the curve R (see section 5).
Over $\mathbb{C}$, $Y \cong X$ then $\mathcal{X}(X) = \mathcal{X}(Y)$.
We have: $\mathcal{X}(X) = 4 + 2g(C)$. Similar to the proof of the proposition 7.2, we have

$$\mathcal{X}(Y) = 8 - \mathcal{X}(R) = 8 - (2 - 2g(R)) = 6 + 2g(R),$$

While $\mathcal{X}(Y) = \mathcal{X}(X)$, then $6 + 2g(R) = 4 + 2g(C)$. Finally, we have $g(R) = g(C) - 1$. ■

Now, we give upper bounds for the number of nested ovals and others ovals of a given real tetragonal curve embedded in Hirzebruch surface in terms of the genus of the curve.

Theorem 7.4. *If R is a real tetragonal curve embedded into a certain Hirzebruch surface $\mathbb{F}_n$. Then,*

$$(i)\ \ a \leq \frac{5}{6}(g(R) + 3),$$

and

$$(ii)\ \ b \leq \frac{5}{6}(g(R) + 3).$$

Proof. (i) Let Y be the ramified double cover of $\mathbb{F}_n$ with ramification locus R. The lift of the ruling on $\mathbb{F}_n$ endows Y with an elliptic fibration with a double section. Thus, the fibration is without multiple fibers. Recall that $h_1(Y(\mathbb{R}))$ must be even [1].
Following the same idea of the proposition 7.2, we get $a \leq \frac{1}{2}h_1(Y(\mathbb{R}))$.
We consider the Jacobian fibration $\Pi : X \longrightarrow \mathbb{P}^1$ associated to the elliptic fibration $\Pi_1 : Y \longrightarrow \mathbb{P}^1$. By construction $\Pi : X \longrightarrow \mathbb{P}^1$ has a real section. We get,

$$h_1(Y(\mathbb{R})) \leq h_1(X(\mathbb{R})).$$

Using the theorem 6.4 we obtain $a \leq 5\mathcal{X}(\mathcal{O}_X)$.
Recall the formula $\mathcal{X}(X) = 6 + 2g(R)$ obtained in the proof of the proposition 7.3. The assertion is obtained by using the Noether Formula.

(ii) In order to prove the statement about the others of ovals of the given tetragonal curve R, we remember that $b \leq \#Y(\mathbb{R})$.
Now using the theorem 6.5, we obtain

$$\#Y(\mathbb{R}) \leq \#X(\mathbb{R}) \leq 5\mathcal{X}(\mathcal{O}_X).$$

We conclude using the same idea of the last part of the proof of (i). ∎

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