Zeros of the derivative of a p-adic meromorphic function and applications *

Kamal Boussaf Alain Escassut Jacqueline Ojeda

Abstract

Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value. We show that if the Wronskian of two entire functions in K is a polynomial, then both functions are polynomials. As a consequence, if a meromorphic function f on all K is transcendental and has finitely many multiple poles, then f' takes all values in K infinitely many times. We then study applications to a meromorphic function f such that $f' + bf^2$ has finitely many zeros, a problem linked to the Hayman conjecture on a p-adic field.

1 Introduction and Main Results

Notation and Definitions. Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $|\ .\ |$. Given $\alpha \in K$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the disk $\{x \in K \mid |x - \alpha| \leq R\}$ and by $d(\alpha, R^-)$ the disk $\{x \in K \mid |x - \alpha| < R\}$, by $\mathcal{A}(K)$ the K-algebra of analytic functions in K (i.e. the set of power series with an infinite radius of convergence), by $\mathcal{M}(K)$ the field of meromorphic functions in K and by $K(x)$ the field of rational functions. Given $f, g \in \mathcal{A}(K)$, we denote by $W(f, g)$ the Wronskian $f'g - fg'$.

We know that any non-constant entire function $f \in \mathcal{A}(K)$ takes all values in K. More precisely, a function $f \in \mathcal{A}(K)$ other than a polynomial takes all values in K infinitely many times (see [5], [8], [9]). Next, a non-constant meromorphic

*Partially supported by CONICYT $N°79090014$ (Inserción de Capital Humano a la Academia)
Received by the editors August 2011.
Communicated by F. Brackx.
2000 *Mathematics Subject Classification* : 12J25; 46S10.
Key words and phrases : zeros of p-adic meromorphic functions, derivative, Wronskian.

function $f \in \mathcal{M}(K)$ takes every value in K, except at most one value. And more precisely, a meromorphic function $f \in \mathcal{M}(K) \setminus K(x)$ takes every value in K infinitely many times except at most one value (see [5], [9]).

Many previous studies were made on Picard's exceptional values for complex and p-adic functions and their derivatives (see [1], [3], [6], [7], [8]). Here we mean to examine precisely whether the derivative of a transcendental meromorphic function in K having finitely many multiple poles, may admit a value that is taken finitely many times and then we will look for applications to Hayman's problem when $m = 2$.

From Theorem 4 [6], we can state the following Theorem A:

Theorem A: *Let $h, l \in \mathcal{A}(K)$ satisfy $W(h,l) = c \in K$, with h non-affine. Then $c = 0$ and $\frac{h}{l}$ is a constant.*

Now we can improve Theorem A:

Theorem 1: *Let $f, g \in \mathcal{A}(K)$ be such that $W(f,g)$ is a non-identically zero polynomial. Then both f, g are polynomials.*

Remark: Theorem 1 does not hold in characteristic $p \neq 0$. Indeed, suppose the characteristic of K is $p \neq 0$. Let $\psi \in \mathcal{A}(K)$. Let $f = x(\psi)^p$ and let $g = (x+1)\psi^{-p}$. Since $p \neq 0$, we have $f' = (\psi)^p$, $g' = \psi^{-p}$ hence $W(f,g) = 1$ and this is true for any function $\psi \in \mathcal{A}(K)$.

Theorem 2: *Let $f \in \mathcal{M}(K) \setminus K(x)$ have finitely many multiple poles. Then f' takes every value $b \in K$ infinitely many times.*

We can easily show Corollary 2.1 from Theorem 2, though it is possible to get it through an expansion in simple elements.

Corollary 2.1: *Let $f \in \mathcal{M}(K) \setminus K(x)$. Then f' belongs to $\mathcal{M}(K) \setminus K(x)$.*

Open question: Do exists transcendental meromorphic functions f such that f' has finitely many zeros? By Theorem 2, such functions should have infinitely many multiple poles.

Now, we can look for some applications to Hayman's problem in a p-adic field. Let $f \in \mathcal{M}(K)$. Recall that in [9], [10] it was shown that if m is an integer ≥ 5 or $m = 1$, then $f' + f^m$ has infinitely many zeros that are not zeros of f. If $m = 3$ or $m = 4$, for many functions $f \in \mathcal{M}(K)$, $f' + f^m$ has infinitely many zeros that are not zeros of f (see [2], [10]) but there remain some cases where it is impossible to conclude, except when the field has residue characteristic equal to zero (see [10]). When $m = 2$, few results are known. Recall also that as far as complex meromorphic functions f are concerned, $f' + f^m$ has infinitely many zeros that are not zeros of f for every $m \geq 3$, but obvious counter-examples show this is wrong for $m = 1$ (with $f(x) = e^x$) and for $m = 2$ (with $f(x) = \tan(-x)$). Here we will particularly examine functions $f' + bf^2$, with $b \in K^*$.

Theorem 3: *Let $b \in K^*$ and let $f \in \mathcal{M}(K)$ have finitely many zeros and finitely many residues at its simple poles equal to $\frac{1}{b}$ and be such that $f' + bf^2$ has finitely many zeros. Then f belongs to $K(x)$.*

Remark: If $f(x) = \frac{1}{x}$, the function $f' + bf^2$ has no zero whenever $b \neq 1$.

Theorem 4: *Let $f \in \mathcal{M}(K) \setminus K(x)$ have finitely many multiple zeros and let $b \in K$. Then $\frac{f'}{f^2} + b$ has infinitely many zeros. Moreover, if $b \neq 0$, every zero α of $\frac{f'}{f^2} + b$ that is not a zero of $f' + bf^2$ is a simple pole of f such that the residue of f at α is equal to $\frac{1}{b}$.*

Corollary 4.1 : *Let $b \in K^*$ and let $f \in \mathcal{M}(K) \setminus K(x)$ have finitely many multiple zeros and finitely many simple poles. Then $f' + bf^2$ has infinitely many zeros that are not zeros of f.*

Remark: In Archimedean analysis, the typical example of a meromorphic function f such that $f' + f^2$ has no zero is $\tan(-x)$ and its residue is 1 at each pole of f. Here we find the same implication but we can't find an example satisfying such properties.

2 The Proofs

Notation: Given $f \in \mathcal{A}(K)$ and $r > 0$, we denote by $|f|(r)$ the norm of uniform convergence on the disk $d(0,r)$. This norm is known to be multiplicative (see [4], [5]).

Lemma 1 is well known (see Theorem 13.5 [4]) :

Lemma 1: *Let $f \in \mathcal{M}(K)$. Then $|f^{(k-1)}|(r) \leq \frac{|f|(r)}{r^{k-1}}$ $\forall r > 0, \forall k \in \mathbb{N}^*$.*

Proof of Theorem 1: First, by Theorem A, we check that the claim is satisfied when $W(f,g)$ is a polynomial of degree 0. Now, suppose the claim holds when $W(f,g)$ is a polynomial of certain degree d. We will show it for $d+1$. Let $f, g \in \mathcal{A}(K)$ be such that $W(f,g)$ is a non-identically zero polynomial P of degree $d+1$.

By hypothesis, we have $f'g - fg' = P$, hence $f''g - fg'' = P'$. We can extract g' and get $g' = \frac{f'g-P}{f}$. Now, consider the function $Q = f''g' - f'g''$ and replace g' by what we just found: we can get $Q = f'(\frac{f''g-fg''}{f}) - \frac{Pf''}{f}$.

Now, we can replace $f''g - fg''$ by P' and obtain $Q = \frac{f'P'-Pf''}{f}$. Thus, in that expression of Q, we can write $|Q|(R) \leq \frac{|f|(R)|P|(R)}{R^2|f|(R)}$, hence $|Q|(R) \leq \frac{|P|(R)}{R^2}$ $\forall R > 0$. But by definition, Q belongs to $\mathcal{A}(K)$ and further, $\deg(Q) \leq \deg(P) - 2$. Consequently, Q is a polynomial of degree at most $d-2$.

Now, suppose Q is not identically zero. Since $Q = W(f', g')$ and since $\deg(Q) < d$, by induction f' and g' are polynomials and so are f and g. Finally, suppose $Q = 0$. Then $P'f' - Pf'' = 0$ and therefore f' and P are two solutions of the differential equation of order 1 for meromorphic functions in K : $(\mathcal{E})\ y' = \psi y$ with $\psi = \frac{P'}{P}$, whereas y belongs to $\mathcal{A}(K)$. The space of solutions of $(\mathcal{E})$ is known to be of dimension 0 or 1 (see for instance Lemma 60.1 in [4]). Consequently, there exists $\lambda \in K$ such that $f' = \lambda P$, hence f is a polynomial. The same holds for g.

Proof of Theorem 2: Suppose f' has finitely many zeros. By classical results (see [4], [5]) we can write f in the form $\dfrac{h}{l}$ with $h,\ l \in \mathcal{A}(K)$, having no common zero. Consequently, each zero of $W(h,l)$ is a zero of f' except if it is a multiple zero of l. But since l only has finitely many multiple zeros, it appears that $W(h,l)$ has finitely many zeros and therefore is a polynomial. Consequently, by Theorem 1, both h and l are polynomials, a contradiction because f does not belong to $K(x)$.

Now, consider $f' - b$ with $b \in K$. It is the derivative of $f - bx$ whose poles are exactly those of f, taking multiplicity into account. Consequently, $f' - b$ also has infinitely many zeros.

Notation: Given $f \in \mathcal{M}(K)$, we will denote by $\text{res}_a(f)$ the residue of f at a.

Lemma 2: *Let* $f = \dfrac{h}{l} \in \mathcal{M}(K)$ *with* $h,\ l \in \mathcal{A}(K)$ *having no common zero, let* $b \in K^*$ *and let* $a \in K$ *be a zero of* $h'l - hl' + bh^2$ *that is not a zero of* $f' + bf^2$. *Then* a *is a simple pole of* f *and* $\text{res}_a(f) = \dfrac{1}{b}$.

Proof: Clearly, if $l(a) \neq 0$, a is a zero of $f' + bf^2$. Hence, a zero a of $h'l - hl' + bh^2$ that is not a zero of $f' + bf^2$ is a pole of f. Now, when $l(a) = 0$, we have $h(a) \neq 0$ hence $l'(a) = bh(a) \neq 0$ and therefore a is a simple pole of f such that $\dfrac{h(a)}{l'(a)} = \dfrac{1}{b}$. But since a is a simple pole of f, we have $\text{res}_a(f) = \dfrac{h(a)}{l'(a)}$ which ends the proof.

Proof of Theorem 3: Let $f = \dfrac{P}{l}$ with P a polynomial, $l \in \mathcal{A}(K)$ having no common zero with P. Then $f' + bf^2 = \dfrac{P'l - l'P + bP^2}{l^2}$. By hypothesis, this function has finitely many zeros. Moreover, if a is a zero of $P'l - l'P + bP^2$ but is not a zero of $f' + bf^2$ then, by Lemma 2, a is a simple pole of f such that $\text{res}_a(f) = \dfrac{1}{b}$. Consequently, $P'l - l'P + bP^2$ has finitely many zeros and so we may write $\dfrac{P'l - l'P + bP^2}{l^2} = \dfrac{Q}{l^2}$ with $Q \in K[x]$, hence $P'l - l'P = -bP^2 + Q$. But then, by Theorem 1, l is a polynomial, which ends the proof.

Proof of Theorem 4: Let $g = \frac{f'}{f^2} + b$. Suppose $b = 0$. Since all zeros of f are simple zeros except maybe finitely many, g has finitely many poles of order ≥ 3, hence a primitive G of g has finitely many multiple poles. Consequently, by Theorem 2, g has infinitely many zeros.

Now, suppose $b \neq 0$. Let α be a zero of g and let $f = \frac{h}{l}$ with $h,\ l \in \mathcal{A}(K)$ having no common zero. Then $\frac{f'}{f^2} + b = \frac{h'l - hl' + bh^2}{h^2}$. Since α is a zero of $\frac{f'}{f^2} + b$, it is not a zero of h and hence it is a zero of $h'l - hl' + bh^2$. Then by Lemma 2, if it is not a zero of $f' + bf^2$, it is a simple pole of f such that $\text{res}_\alpha(f) = \frac{1}{b}$, which ends the proof of Theorem 4.

Acknowledgement: The authors are very grateful to Jean-Paul Bezivin for an important improvement concerning Theorem 1.

References

[1] **W. Bergweiler and X. C. Pang** *On the derivative of meromorphic functions with multiple zeros,* J. Math. Anal. Appl. 278 , pp. 285-292 (2003).

[2] **K. Boussaf and J. Ojeda** *Value distribution of p-adic meromorphic functions*, Bulletin of the Belgian Mathematical Society - Simon Stevin, Vol 18, n.4, p. 667-678 (2011).

[3] **K. Boussaf** *Picard values of p-adic meromorphic function*, p-adic Numbers, Ultrametric Analysis and Applications. Vol. 2, N. 4, pp. 285-292 (2010).

[4] **A. Escassut** *Analytic Elements in p-adic Analysis*. World Scientific Publishing Co. Pte. Ltd. (Singapore, 1995).

[5] **A. Escassut** *p-adic Value Distribution*. Some Topics on Value Distribution and Differentiability in Complex and P-adic Analysis, pp. 42-138. Mathematics Monograph, Series 11. Science Press (Beijing, 2008).

[6] **A. Escassut and J. Ojeda** *Exceptional values of p-adic analytic functions and derivatives*. Complex Variables and Elliptic Equations. Vol 56, N. 1-4, pp. 263-269 (2011).

[7] **W. Hayman** *Picard values of meromorphic functions and their derivatives*, Annals of Mathematics. Vol. 70, N. 1, pp. 9-42 (1959).

[8] **P. C. Hu and C. C. Yang** *Meromorphic Functions over non-Archimedean Fields*. Kluwer Academic Publishers (2000).

[9] **J. Ojeda** *Distribution de valeurs des fonctions méromorphes ultramétrique, applications de la Théorie de Nevanlinna*. Thèse (N. d'ordre D.U. 1865). Université Blaise Pascal, Clermont-Ferrand, France (2008).

[10] **J. Ojeda** *Hayman's Conjecture over a p-adic field*. Taiwanese Journal of Mathematics. Vol. 12, N. 9, pp. 2295-2313 (2008).

Laboratoire de Mathématiques, CNRS UMR 6620
Université Blaise Pascal
24 Avenue des Landais BP 80026
63171 Aubière Cedex, France
email: Kamal.Boussaf,Alain.Escassut@math.univ-bpclermont.fr

Departamento de Matemática
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción
Concepción, Chile
email: jacqojeda@udec.cl