Weighted L^p-spaces on locally compact groups

Fatemeh Abtahi

Abstract

Let G be a locally compact group, ω be a weight function on G and $0 < p < 1$. We investigate the space $L^p(G,\omega)\star L^p(G,\omega)$ as a subset of some special spaces such as $L^p(G,\omega)$, $L^\infty(G,1/\widetilde{\omega})$ and $C_0(G,1/\widetilde{\omega})$. As the main result we show that all of the mentioned inclusions are equivalent to the discreteness of G.

1 Introduction

Throughout the paper, let G be a locally compact group with a fixed left Haar measure λ and ω be a weight function on G; that is, a positive Borel measurable function on G. Set $\widetilde{\omega}(x) = \omega(x^{-1})$, for all $x \in G$. For $0 < p < \infty$, the space $L^p(G,\omega)$ with respect to λ is the set of all complex valued measurable functions f on G such that $f\omega \in L^p(G)$, the usual Lebesgue space as defined in [7]. Let us remark that

$$\|f\|_{p,\omega} := \int_G |f(x)|^p \omega(x)^p d\lambda(x) \qquad (f \in L^p(G,\omega)),$$

defines a quasi norm on $L^p(G,\omega)$ to make it a complete metric space. We denote this space by $\ell^p(G,\omega)$ when G is discrete. Notice that $L^\infty(G,1/\omega)$ is the Banach space consisting of (equivalent classes) all complex valued measurable functions f on G such that $f/\omega \in L^\infty(G)$, with norm $\|f\|_{\infty,1/\omega} := \|f/\omega\|_\infty$, for each $f \in L^\infty(G,1/\omega)$. We also introduce the space $C_0(G,1/\omega)$, closed linear subspace of $L^\infty(G,1/\omega)$ such that $f/\omega \in C_0(G)$, the closed subalgebra of $L^\infty(G)$ of all continuous functions vanishing at infinity. The convolution product of two

Received by the editors August 2011 - In revised form in September 2011.
Communicated by F. Bastin.
2000 *Mathematics Subject Classification* : 43A15, 46H05.
Key words and phrases : Locally compact group, Weighted L^p-spaces, Weight function.

measurable functions f and g on G, $f \star g$, is said to exist as a function if $(f \star g)(x)$ exists for almost all $x \in G$. For $p > 1$, topological and algebraic properties related to L^p-spaces have been investigated a lot; see [1] and [9] and also [2], [3], [5] and [8], for the weighted case. Here, we show that for $0 < p < 1$, there are intimate relations between the following assertions, as the main purpose of the present work,

$$\begin{array}{ll} (i) & L^p(G,\omega)\star L^p(G,\omega) \subseteq L^p(G,\omega), \\ (ii) & L^p(G,\omega)\star L^p(G,\omega) \subseteq L^\infty(G,1/\widetilde{\omega}), \\ (iii) & L^p(G,\omega)\star L^p(G,\omega) \subseteq C_0(G,1/\widetilde{\omega}), \\ (iv) & L^p(G,\omega)\star L^p(G,\omega) \subseteq L^1(G,\omega), \\ (v) & f\star g \text{ exists for all } f,g \in L^p(G,\omega), \\ (vi) & G \text{ is discrete}, \\ (vii) & L^p(G,\omega)\star L^p(G,\omega) \subseteq L^1(G). \end{array} \tag{1}$$

2 Main Results

Before proceeding to the main theorem, we give the following result that is essentially due to [2] and also [8], but we shall discuss it here again for the sake of later use.

Proposition 2.1. *Let G be a locally compact group, α and β be weight functions on G, $0 < p < \infty$ and $0 < r \leq \infty$. If $L^p(G,\alpha) \star L^p(G,\alpha) \subseteq L^r(G,\beta)$, then*
(i) *the map*

$$L^p(G,\alpha) \times L^p(G,\alpha) \to L^r(G,\beta),$$

defined by $(f,g) \mapsto f\star g$ is jointly continuous;
(ii) *$\beta \in L^r(A)$, for all compact subsets $A \subseteq G$ of positive measure.*

Proof. (i). We first show that the map is separately continuous. Suppose on the contrary that there is $g \in L^p(G,\alpha)$ positive such that the map $f \mapsto f \star g$ from $L^p(G,\alpha)$ into $L^r(G,\beta)$ is not continuous. Hence there exists some sequence (f_n) of positive functions in $L^p(G,\alpha)$ with $\|f_n\|_{p,\alpha} \leq 1$ and $\|f_n \star g\|_{r,\beta} \geq n^3$. Set $f := \sum_{n=1}^{\infty} \frac{f_n}{n^2}$. It is obvious that $f \star g \notin L^r(G,\beta)$ what makes a contradiction. Now, joint continuity of the map follows from Baire category theorem [10].

(ii). Take a compact set $A \subseteq G$ of positive measure and consider the functions $\varphi = \chi_A/\max\{1,\alpha\}$ and $\psi = \chi_{A^{-1}A}/\max\{1,\alpha\}$, where χ_A denotes the characteristic function of a set A. An argument similar to [8, Lemma 2.1] implies that $\beta \in L^r(A)$. ∎

Proposition 2.2. *Suppose that G is a locally compact group, ω is a weight function on G bounded on a relatively compact open neighborhood U of the identity element of G, and $0 < p < 1$. If $f \star g$ exists for all $f,g \in L^p(G,\omega)$, then G is discrete.*

Proof. Suppose on the contrary that G is not discrete. By the hypothesis, $\omega(x) \leq K$, for all $x \in U$ and some constant $K > 0$. There is an open relatively compact neighborhood O of the identity element of G with $O^2 \subseteq U$. We can choose a sequence (O_n) of pairwise disjoint open subsets of O such that $\overline{O_n}$ is compact and $\lambda(O_n) < 2^{-n}$, for all $n \geq 1$. Define the functions f and g on G by

$$f(x) = \sum_{n=1}^{\infty} \frac{\chi_{O_n}(x)}{\lambda(O_n)^{1/p}\, n^{1+1/p}} \qquad \text{and} \qquad g(x) = \chi_{O^2}(x),$$

for all $x \in G$. It is easy to see that $f, g \in L^p(G, \omega)$. So for each $x \in O$, we have

$$(f{\star}g)(x) \geq \sum_{n=1}^{\infty} \frac{2^{n(1/p-1)}}{n^{1+1/p}} = \infty.$$

This contradiction completes the proof. ■

Remark 2.3. The proof of Proposition 2.2 shows that boundedness of ω on compact subsets of G is considerable.

(i). If ω is submultiplicative, i.e. $\omega(xy) \leq \omega(x)\omega(y)$, for all $x, y \in G$, then it is bounded and bounded away from zero on every compact subset of G [4, Proposition 1.16].

(ii). Let ω_1 be of moderate growth, i.e. for all $x \in G$,

$$\text{ess sup}_{\,y\in G} \frac{\omega_1(xy)}{\omega_1(y)} < \infty,$$

and $\omega_1 \in L^p(A)$ for each compact subset A of G of positive measure. Since ω_1^p is also of moderate growth and $\omega_1^p \in L^1(A)$, then ω_1^p is equivalent to a continuous weight function ω_2, i.e. for some constants C_1, C_2,

$$C_1 \leq \frac{\omega_1^p}{\omega_2} \leq C_2,$$

locally almost everywhere on G ([6, Theorem 2.7]). It follows that ω_1 is bounded and bounded away from zero on every compact subset of G.

(iii). For all $0 < p < \infty$, on a discrete group, a weight function of any $\ell^p(G, \omega)$ that is closed under convolution, is quasi submultiplicative, i.e. for some constant $C > 0$,

$$\omega(xy) \leq C\omega(x)\omega(y) \qquad (x, y \in G);$$

indeed, Proposition 2.1 implies that there exists a constant $C > 0$,

$$\omega(xy) = \|\delta_x \star \delta_y\|_{p,\omega}^{1/p} \leq C\|\delta_x\|_{p,\omega}^{1/p}\|\delta_y\|_{p,\omega}^{1/p} = C\omega(x)\omega(y) \qquad (x, y \in G),$$

where δ_x is the Dirac measure at x.

It is of interest to discuss the converse of Proposition 2.2. The following proposition confirms a stronger result, for the weight functions of moderate growth.

Proposition 2.4. *Let G be a locally compact group, ω be a weight function on G of moderate growth and $0 < p < 1$. Then $L^p(G, \omega)$ is closed under convolution if and only if G is discrete and ω is quasi submultiplicative.*

Proof. Let G be discrete and ω be quasi submultiplicative. So there exists the constant $C > 0$ such that for all $f, g \in \ell^p(G,\omega)$ we have

$$\begin{aligned} \|f \star g\|_{p,\omega} &= \sum_{x\in G} \left(\sum_{y \in G} f(y) g(y^{-1}x) \right)^p \omega(x)^p \\ &\leq C \sum_{x\in G} \sum_{y\in G} f(y)^p \omega(y)^p g(y^{-1}x)^p \omega(y^{-1}x)^p \\ &= C\|f\|_{p,\omega} \|g\|_{p,\omega}. \end{aligned}$$

The converse is obtained from Propositions 2.1 and 2.2 and also Remark 2.3. ∎

Here we state the main result of this note.

Theorem 2.5. *Let G be a locally compact group, ω be a submultiplicative weight function on G and $0 < p < 1$. Then the assertions $(i) - (vi)$ given in (1) are equivalent.*

Proof. By Remark 2.3 part (i), ω is bounded and bounded away from zero on every compact subset of G and so (i) is equivalent to (vi) by Proposition 2.4. (vi) is concluded from (v), by Proposition 2.2 and (i) implies (v) immediately. Thus (i), (v) and (vi) are equivalent. Suppose that (ii) holds. Thus $f \star g$ exists as a function for all $f, g \in L^p(G,\omega)$ and so G is discrete by the equivalence of (v) and (vi). Now let G be discrete, then we have the following inclusions, by the elementary calculations

$$\ell^p(G,\omega) \star \ell^p(G,\omega) \subseteq \ell^1(G,\omega) \subseteq \ell^\infty(G, 1/\tilde{\omega}).$$

It follows that (ii) and (vi) are equivalent. Similarly, the equivalence of (iii) and (iv) to (vi) are concluded. ∎

We end this work with the following result that gives us an equivalent condition to part (vii) of (1)

Proposition 2.6. *Let G be a locally compact group, ω be a submultiplicative weight function on G and $0 < p < 1$. Then $L^p(G,\omega) \star L^p(G,\omega) \subseteq L^1(G)$ if and only if G is discrete and ω is bounded away from zero.*

Proof. Let $L^p(G,\omega) \star L^p(G,\omega) \subseteq L^1(G)$. Proposition 2.2 and Remark 2.3 part (i), imply that G is discrete and so $\ell^p(G,\omega) = \ell^p(G,\omega) \star \ell^p(G,\omega) \subseteq \ell^1(G)$. If ω is not bounded away from zero, then for each $n > 0$, there exists $x_n \in G$ such that $\omega(x_n) \leq \frac{1}{n^{2/p-1}}$. Then the function $f = \sum_{n=1}^{\infty} \frac{\chi_{x_n}}{n}$ belongs to $\ell^p(G,\omega)$, whereas $f \notin \ell^1(G)$. For the converse, note that if G is discrete and ω is bounded away from zero, then $\ell^p(G,\omega) \subseteq \ell^1(G,\omega) \subseteq \ell^1(G)$ and the inclusion is concluded. ∎

Acknowledgment. The author thanks the referee for several helpful comments that have improved the quality of the paper. This research was supported by the Center of Excellence for Mathematics at the University of Isfahan.

References

[1] Abtahi, F., Nasr Isfahani, R., and Rejali, A., *On the L^p-conjecture for locally compact groups*, Arch. Math. **89** (Basel) (2007) 237-242.

[2] Abtahi, F., Nasr Isfahani, R., and Rejali, A., *On the weighted ℓ^p-space of a discrete group*, Publ. Math. Debrecen. **75/3-4** (2009) 365-374.

[3] Abtahi, F., Nasr Isfahani, R., and Rejali, A., *Weighted L^p-conjecture for locally compact groups*, Periodica Mathematica Hungarica **60/1**, (2010), 1-11.

[4] Edwards, R. E., *The stability of weighted Lebesgue spaces*, Trans. Amer. Math. Soc. **93** (1959) 369-394.

[5] El Kinani, A., and Benazzouz, A., *Structure m-convex dans l'espace à poids $L^p_\Omega(\mathbb{R}^n)$*, Bull. Belg. Math. Soc. **10** (2003) 49-57.

[6] Feichtinger, H. G., Gewichtsfunktionen auf, lokalkompakten Gruppen, *Sitzber. Österr. Akad. Wiss. Abt. II,* **188** No. 8-10 (1979), 451-471.

[7] Hewitt, E., and Ross, B. I, II, Abstract harmonic analysis I,II, Springer-Verlag, New York, 1970.

[8] Kuznetsova, Yu. N., *Invariant weighted algebras $L^p(G,\omega)$*, Math. Notes, **84/3-4** (2008), 529-537

[9] Saeki, S., *The L^p-conjecture and Young's inequality*, Illinois J. Math. **34** (1990) 615-627.

[10] Schaefer, H. H., Topological vector spaces, Springer, 1971.

Department of Mathematics, University of Isfahan,
Isfahan 81746-73441, Iran
email:f.abtahi@sci.ui.ac.ir