Weak compactness of AM-compact operators

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Abstract

We characterize Banach lattices under which each AM-compact operator (resp. the second power of a positive AM-compact operator) is weakly compact. Also, we give some interesting results about b-weakly compact operators and operators of strong type B.

1 Introduction and notation

Recall that an operator T from a Banach lattice E into a Banach space X is called AM-compact if the image of each order bounded subset of E is a relatively compact subset of X. Note that an AM-compact operator is not necessary weakly compact. In fact, the identity operator of the Banach lattice ℓ^1, is AM-compact but it is not weakly compact. Conversely, a weakly compact operator is not necessary AM-compact. For an example, the identity operator of the Banach lattice $L^2([0,1])$ is weakly compact but it is not AM-compact. If not, for each $x \in L^2([0,1])$, the order interval $[0,x]$ would be norm compact, and hence $L^2([0,1])$ would be discrete, and this is false.

Note that none of the two classes satisfies the problem of domination [2, 7], but while the class of weakly compact operators satisfies the problem of duality that of AM-compact operators does not satisfy it [8, 17].

Our objective in this paper is to investigate Banach lattices on which each AM-compact operator is weakly compact and in another paper, we will look at the

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reciprocal problem. In fact, in this paper, we will establish that if E is a Banach lattice and X is a Banach space such that each AM-compact operator $T : E \longrightarrow X$ is weakly compact, then the norm of E' is order continuous or X is reflexive. And conversely, if E is a KB-space, then each AM-compact operator $T : E \longrightarrow X$ is weakly compact if E' is order continuous or X is reflexive. Next, we will give a necessary and sufficient condition for which the second power of an AM-compact operator (resp. operator of strong type B) is weakly compact.

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice.

We refer to [1] for unexplained terminology on Banach lattice theory.

2 Main results

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Note that each positive linear mapping on a Banach lattice is continuous.

Let us recall that a norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space.
Our following result gives necessary conditions under which each AM-compact operator is weakly compact.

Theorem 2.1. *Let E be a Banach lattice and X a Banach space. If each AM-compact operator $T : E \longrightarrow X$ is weakly compact, then one of the following assertions is valid:*

1. *the norm of E' is order continuous,*
2. *X is reflexive.*

Proof. Assume that the norm of E' is not order continuous. It follows from Theorem 2.4.14 and Proposition 2.3.11 of [14] that E contains a sublattice isomorphic to ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$.

To finish the proof we have to show that X is reflexive. By the Eberlein-Smulian's Theorem it suffices to show that every sequence (x_n) in the closed unit ball of X has a subsequence, that we note also by (x_n), which converges weakly to an element of X. Consider the operator $T : \ell^1 \longrightarrow X$ defined by $T((\lambda_i)) = \sum_{i=1}^{\infty} \lambda_i x_i$ for each $(\lambda_i) \in \ell^1$. The composed operator $T \circ P : E \longrightarrow \ell^1 \longrightarrow X$ is AM-compact (because $T \circ P = T \circ Id_{\ell^1} \circ P$ and Id_{ℓ^1} is AM-compact) and hence by our hypothesis $T \circ P$ is weakly compact. If we note by (e_n) the sequence with all terms zero and the nth equals 1, then the sequence $(x_n) = ((T \circ P)(e_n))$ has a subsequence which converges weakly to an element of X. This ends the proof. ∎

Remark 2.2. *The necessary condition (2) in Theorem 2.1 is sufficient, but the condition (1) is not. In fact, the identity operator Id_{c_0} of the Banach lattice c_0 is AM-compact and the norm of $(c_0)' = \ell^1$ is order continuous, but Id_{c_0} is not weakly compact.*

Recall from [3] that a subset A of a Banach lattice E is called b-order bounded if it is order bounded in the topological bidual E''. It is clear that every order bounded subset of E is b-order bounded. However, the converse is not true in general.

A Banach lattice E is said to have the (b)-property if $A \subset E$ is order bounded in E whenever it is order bounded in its topological bidual E''.

An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X. Note that each weakly compact operator is b-weakly compact but the converse may be false in general. For an example, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \longrightarrow L^1[0,1]$ is b-weakly compact but it is not weakly compact. For more information on b-weakly compact operators see [9],[10],[3],[6],[4].

Conversely, we have the following result.

Theorem 2.3. *Let E be a Banach lattice and X a Banach space. Then each AM-compact operator $T : E \longrightarrow X$ is weakly compact if one of the following assertions is valid:*

1. *E is reflexive,*
2. *X is reflexive.*

Whenever E is a KB-space, then by using Theorem 2.1 and Theorem 2.3, we establish that the two necessary conditions of Theorem 2.1 become sufficient.

Theorem 2.4. *Let E be a KB-space and let X be a Banach space. Then the following assertions are equivalent:*

1. *each AM-compact operator $T : E \longrightarrow X$ is weakly compact,*
2. *one of the following assertions holds:*
   - *(a) the norm of E' is order continuous,*
   - *(b) X is reflexive.*

Proof. (1)$\Longrightarrow$(2) Follows from Theorem 2.1.
(2)$\Longrightarrow$(1) Follows from Theorem 2.3. ∎

Remark 2.5. *The two properties "the norm of E' is order continuous" and "E is a KB-space" are independent. In fact, there exists a KB-space E such that the norm of its topological dual E' is not order continuous. For example, the Banach lattice ℓ^1 is a KB-space but $(\ell^1)' = \ell^\infty$ does not have an order continuous norm. And conversely, there exists a Banach lattice E which is not a KB-space but the norm of its topological dual E' is order continuous. For example, the Banach lattice c_0 is not a KB-space but the norm of $(c_0)' = \ell^1$ is order continuous norm.*

Remark 2.6. *The assumption "E is a KB-space" is essential in Theorem 2.4. For instance, for $p > 1$ the operator $T_p : X_P \longrightarrow c_0$ constructed in [13] is AM-compact which is not weakly compact where the Banach lattice X_P as defined in [13]. However, the norm of $(X_P)'$ is order continuous. Note that the Banach lattice X_P is not a KB-space. Otherwise, the operator $T_p : X_P \longrightarrow c_0$ would be weakly compact.*

Now, from Theorem 2.4, we derive two characterizations. The first one concerns Banach lattices whose topological duals have order continuous norms:

Corollary 2.7. *Let E be a KB-space and X a non reflexive Banach space. Then the following assertions are equivalent:*

1. *each AM-compact operator $T : E \longrightarrow X$ is weakly compact.*
2. *the norm of E' is order continuous.*

The second one concerns reflexive Banach spaces:

Corollary 2.8. *Let X be a Banach space. Then the following assertions are equivalent:*

1. *each operator $T : \ell^1 \longrightarrow X$ is weakly compact.*
2. *X is reflexive.*

If in Theorem 2.4, we take E and F are two Banach lattices, then we obtain the following characterization:

Theorem 2.9. *Let E and F be two Banach lattices such that E is a KB-space. Then the following assertions are equivalent:*

1. *each AM-compact operator $T : E \longrightarrow F$ is weakly compact,*
2. *each positive AM-compact operator $T : E \longrightarrow F$ is weakly compact,*
3. *one of the following assertions holds:*
   (a) *the norm of E' is order continuous,*
   (b) *F is reflexive.*

On the other hand, we observe that if E is a Banach lattice, the second power of an AM-compact operator $T : E \longrightarrow E$ is not necessary weakly compact. In fact, the identity operator Id_{ℓ^1} is AM-compact but its second power $(Id_{\ell^1})^2 = Id_{\ell^1}$ is not weakly compact.

In the following, we give a necessary and sufficient condition for which the second power operator of an AM-compact operator is always weakly compact.

Theorem 2.10. *Let E be a KB-space. Then the following assertions are equivalent:*

1. *for all positive operators S and T from E into E with $0 \leq S \leq T$ and T is AM-compact, S is weakly compact,*

2. *each positive AM-compact operator $T : E \longrightarrow E$ is weakly compact,*

3. *for each positive AM-compact operator $T : E \longrightarrow E$, the second power T^2 is weakly compact,*

4. *the norm of E' is order continuous.*

Proof. (1)$\Longrightarrow$(2) Let $T : E \longrightarrow E$ be a positive AM-compact operator. Since $0 \leq T \leq T$, then by our hypothesis T is weakly compact.
(2)$\Longrightarrow$(3) By our hypothesis T is weakly compact and hence T^2 is weakly compact.

(3)$\Longrightarrow$(4) By way of contradiction, suppose that the norm of E' is not order continuous. We have to construct a positive AM-compact operator such that its second power is not weakly compact.

Since the norm of E' is not order continuous,it follows from Theorem 2.4.14 and Proposition 2.3.11 of [14] that E contains a complemented copy of ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$.
Consider the operator $T = i \circ P$ with i is the canonical injection of ℓ^1 in E. Clearly the operator T is AM-compact but it is not weakly compact. Otherwise, the operator $P \circ T \circ i = Id_{\ell_1}$ would be weakly compact, and this is impossible. Hence, the operator $T^2 = T$ is not weakly compact.
(4)$\Longrightarrow$(1) Follows from Theorem 14.22 of [1]. ∎

Let us recall from [15] that an operator T from a Banach lattice E into a Banach space X is called of strong type B whenever T carries the band B_E, generated by E in E'', into X. Note that each weakly compact operator is of strong type B but the converse is false in general. In fact, the identity operator of the Banach lattice $L^1[0, 1]$ is of strong type B but it is not weakly compact. And in [5] Alpay studied the weak compactness of operators of strong type B.

Also, if E is a Banach lattice, the second power of an operator of strong type B, $T : E \longrightarrow E$, is not necessary weakly compact. In fact, the identity operator Id_{ℓ^1} is of strong type B but its second power $(Id_{\ell^1})^2 = Id_{\ell^1}$ is not weakly compact.

In the following result, we characterize Banach lattices on which the second power of each operator of strong type B is weakly compact.

Theorem 2.11. *Let E be a Banach lattice. Then the following assertions are equivalent:*

1. *for all positive operators S and T from E into E with $0 \leq S \leq T$ and T is of strong type B, S is weakly compact,*

2. *each positive operator, of strong type B, is weakly compact,*

3. *for each positive operator of strong type B, $T : E \longrightarrow E$, its second power T^2 is weakly compact,*

4. *the norm of E' is order continuous.*

Proof. (1)$\Longrightarrow$(2) Let $T : E \longrightarrow E$ be a positive operator of strong type B. Since $0 \leq T \leq T$, then by our hypothesis T is weakly compact.

(2)$\Longrightarrow$(3) Let $T : E \longrightarrow E$ be an operator of strong type B. By our hypothesis T is weakly compact and hence T^2 is weakly compact.

(3)$\Longrightarrow$(4) Suppose that the norm of E' is not order continuous. Then it follows from Theorem 2.4.14 and Proposition 2.3.11 of [14] that E contains a complemented copy of ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$.
Consider the operator $T = i \circ P$ with i is the canonical injection of ℓ^1 in E. The operator T is of strong type B but it is not weakly compact. Otherwise, the operator $P \circ T \circ i = Id_{\ell_1}$ would be weakly compact, and this is impossible. Hence, the operator $T^2 = T$ is not weakly compact.
(4)$\Longrightarrow$(1) Follows from Proposition 3.2 of [5] and Theorem 5.31 of [1]. ∎

Recall from [11] that an operator T defined from a Banach lattice E into a Banach space X is said to be b-AM-compact provided that T maps b-order bounded subsets of E into relatively compact subsets of X. Note that this class of operators is larger than that of compact operators but smaller than that of AM-compact operators.

On the other hand, there exists an operator which is b-AM-compact but not weakly compact. In fact, the identity operator of the Banach lattice l^1 is b-AM-compact but it is not weakly compact.

The following result was claimed for b-weakly compact operators in [10] and for b-AM-compact in [12]. In one part of the proof we were misguided by an erroneous part of Proposition 2 in [4]. However, our claim is still true under the condition "the norm of E is order continuous".

Theorem 2.12. *Let E be a Banach lattice with an order continuous norm and let X be a Banach space. Then the following assertions are equivalent:*

1. *each b-weakly compact operator $T : E \longrightarrow X$ is weakly compact,*
2. *each b-AM-compact operator $T : E \longrightarrow X$ is weakly compact,*
3. *one of the following assertions holds:*
   (a) *the norm of E' is order continuous,*
   (b) *X is reflexive.*

Proof. (1)$\Longrightarrow$(2) Since each b-AM-compact operator is b-weakly compact, it follows from the assertion 1 that each b-AM-compact operator is weakly compact.
(2)$\Longrightarrow$(3) The proof of this implication follows by the same lines as in [10], it suffices to remark that the operator constructed in [10] is b-AM-compact but it is not weakly compact.

(3)$\Longrightarrow$(1) Let $T : E \longrightarrow X$ be a b-weakly compact operator. Since the norm of E is order continuous, then $T : E \longrightarrow X$ is of strong type B. As the norm of E' is order continuous, it follows from Proposition 3.2 of [5] that $T : E \longrightarrow X$ is weakly compact. ∎

Remark 2.13. *The assumption "the norm of E is order continuous" is essential in Theorem 2.12. For instance, for $p > 1$ the operator $T_p : X_P \longrightarrow c_0$ constructed in [13] is b-weakly compact but it is not weakly compact. However, the norm of $(X_P)'$ is order continuous. Note that the norm of the Banach lattice X_P is not order continuous. Otherwise, the operator $T_p : X_P \longrightarrow c_0$ would be of strong type B and since the norm of $(X_P)'$ is order continuous, it follows from Proposition 3.2 of [5] that the operator $T_p : X_P \longrightarrow c_0$ is weakly compact.*

Whenever E and F are two Banach lattices, then we obtain the following result:

Theorem 2.14. *Let E and F be two Banach lattices such that the norm of E is order continuous. Then the following assertions are equivalent:*

1. *each b-weakly compact operator $T : E \longrightarrow F$ is weakly compact,*
2. *each b-AM-compact operator $T : E \longrightarrow F$ is weakly compact,*
3. *each positive b-AM-compact operator $T : E \longrightarrow F$ is weakly compact,*
4. *one of the following assertions holds:*
   - *(a) the norm of E' is order continuous,*
   - *(b) F is reflexive.*

On the other hand, if E is a Banach lattice, the second power of a b-weakly compact operator $T : E \longrightarrow E$ is not necessary weakly compact. In fact, the identity operator Id_{ℓ^1} is b-weakly compact but its second power $(Id_{\ell^1})^2 = Id_{\ell^1}$ is not weakly compact.

The following result was stated as Theorem 2.8 in [10]:

Theorem 2.15. *Let E be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:*

1. *for all positive operators S and T from E into E with $0 \leq S \leq T$ and T is b-weakly compact, S is weakly compact,*
2. *each positive b-weakly compact operator $T : E \longrightarrow E$ is weakly compact,*
3. *for each positive b-weakly compact operator $T : E \longrightarrow E$, the second power T^2 is weakly compact,*
4. *the norm of E' is order continuous.*

Proof. (1)$\Longrightarrow$(2) Let $T : E \longrightarrow E$ be a positive b-weakly compact operator. Since $0 \leq T \leq T$, then by our hypothesis T is weakly compact.
(2)$\Longrightarrow$(3) Let $T : E \longrightarrow E$ be a b-weakly compact operator. By our hypothesis T is weakly compact and hence T^2 is weakly compact.

(3)$\Longrightarrow$(4) Suppose that the norm of E' is not order continuous. Then it follows from Theorem 2.4.14 and Proposition 2.3.11 of [14] that E contains a complemented copy of ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$.

Consider the operator $T = i \circ P$ with i is the canonical injection of ℓ^1 in E. The operator T is b-weakly compact but it is not weakly compact. Otherwise, the operator $P \circ T \circ i = Id_{\ell_1}$ would be weakly compact, and this is impossible. Hence, the operator $T^2 = T$ is not weakly compact.
(4)$\Longrightarrow$(1) Let S and T be two positive operators from E into E with $0 \leq S \leq T$ and T is b-weakly compact. It follows from Corollary 2.9 of [3] that S is b-weakly compact. Since the norm of E is order continuous, then the operator S is of strong type B and since the norm of E' is order continuous, it follows from Proposition 3.2 of [5] that S is weakly compact. ■

We end this paragraph by proving a necessary condition for which a positive AM-compact operator is compact. In fact, we have the following Theorem:

Theorem 2.16. *Let E be a Banach lattice. If each positive AM-compact operator T from E into E is compact, then E′ has an order continuous norm.*

Proof. Assume that the norm of E' is not order continuous, then it follows from Theorem 2.4.14 and Proposition 2.3.11 of [14] that E contains a sublattice isomorphs to ℓ^1 and there exists a positive projection P from E into ℓ^1.
Consider the operator product
$i \circ P : E \longrightarrow l^1 \longrightarrow E$
where i is the inclusion operator of ℓ^1 in E. Since $i \circ P = i \circ Id_{l^1} \circ P$, the operator $i \circ P$ is AM-compact which is not compact. If not its restriction to l^1, that we denote by $(i \circ P)_{|\ell^1}$, would be compact and the product operator $P \circ \left((i \circ P)_{|\ell^1}\right) = Id_{l^1}$ will be compact. This presents a contradiction. ■

Remark 2.17. *Note that there exist Banach lattices E and F and an AM-compact operator T from E into F which is not weakly compact, however*

1. *the norms of E′ and F are order continuous,*
2. *E′ is discrete and its norm is order continuous,*
3. *F is discrete and its norm is order continuous.*

In fact, if we take $E = F = c_0$, the identity operator of c_0, is AM-compact but is not weakly compact.

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References

[1] Aliprantis, C.D. and Burkinshaw, O., Positive operators. Reprint of the 1985 original. Springer, Dordrecht, 2006.

[2] Aliprantis, C.D. and Burkinshaw, O., On weakly compact operators on Banach lattices, Pro. Amer. Math. Soc. 83(3), (1981) 573-578.

[3] Alpay, S., Altin, B. and Tonyali, C. On property (b) of vector lattices. Positivity 7, No. 1-2, (2003), 135-139.

[4] Alpay, S., Altin, B. A note on b-weakly compact operators. Positivity 11 (2007), 575-582.

[5] Alpay, S. On operators of strong type B. Preprint.

[6] Altin, B., Some property of b-weakly compact operators. G.U. Journal of science. 18(3) (2005), 391-395.

[7] Aqzzouz B., Nouira R. and Zraoula L., Compactness properties of operators dominated by AM-compact operators. Proc. Amer. Math. Soc. 135 (2007), no. 4, 1151-1157.

[8] Aqzzouz B., Nouira R. and Zraoula L., The duality problem for the class of AM-compact operators on Banach lattices, Can. Math. Bull. 51 (1) (2008) 15-20.

[9] Aqzzouz, B., Elbour, A. and Hmichane, J., The duality problem for the class of b-weakly compact operators. Positivity 13 (2009), no. 4, 683–692.

[10] Aqzzouz, B. and Elbour, A., On the weak compactness of b-weakly compact operators. Positivity, vol. 14, Number 1 (2010), 75-81.

[11] Aqzzouz, B. and Hmichane, J., The class of b-AM-compact operators. To appear in Quaestiones Mathematicae in 2011.

[12] Aqzzouz, B., Aboutafail, O. and Hmichane, J.,Compactness of b-weakly compact operators. To appear in Acta Szeged in 2011.

[13] Ghoussoub, N. and Johnson, W. B. Counterexamples to several problems on the factorization of bounded linear operators. Proceedings of the American mathematical Society. Volume 92, Number 2, October 1984.

[14] Meyer-Nieberg, P., Banach lattices. Universitext. Springer-Verlag, Berlin, 1991.

[15] Niculescu, C. Order σ-continuous operators on Banach lattices, Lecture Notes in Math. Springer-Verlag 991 (1983), 188-201.

[16] Wnuk, W., Banach lattices with order continuous norms. Polish Scientific Publishers, Warsaw 1999.

[17] Zaanen A.C., Riesz-spaces II (North Holland Publishing company, 1983).

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