New common fixed point theorems and invariant approximation in convex metric spaces

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Abstract

In this paper, we use new concepts of subcompatibility and subsequential continuity contained in (Bouhadjera, Godet-Thobie, Common fixed theorems for pairs of subcompatible maps, 17 June 2009. [math.FA]) to prove common fixed point theorems for a pair of maps in metric as well as convex metric spaces which are essentially patterned after a theorem of Huang and Li (Fixed point theorems of compatible mappings in convex metric spaces, Soochow J. Math. 22(3) (1996), 439–447). We also prove some related fixed point theorems and utilize certain such results to prove theorems on best approximation.

1 Introduction

In 1970, Takahashi [17] introduced a noted and useful notion of convexity in metric spaces and utilize the same to prove some fixed point theorems for nonexpansive mappings in such spaces which are often inspired by similar looking theorems in Banach spaces. To describe this Takahashi convex structure, let $(\mathcal{X}, d)$ be a metric space and $\mathcal{I} = [0, 1]$. A mapping $\mathcal{W} : \mathcal{X} \times \mathcal{X} \times \mathcal{I} \to \mathcal{X}$ is said to be a convex structure on $\mathcal{X}$ if for each $(x, y, \lambda) \in \mathcal{X} \times \mathcal{X} \times \mathcal{I}$ and $u \in \mathcal{X}$,

$$d(u, \mathcal{W}(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

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A metric space $(\mathcal{X}, d)$ together with a convex structure $\mathcal{W}$ is called a convex metric space. Obviously, every Banach space and all of its convex subsets are simple examples of convex metric spaces with respect to the convex structure $\mathcal{W}(x, y, \lambda) = \lambda x + (1-\lambda)y$ but a *Fréchet* space need not be a convex metric space. There are many examples of convex metric spaces which can not be embedded in any Banach space.

In the course of last forty years, numerous researchers discussed the existence of fixed points as well as the convergence of various iterative processes for nonexpansive mappings in convex metric spaces (e.g. [4, 5, 7]). Recently, Beg et al. [2] proved results on the existence of common fixed point in convex metric spaces and utilize the same to prove results on the existence of best approximant for relatively contractive commuting mappings which also covers the core result of Sahab et al. [13]. Here it may be pointed out that the result of Sahab et al. [13] engineered an intense research activity in this direction in the preceding years.

With a view to improve commutativity conditions in common fixed point theorems, Sessa [14] introduced the concept of weakly commuting pair of maps. He terms the maps $\mathcal{T}$ and $\mathcal{I}$ to be weakly commuting if

$$d(\mathcal{T}\mathcal{I}x, \mathcal{I}\mathcal{T}x) \leq d(\mathcal{T}x, \mathcal{I}x)$$

for all $x \in \mathcal{X}$. Jungck [10] further enlarged the class of weakly compatible pairs by introducing the notion of compatible mappings. Inspired by the definition of Jungck [10], researchers of this domain introduced several definitions of compatible-like conditions such as: compatible mappings of type (A), (B), (C) and (P), biased mappings, weakly compatible mappings, occasionally weakly compatible mappings, and some others whose systematic survey (only up to 2001) is available in Murthy [11]. In 2009 Bouhadjera and Godet-Thobie [3] further enlarged the class of compatible (reciprocally continuous) pairs by introducing the concept of subcompatible (subsequential continuous) pair which is substantially weaker than compatibility (reciprocal continuity).

The following theorem due to Huang and Li [7] has inspired our studies in this paper.

Theorem 1.1. **[7]** Let $(\mathcal{X}, d)$ be a convex metric space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$. If $(\mathcal{T}, \mathcal{I})$ is a compatible pair of self mapping defined on $\mathcal{K}$ such that for all $x, y \in \mathcal{K}$,

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &\leq a\, d(\mathcal{I}x, \mathcal{I}y) + b \max\{d(\mathcal{I}x, \mathcal{T}x), d(\mathcal{I}y, \mathcal{T}y)\} \\ &+ c \max\{d(\mathcal{I}x, \mathcal{I}y), d(\mathcal{I}x, \mathcal{T}x), d(\mathcal{I}y, \mathcal{T}y), \frac{1}{2}(d(\mathcal{I}x, \mathcal{T}y) + d(\mathcal{I}y, \mathcal{T}x))\} \end{aligned} \quad (1.1)$$

where a, b and c are nonnegative real numbers such that $a + b + c = 1$ and $c < 2b(1-b)/(2+b)$. If $\mathcal{T}(\mathcal{K}) \subset \mathcal{I}(\mathcal{K})$ and $\mathcal{I}$ is $\mathcal{W}$-affine and continuous, then there exists a unique common fixed point z of $\mathcal{T}$ and $\mathcal{I}$ and $\mathcal{T}$ is continuous at z.

In this paper, we prove some existence results on common fixed point for a pair of weakly compatible mappings in the setting of convex metric space. Similar results for subcompatible (resp. compatible) and reciprocally continuous (resp.

subsequentially continuous) pair of mappings are proved in metric space which are essentially inspired by Imdad et al. [8, 9]. Finally we prove some related results and utilize the same to prove results on invariant approximation.

2 Preliminaries

In this section, we prepare the background material for the results to be presented in this paper. We begin with some examples of convex metric spaces.

Example 2.1.(cf. [17]) Let $\mathcal{I}$ be the unit interval $[0,1]$ and $\mathcal{X}$ be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $\mathcal{I}_i = [a_i, b_i], \mathcal{I}_j = [a_j, b_j]$ and $\lambda (0 \leq \lambda \leq 1)$, we define a mapping $\mathcal{W}$ by $\mathcal{W}(\mathcal{I}_i, \mathcal{I}_j; \lambda) = [\lambda a_i + (1-\lambda)a_j, \lambda b_i + (1-\lambda)b_j]$ whereas a metric d in $\mathcal{X}$ by the Hausdorff distance, i.e.

$$d(\mathcal{I}_i, \mathcal{I}_j) = \sup_{a \in \mathcal{I}} \{ | \inf_{b \in \mathcal{I}_i} \{|a-b|\} - \inf_{c \in \mathcal{I}_j} \{|a-c|\} | \}.$$

Example 2.2.(cf.[17]) A linear space $\mathcal{L}$ equipped with the following two properties is a natural convex metric space:

(1) For $x, y \in \mathcal{L}$, $d(x,y) = d(x-y, 0)$;
(2) For $x, y \in \mathcal{L}$ and $\lambda (0 \leq \lambda \leq 1)$,

$$d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x,0) + (1-\lambda)d(y,0).$$

Definition 2.1.(cf.[17]) A subset $\mathcal{K}$ of a convex metric space $(\mathcal{X}, d)$ is said to be convex, if $\mathcal{W}(x,y,\lambda) \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and $\lambda \in [0,1]$. The set $\mathcal{K}$ is said to q-starshaped if there exists $q \in \mathcal{K}$ such that $\mathcal{W}(x,q,\lambda) \in \mathcal{K}$ for all $x \in \mathcal{K}$ and $\lambda \in [0,1]$. Clearly q-starshaped subsets of $\mathcal{X}$ contain all convex subsets of $\mathcal{X}$ as a proper subclass.
Takahashi [17] has shown that open spheres $\mathcal{B}(x,r) = \{y \in \mathcal{X} : d(y,x) < r\}$ and closed spheres $\mathcal{B}[x,r] = \{y \in \mathcal{X} : d(y,x) \leq r\}$ are convex in a convex metric space $(\mathcal{X}, d)$.

Definition 2.2.(cf.[17]) A convex metric space$(\mathcal{X}, d)$ is said to satisfy the property (I), if for all $x, y, z \in \mathcal{X}$ and $\lambda \in [0,1]$,

$$d(\mathcal{W}(x,z,\lambda), \mathcal{W}(y,z,\lambda)) \leq \lambda d(x,y).$$

For motivation and further details in respect of the Property (I), one can be referred to Guay et al. [6] (e.g. Definition 3.2).

Definition 2.3.(cf.[7],[12]) A map $\mathcal{I}$ from a closed convex subset $\mathcal{K}$ of a convex metric space $(\mathcal{X}, d)$ into itself is said to be $\mathcal{W}$-affine if $\mathcal{I}(\mathcal{W}(x,y,\lambda)) = \mathcal{W}(\mathcal{I}x, \mathcal{I}y, \lambda)$ whenever $\lambda \in [0,1] \cap \mathcal{Q}$ and $x, y \in \mathcal{K}$, where $\mathcal{Q}$ stands for the set of rational numbers.

Definition 2.4.(cf.[9]) A pair $(\mathcal{T}, \mathcal{I})$ of self-mappings of a metric space $(\mathcal{X}, d)$ is said to be compatible, if $d(\mathcal{T}\mathcal{I}x_n, \mathcal{I}\mathcal{T}x_n) \to 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{T}x_n, \mathcal{I}x_n \to t \in \mathcal{X}$.

Definition 2.5.(cf.[1]) A pair $(\mathcal{T},\mathcal{I})$ of self-mappings of a metric space $(\mathcal{X},d)$ is said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{T}x_n,\mathcal{I}x_n \to t \in \mathcal{X}$.

Definition 2.6.(cf.[9]) A pair $(\mathcal{T},\mathcal{I})$ of self mappings of a metric space $(\mathcal{X},d)$ is said to be subcompatible iff there exists a sequence $\{x_n\} \in \mathcal{X}$ such that $d(\mathcal{T}\mathcal{I}x_n,\mathcal{I}\mathcal{T}x_n) \to 0$, with $\mathcal{T}x_n,\mathcal{I}x_n \to t \in \mathcal{X}$.

Clearly a pair of noncompatible or subcompatible mapping satisfies the property $(E.A)$.

Obviously, compatible maps which satisfy the property $(E.A)$ are subcompatible but the converse statement does not hold in general as substantiated by the following example.

Example 2.3.(cf.[9]) Consider $\mathcal{X} = \mathbb{R}$ equipped with the usual metric. Define $\mathcal{T},\mathcal{I} : \mathcal{X} \to \mathcal{X}$ as follows:

$$\mathcal{T}x = \begin{cases} \frac{x}{2}, & \text{if } x \in (-\infty,1) \\ 3x-2, & \text{if } x \in [1,\infty) \end{cases} \quad \text{and} \quad \mathcal{I}x = \begin{cases} x+1, & \text{if } x \in (-\infty,1) \\ 2x-1, & \text{if } x \in [1,\infty). \end{cases}$$

In respect of the sequence $x_n = 1+\frac{1}{n}$, $\lim_{n\to\infty}\mathcal{T}(x_n) = 1$ and $\lim_{n\to\infty}\mathcal{I}(x_n) = 1$. Also $\lim_{n\to\infty}\mathcal{T}\mathcal{I}(x_n) = 1 = \mathcal{T}(1)$ and $\lim_{n\to\infty}\mathcal{I}\mathcal{T}(x_n) = 1 = \mathcal{I}(1)$ and $\lim_{n\to\infty} d(\mathcal{T}\mathcal{I}x_n,\mathcal{I}\mathcal{T}x_n) = 0$.

Consider another sequence $x_n = \frac{1}{n} - 2$, then $\lim_{n\to\infty}\mathcal{T}(x_n) = -1$ and $\lim_{n\to\infty}\mathcal{I}(x_n) = -1$. Also, $\lim_{n\to\infty}\mathcal{T}\mathcal{I}(x_n) = \frac{-1}{2} = \mathcal{T}(-1)$ and $\lim_{n\to\infty}\mathcal{I}\mathcal{T}(x_n) = 0 = \mathcal{I}(-1)$ and $\lim_{n\to\infty} d(\mathcal{T}\mathcal{I}x_n,\mathcal{I}\mathcal{T}x_n) \neq 0$.

Thus, the pair $(\mathcal{T},\mathcal{I})$ is subcompatible, but not compatible.

Definition 2.7.(cf.[8, 9]) A pair $(\mathcal{T},\mathcal{I})$ of self mappings of a metric space $(\mathcal{X},d)$ is said to be reciprocally continuous iff $\lim_{n\to\infty}\mathcal{T}\mathcal{I}x_n = \mathcal{T}(t)$ and $\lim_{n\to\infty}\mathcal{I}\mathcal{T}x_n = \mathcal{I}(t)$, for every sequence x_n in $\mathcal{X}$ satisfying

$$\lim_{n\to\infty}\mathcal{T}x_n = \lim_{n\to\infty}\mathcal{I}x_n = t$$

for some $t \in \mathcal{X}$.

Clearly, any pair of continuous mappings is reciprocally continuous, but the converse need not be true in general (e.g. Example 2.3).

Definition 2.8.(cf.[8, 9]) A pair $(\mathcal{T},\mathcal{I})$ of self mappings of a metric space $\mathcal{X}$ is said to be subsequentially continuous iff there exists a sequence $\{x_n\} \in \mathcal{X}$ such that $\lim_{n\to\infty}\mathcal{T}\mathcal{I}x_n = \mathcal{T}(t)$ and $\lim_{n\to\infty}\mathcal{I}\mathcal{T}x_n = \mathcal{I}(t)$ with $\lim_{n\to\infty}\mathcal{T}x_n = \lim_{n\to\infty}\mathcal{I}x_n = t$ for some $t \in \mathcal{X}$.

If $\mathcal{T}$ and $\mathcal{I}$ are both continuous or reciprocally continuous, then they are obviously subsequentially continuous. But there do exist pairs of subsequentially continuous mappings which are neither continuous nor reciprocally continuous as exhibited by the following example.

Example 2.4. Consider $\mathcal{X} = [0,\infty)$ endowed with the usual metric d and define $\mathcal{T}$ and $\mathcal{I} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}x = \begin{cases} x+1\,, & \text{if } x \in [0,1] \\ 2x-1\,, & \text{if } x \in (1,\infty) \end{cases} \quad \text{and} \quad \mathcal{I}x = \begin{cases} 1-x\,, & \text{if } x \in [0,1) \\ 3x-2\,, & \text{if } x \in [1,\infty). \end{cases}$$

Obviously, $\mathcal{T}$ and $\mathcal{I}$ are discontinuous at $x = 1$.
In respect of the sequence $x_n = \frac{1}{n}$ for $n = 1, 2, \cdots$, we have

$$\mathcal{T}x_n = 1 + x_n \to 1 = t \text{ and } \mathcal{I}x_n = 1 - x_n \to 1 \text{ when } n \to \infty,$$

and

$$\mathcal{T}\mathcal{I}x_n = \mathcal{T}(1 - x_n) = 2 - x_n \to 2 = \mathcal{T}(1)$$
$$\mathcal{I}\mathcal{T}x_n = \mathcal{I}(1 + x_n) = 1 + 3x_n \to 1 = \mathcal{I}(1)$$

which show that $\mathcal{T}$ and $\mathcal{I}$ are subsequentially continuous.
Now, in respect of the sequence $x_n = 1 + \frac{1}{n}$ for $n = 1, 2, \cdots$, we have

$$\mathcal{T}x_n = 2x_n - 1 \to 1 = t \quad \text{and} \quad \mathcal{I}x_n = 3x_n - 2 \to 1 = t$$

when $n \to \infty$, and

$$\mathcal{T}\mathcal{I}x_n = \mathcal{T}(3x_n - 2) = 6x_n - 5 \to 1 \neq 2 = \mathcal{T}(1)$$

when $n \to \infty$ which show that the maps $\mathcal{T}$ and $\mathcal{I}$ are not reciprocally continuous.

The example below shows that there exist subcompatible maps which are reciprocally continuous but are neither continuous nor compatible.

Example 2.5. Consider $\mathcal{X} = [0,\infty)$ with the usual metric d. Define $\mathcal{T}$ and $\mathcal{I}$ as follows:

$$\mathcal{T}(x) = \begin{cases} 2x-1 & \text{if } x \in [0,4] \cup (9,\infty) \\ x+12 & \text{if } x \in (4,9] \end{cases}, \quad \mathcal{I}(x) = \begin{cases} x^2 & \text{if } x \in [0,4] \cup (9,\infty) \\ 2x+4 & \text{if } x \in (4,9] \end{cases},$$

Obviously, $\mathcal{T}$ and $\mathcal{I}$ are discontinuous at $x = 4$.
In respect of the sequence $x_n = 8 + \frac{1}{n}$ for $n = 1, 2, \cdots$, we have

$$\mathcal{T}x_n = 20 + \tfrac{1}{n} \to 20 = t \quad \text{and} \quad \mathcal{I}x_n = 20 + \tfrac{2}{n} = 20 \text{ when } n \to \infty$$

and

$$\mathcal{T}\mathcal{I}x_n \to 39 \qquad \text{and} \qquad \mathcal{I}\mathcal{T}x_n \to 400.$$

Thus $\lim_{n\to\infty} d(\mathcal{T}\mathcal{I}x_n, \mathcal{I}\mathcal{T}x_n) \neq 0$ so that the pair of maps $\mathcal{T}$ and $\mathcal{I}$ are not compatible.

Now, in respect of the sequence $x_n = 1 + \frac{1}{n}$ for n = 1, 2, ... , we have

$$\mathcal{T}x_n = 2x_n - 1 \to 1 = t \qquad \text{and} \qquad \mathcal{I}x_n = x_n^2 \to 1 = t$$

when $n \to \infty$ and

$$\mathcal{T}\mathcal{I}x_n = \mathcal{T}x_n^2 = 2x_n^2 - 1 \to 1 \quad \text{and} \quad \mathcal{I}\mathcal{T}x_n = \mathcal{I}(2x_n - 1) = (2x_n - 1)^2 \to 1$$

when $n \to \infty$.

Thus $\lim_{n\to\infty} d(\mathcal{T}\mathcal{I}x_n, \mathcal{I}\mathcal{T}x_n) = 0$ so that the pair of maps $\mathcal{T}$ and $\mathcal{I}$ are sub-compatible. Notice that the pair of maps $\mathcal{T}$ and $\mathcal{I}$ are discontinuous but still they are reciprocally continuous.

Definition 2.9. (cf. [15]) Let $\mathcal{K}$ be a closed subset of a metric space $(\mathcal{X}, d)$. Let $x_0 \in \mathcal{X}$. An element $y \in \mathcal{K}$ is called a best approximant to $x_0 \in \mathcal{X}$, if

$$d(x_0, y) = d(x_0, \mathcal{K}) = \inf\{d(x_0, z) : z \in \mathcal{K}\}.$$

We denote by $\mathcal{P}_{\mathcal{K}}(x_0)$, the set of best $\mathcal{K}-$approximants to x_0.

Example 2.6.(cf.[16]) Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{K} = [0, \frac{1}{2}]$. Define $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\mathcal{T}x = \begin{cases} x - 1 & \text{if } x < 0, \\ x & \text{if } 0 \le x \le \frac{1}{2}, \\ \frac{x+1}{2} & \text{if } x > \frac{1}{2}. \end{cases}$$

Clearly, $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ and $\mathcal{T}(1) = 1$ (i.e.$x_0 = 1$). Also

$$\mathcal{P}_{\mathcal{K}}(x_0) = \{\frac{1}{2}\}.$$

Hence, $\mathcal{T}$ has a fixed point in $\mathcal{P}_{\mathcal{K}}(x_0)$ which is a best approximation to x_0 in $\mathcal{K}$. Thus, $\frac{1}{2}$ is an invariant approximation.

Definition 2.10. Let $\mathcal{K}$ be a subset of metric space $\mathcal{X}$. The map $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is said to be compact if for every bounded sequence $\{x_n\}$ in $\mathcal{K}$, $\{\mathcal{T}x_n\}$ admits a convergent subsequence $\{\mathcal{T}x_m\}$ in $\mathcal{K}$.

Definition 2.11. Let $\mathcal{T}$ and $\mathcal{I}$ be two self-maps defined on a set $\mathcal{X}$. Then $\mathcal{T}$ and $\mathcal{I}$ are said to be weakly compatible, if they commute at every coincidence point.

3 Main Results

We begin with the following observation.
In view of the definition of the property $(E.A.)$, a carefully examination of the proof of Theorem 3.1 of Huang and Li [7] enables us to fish out the following lemma:

Lemma 3.1. Let $(\mathcal{X}, d)$ be a convex metric space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$. If the pair of mappings $\mathcal{T}$ and $\mathcal{I}$ satisfy (1.1) wherein $\mathcal{T}\mathcal{K} \subset \mathcal{I}\mathcal{K}$, $\mathcal{I}$ is $\mathcal{W}$-affine and $\mathcal{I}(\mathcal{K})$ (or $\mathcal{T}(\mathcal{K})$) is a complete subset of $\mathcal{X}$, then the maps $\mathcal{T}$ and $\mathcal{I}$ share the property $(E.A)$.

The following theorem generalizes Theorem 1.1 in the considerations of compatibility and continuity of the involved pair of mappings besides replacing completeness of the space with completeness of two alternate subspaces.

Theorem 3.1. Let $\mathcal{K}$ be a nonempty closed convex subset of a convex metric space $(\mathcal{X}, d)$. If the maps $\mathcal{T}$ and $\mathcal{I}$ are self mappings defined on $\mathcal{K}$ which satisfy the inequality (1.1) with $\mathcal{I}$ is $\mathcal{W}$-affine, $\mathcal{T}\mathcal{K} \subset \mathcal{I}\mathcal{K}$ and $\mathcal{I}(\mathcal{K})$ (or $\mathcal{T}(\mathcal{K})$) is a complete subspace of $\mathcal{X}$, then

(i) the maps $\mathcal{T}$ and $\mathcal{I}$ have a coincidence point v,
(ii) $\mathcal{T}v = u$ is a unique common fixed point of $\mathcal{T}$ and $\mathcal{I}$ provided the maps $\mathcal{T}$ and $\mathcal{I}$ are weakly compatible,
(iii) the mapping $\mathcal{T}$ is continuous at u provided $\mathcal{I}$ is continuous at u.

Proof. Notice that $c < \frac{2b(1-b)}{2+b}$ implies $a > 0, b > 0$ as $a + b + c = 1$ and a, b, c are nonnegative real numbers. In view of Lemma 3.1, there exists a sequence $\{x_n\}$ and $u \in \mathcal{K}$ such that

$$\lim_n \mathcal{T}x_n = \lim_n \mathcal{I}x_n = u. \tag{3.1}$$

Now, suppose that $\mathcal{I}(\mathcal{K})$ is a complete subspace of $\mathcal{X}$, then $u \in \mathcal{I}(\mathcal{K})$ and henceforth one can find a $v \in \mathcal{K}$ such that

$$\mathcal{I}v = u. \tag{3.2}$$

Firstly, we show that $\mathcal{T}v = u$. To accomplish this, on setting $x = v$ and $y = x_n$ in (1.1) and making use of (3.1) and (3.2), one gets

$$\begin{aligned} d(\mathcal{T}v, \mathcal{T}x_n) \leq a\, d(\mathcal{I}v, \mathcal{I}x_n) + b \max\{d(\mathcal{I}v, \mathcal{T}v), d(\mathcal{I}x_n, \mathcal{T}x_n)\} \\ +c \max\{d(\mathcal{I}v, \mathcal{I}x_n), d(\mathcal{I}v, \mathcal{T}v), d(\mathcal{I}x_n, \mathcal{T}x_n), \\ \frac{1}{2}(d(\mathcal{I}v, \mathcal{T}x_n) + d(\mathcal{I}x_n, \mathcal{T}v))\} \end{aligned}$$

which on letting $n \to \infty$, gives rise

$$d(u, \mathcal{T}v) \leq (b + c)\, d(u, \mathcal{T}v)$$

yielding thereby $d(u, \mathcal{T}v) = 0$ so that

$$\mathcal{T}v = u \tag{3.3}$$

as $a > 0$ and $a + b + c = 1$. Owing to (3.2) and (3.3), one can write

$$\mathcal{I}v = \mathcal{T}v = u \tag{3.4}$$

which shows that v is a coincidence point of the maps $\mathcal{T}$ and $\mathcal{I}$. Moreover, if the maps $\mathcal{T}$ and $\mathcal{I}$ are weakly compatible, from (3.4), we have

$$\mathcal{I}u = \mathcal{I}(\mathcal{T}v) = \mathcal{T}(\mathcal{I}v) = \mathcal{T}u. \tag{3.5}$$

In order to show that u is common fixed point of $\mathcal{T}$ and $\mathcal{I}$, on taking $x = u$ and $y = x_n$ in (1.1) and making use of (3.5), one gets

$$d(\mathcal{T}u, \mathcal{T}x_n) \leq a\, d(\mathcal{I}u, \mathcal{I}x_n) + b \max\{d(\mathcal{I}u, \mathcal{T}u), d(\mathcal{I}x_n, \mathcal{T}x_n)\}$$

$$+c\max\{d(\mathcal{I}u,\mathcal{I}x_n),d(\mathcal{I}u,\mathcal{T}u),d(\mathcal{I}x_n,\mathcal{T}x_n),\\ \frac{1}{2}(d(\mathcal{I}u,\mathcal{T}x_n)+d(\mathcal{I}x_n,\mathcal{T}u))\}$$

which on letting $n\to\infty$, gives rise

$$d(u,\mathcal{T}u)\le(a+c)\,d(u,\mathcal{T}u)\le(1-b)\,d(u,\mathcal{T}u)<d(u,\mathcal{T}u)$$

so that

$$\mathcal{I}u=\mathcal{T}u=u \tag{3.6}$$

as $b>0$ and $a+b+c=1$.

To prove the uniqueness of common fixed point u, let u_1 be another common fixed point of $\mathcal{T}$ and $\mathcal{I}$ so that $d(u,u_1)>0$, it follows from (1.1) that

$$d(u,u_1)=d(\mathcal{T}u,\mathcal{T}u_1)\le a\,d(\mathcal{I}u,\mathcal{I}u_1)+b\max\{d(\mathcal{I}u,\mathcal{T}u),d(\mathcal{I}u_1,\mathcal{T}u_1)\}\\ +c\max\{d(\mathcal{I}u,\mathcal{I}u_1),d(\mathcal{I}u,\mathcal{T}u),d(\mathcal{I}u_1,\mathcal{T}u_1),\\ \frac{1}{2}(d(\mathcal{I}u,\mathcal{T}u_1)+d(\mathcal{I}u_1,\mathcal{T}u))\}=(a+c)d(u,u_1)$$

i.e.

$$d(u,u_1)\le(a+c)d(u,u_1) \tag{3.7}$$

which is a contradiction as $a+c=1-b<1$. Hence, u is the unique common fixed point of $\mathcal{T}$ and $\mathcal{I}$.
Next, if $\mathcal{T}\mathcal{K}$ is complete subspace of $\mathcal{X}$, then $u\in\mathcal{T}\mathcal{K}$. Since $\mathcal{T}\mathcal{K}\subset\mathcal{I}\mathcal{K}$, therefore $u\in\mathcal{I}\mathcal{K}$. The rest of the arguments can be completed on the preceding lines.

Now, we proceed to show that $\mathcal{T}$ is continuous at u provided $\mathcal{I}$ is continuous at u.
Let $\{u_n\}\subset\mathcal{K}$ such that $u_n\to u$, then owing to continuity of $\mathcal{I}$ at u, $\mathcal{I}u_n\to\mathcal{I}u$. On using (1.1), we have

$$d(\mathcal{T}u_n,\mathcal{T}u)\le a\,d(\mathcal{I}u_n,\mathcal{I}u)+b\max\{d(\mathcal{I}u_n,\mathcal{T}u_n),d(\mathcal{I}u,\mathcal{T}u)\}\\ +c\max\{d(\mathcal{I}u_n,\mathcal{I}u),d(\mathcal{I}u_n,\mathcal{T}u_n),d(\mathcal{I}u,\mathcal{T}u),\\ \frac{1}{2}(d(\mathcal{I}u_n,\mathcal{T}u)+d(\mathcal{I}u,\mathcal{T}u_n))\} \tag{3.8}$$

which, in turn, yields

$$\lim_{n\to\infty}\sup d(\mathcal{T}u_n,\mathcal{T}u)\le(b+c)\lim_{n\to\infty}\sup d(\mathcal{T}u_n,\mathcal{T}u)$$

so that $\lim\limits_{n\to\infty}d(\mathcal{T}u_n,\mathcal{T}u)=0$, as $b+c=1-a<1$. Therefore, $\mathcal{T}$ is continuous at u. This completes the proof.

Example 3.1. Consider $\mathcal{X}=\mathbb{R}$ equipped with usual metric and $\mathcal{K}=[0,\infty)$. Define a mapping $\mathcal{W}$ by $\mathcal{W}(x,y;\lambda)=\lambda x+(1-\lambda)y$ whereas the self maps $\mathcal{T}$ and $\mathcal{I}$

as $\mathcal{I}x = \frac{x+1}{2}$ and

$$\mathcal{T}x = \begin{cases} 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$$

Clearly, $\mathcal{I}$ is continuous and $\mathcal{W}$-affine. Also, notice that the pair $\mathcal{T}$ and $\mathcal{I}$ are weakly compatible as $\mathcal{I}1 = \mathcal{T}1$ implies

$$\mathcal{T}\mathcal{I}1 = \mathcal{I}\mathcal{T}1 = 1.$$

Observe that $\mathcal{T}(\mathcal{K}) \subset \mathcal{I}(\mathcal{K})$ and $\mathcal{T}(\mathcal{K})$ is a complete subspace of $\mathcal{X}$. By a routine calculation, one can verify inequality (1.1) with $a = \frac{3}{10}$, $b = \frac{23}{40}$, $c = \frac{1}{8}$. Therefore all the conditions of Theorem 3.1 are satisfied. Notice that 1 is a unique common fixed point the pair $\mathcal{T}$ and $\mathcal{I}$. Also, $\mathcal{T}$ and $\mathcal{I}$ are continuous at 1.

The following example exhibits that condition (1.1) is necessary in Theorem 3.1.

Example 3.2. Consider $\mathcal{X} = \mathbb{R}$ equipped with usual metric and $\mathcal{K} = [0,1]$. Define a mapping $\mathcal{W}$ by $\mathcal{W}(x,y;\lambda) = \lambda x + (1-\lambda)y$ besides defining self maps $\mathcal{T}$ and $\mathcal{I}$ as $\mathcal{I}x = x$ and

$$\mathcal{T}x = \begin{cases} \frac{x}{2} & \text{if } x > 0 \\ 1 & \text{if } x = 0. \end{cases}$$

By a routine calculation, one can verify all conditions of Theorem 3.1 except inequality (1.1). Notice that $\mathcal{T}$ and $\mathcal{I}$ do not have any common fixed point.

Setting c=0 in Theorem 3.1, we deduce the following result.

Corollary 3.1. Let $\mathcal{K}$ be a nonempty closed convex subset of a convex metric space $(\mathcal{X}, d)$. Let the maps $\mathcal{T}$ and $\mathcal{I}$ be self mappings defined on $\mathcal{K}$ which satisfy the inequality

$$d(\mathcal{T}x, \mathcal{T}y) \leq a\, d(\mathcal{I}x, \mathcal{I}y) + (1-a)\max\{d(\mathcal{I}x, \mathcal{T}x), d(\mathcal{I}y, \mathcal{T}y)\} \tag{3.9}$$

for all $x, y \in \mathcal{K}$ wherein $0 < a < 1$, $\mathcal{I}$ is $\mathcal{W}$-affine, $\mathcal{T}\mathcal{K} \subset \mathcal{I}\mathcal{K}$ and $\mathcal{I}(\mathcal{K})$ (or $\mathcal{T}(\mathcal{K})$) is complete subspace of $\mathcal{X}$, then

(i) the maps $\mathcal{T}$ and $\mathcal{I}$ have a coincidence point v,
(ii) $\mathcal{T}v = u$ is a unique common fixed point of $\mathcal{T}$ and $\mathcal{I}$ provided the maps $\mathcal{T}$ and $\mathcal{I}$ are weakly compatible,
(iii) the mapping $\mathcal{T}$ is continuous at u provided $\mathcal{I}$ is a continuous at u.

Remark 3.1. In Theorem 3.1, as compared to Theorem 1.1, the class of compatible mappings is significantly enlarged to class of weakly compatible mappings besides reducing the continuity requirement of the mapping $\mathcal{I}$. Moreover, the completeness of the space is alternately replaced by the completeness of the subspace $\mathcal{I}(\mathcal{K})$ (or $\mathcal{T}(\mathcal{K})$).

In what follows, on the lines of Imdad et.al [9], we prove two common fixed point theorems in metric spaces without convexity structure employing some relatively recent weak definitions on commutativity and continuity of the involved pair. Here, it can be pointed out that such theorems never require any condition on completeness (or closedness) of the underlying space (or subspace). Our first such a result runs as follows.

The following theorem is a result in a metric space which is similar to Theorem 3.1.

Theorem 3.2. Let $\mathcal{K}$ be a nonempty subset of a metric space $(\mathcal{X}, d)$. If the pair of self-mappings $\mathcal{T}$ and $\mathcal{I}$ defined on $\mathcal{K}$ are subcompatible and reciprocally continuous besides satisfying inequality (1.1), then
(i) there exists a unique common fixed point u of $\mathcal{T}$ and $\mathcal{I}$,

(ii) $\mathcal{T}$ is continuous at u, provided $\mathcal{I}$ is continuous at u.

Proof: As the maps $\mathcal{T}$ and $\mathcal{I}$ are subcompatible, there exists a sequence $\{x_n\}$ and $u \in \mathcal{K}$ such that

$$\mathcal{T}x_n \to u, \mathcal{I}x_n \to u \text{ and } d(\mathcal{T}\mathcal{I}x_n, \mathcal{I}\mathcal{T}x_n) \to 0 \text{ when } n \to \infty. \tag{3.10}$$

Since the pair $\mathcal{T}$ and $\mathcal{I}$ are reciprocally continuous, so

$$\mathcal{T}\mathcal{I}x_n \to \mathcal{T}u \text{ and } \mathcal{I}\mathcal{T}x_n \to \mathcal{I}u \text{ when } n \to \infty\,. \tag{3.11}$$

Now, in view of (3.10) and (3.11), one gets

$$\mathcal{T}u = \mathcal{I}u\ . \tag{3.12}$$

On taking $x = u$ and $y = x_n$ in condition (1.1), one gets

$$\begin{aligned} d(\mathcal{T}u, \mathcal{T}x_n) \le a\, d(\mathcal{I}u, \mathcal{I}x_n) + b \max\{d(\mathcal{I}u, \mathcal{T}u), d(\mathcal{I}x_n, \mathcal{T}x_n)\} \\ +c \max\{d(\mathcal{I}u, \mathcal{I}x_n), d(\mathcal{I}u, \mathcal{T}u), d(\mathcal{I}x_n, \mathcal{T}x_n) \\ \frac{1}{2}(d(\mathcal{I}u, \mathcal{T}x_n) + d(\mathcal{I}x_n, \mathcal{T}u))\} \end{aligned}$$

which on letting $n \to \infty$, gives rise

$$d(u, \mathcal{T}u) \le (a + c)\, d(u, \mathcal{T}u)$$

so that

$$\mathcal{I}u = \mathcal{T}u = u$$

as $b > 0$ and $a + b + c = 1$ which shows that u is a common fixed point of the pair of mappings.
The rest of the proof can be completed on the lines of the proof of Theorem 3.1, hence details are omitted. This concludes the proof.

The following example demonstrates Theorem 3.2.

Example 3.3. Consider $\mathcal{X} = \mathbb{R}$ equipped with usual metric and $\mathcal{K} = [0, \infty)$. Define self maps $\mathcal{T}$ and $\mathcal{I}$ by $\mathcal{T}x = x$ and $\mathcal{I}x = 3x$.

Notice that the pair $\mathcal{T}$ and $\mathcal{I}$ are subcompatible and reciprocally continuous. By a routine calculation one can verify condition (1.1) with $a = \frac{1}{3}, b = \frac{1}{2}$ and $c =$

$\frac{1}{6}$. Thus all the conditions of Theorem 3.2 are satisfied . Notice that $x = 0$ is common fixed point $\mathcal{T}$ and $\mathcal{I}$.

The other result, similar to Theorem 3.2, runs as follows.

Theorem 3.3. Let $\mathcal{K}$ be a nonempty subset of a metric space$(\mathcal{X}, d)$. If the pair of self-mappings $\mathcal{T}$ and $\mathcal{I}$ defined on $\mathcal{K}$ are compatible and subsequentially continuous satisfying the inequality (1.1), then

(i) the maps $\mathcal{T}$ and $\mathcal{I}$ have a unique common fixed point u,
(ii) $\mathcal{T}$ is continuous at u, provided $\mathcal{I}$ is continuous at u.

Proof: As the pair $\mathcal{T}$ and $\mathcal{I}$ are subsequentially continuous, there exists a sequence $\{x_n\}$ and $u \in \mathcal{K}$ such that

$$\mathcal{T}x_n \to u, \mathcal{I}x_n \to u \ , \ \mathcal{T}\mathcal{I}x_n \to \mathcal{T}u \text{ and } \mathcal{I}\mathcal{T}x_n \to \mathcal{I}u \text{ as } n \to \infty. \tag{3.13}$$

Also, the pair $\mathcal{T}$ and $\mathcal{I}$ are compatible, therefore

$$d(\mathcal{T}\mathcal{I}x_n, \mathcal{I}\mathcal{T}x_n) \to 0, \text{ as } n \to \infty. \tag{3.14}$$

Now, in view of (3.13) and (3.14), we have

$$\mathcal{I}u = \mathcal{T}u.$$

The rest of the proof can be completed on the lines of Theorems 3.1 and 3.2, therefore details are avoided. This concludes the proof of the theorem.

The following example demonstrate Theorem 3.3.

Example 3.4. Confider $\mathcal{X} = \mathbb{R}$ endowed with the natural metric and $\mathcal{K} = [1, \infty)$. Define

$$\mathcal{T}x = \begin{cases} x+1, & \text{if } x \in (-\infty, 1) \\ 2x-1, & \text{if } x \in [1, \infty) \end{cases} \quad \text{and} \quad \mathcal{I}x = \begin{cases} x-1, & \text{if } x \in (-\infty, 1) \\ 3x-2, & \text{if } x \in [1, \infty). \end{cases}$$

Notice that $\mathcal{T}$ and $\mathcal{I}$ are discontinuous at $x = 1$. In respect of the sequence $x_n = 1 + \frac{1}{n}$, $\mathcal{T}x_n \to 1$ and $\mathcal{I}x_n \to 1$ as $n \to \infty$, also

$$\mathcal{T}\mathcal{I}x_n = 2 + \frac{6}{n} - 1 \to 1 = \mathcal{T}(1)$$

$$\mathcal{I}\mathcal{T}x_n = 3 + \frac{6}{n} - 2 \to 1 = \mathcal{I}(1)$$

as $n \to \infty$. Also $d(\mathcal{T}\mathcal{I}x_n, \mathcal{I}\mathcal{T}x_n) \to 0$, whenever $\mathcal{T}x_n \to t$ and $\mathcal{I}x_n \to t$ as $n \to \infty$. Thus the pair $\mathcal{T}$ and $\mathcal{I}$ is compatible as well as subsequentially continuous. One can also easily verify inequality (1.1) with $a = \frac{2}{3}, b = \frac{1}{4} and c = \frac{1}{12}$. Thus, all the conditions of Theorem 3.3 are satisfied and $x = 1$ is a common fixed point of the pair $\mathcal{T}$ and $\mathcal{I}$.

Setting $c = 0$ in Theorem 3.3, we deduce the following :

Corollary 3.2. Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$. If the maps $\mathcal{T}$ and $\mathcal{I}$ are compatible as well as subsequentially continuous on $\mathcal{K}$ such that for all $x, y \in \mathcal{K}$

$$d(\mathcal{T}x, \mathcal{T}y) \leq a\, d(\mathcal{I}x, \mathcal{I}y) + (1-a) \max\{d(\mathcal{I}x, \mathcal{T}x), d(\mathcal{I}y, \mathcal{T}y)\}$$

where $0 < a < 1$, then
(i) the maps $\mathcal{T}$ and $\mathcal{I}$ have a unique common fixed point u,
(ii) $\mathcal{T}$ is continuous at u provided $\mathcal{I}$ is continuous at u.

The following lemma is required in our next theorem.

Lemma 3.2. Let $\mathcal{K}$ be a nonempty subset of a metric space $(\mathcal{X}, d)$. If the maps $\mathcal{T}$ and $\mathcal{I}$ are self-compatible on $\mathcal{K}$ such that (for all $x, y \in \mathcal{K}$)

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{I}x, \mathcal{I}y) + \frac{1-k}{k} \max\{d(\mathcal{I}x, \mathcal{T}x), d(\mathcal{I}y, \mathcal{T}y)\} \tag{3.15}$$

where $\frac{1}{2} < k < 1$ and $\mathcal{I}$ is nonexpansive, then the maps $\mathcal{T}$ and $\mathcal{I}$ are reciprocally continuous.

Proof: Since the maps $\mathcal{T}$ and $\mathcal{I}$ are compatible, therefore for sequences $\{x_m\} \subset \mathcal{K}$ with $\mathcal{T}x_m \to u$ and $\mathcal{I}x_m \to u$, we have

$$d(\mathcal{T}\mathcal{I}x_m, \mathcal{I}\mathcal{T}x_m) \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Since $\mathcal{I}$ is nonexpansive, we have $d(\mathcal{I}\mathcal{T}x_m, \mathcal{I}u) \leq d(\mathcal{T}x_m, u)$, so that

$$\mathcal{I}\mathcal{T}x_m \to \mathcal{I}u \text{ as } m \to \infty. \tag{3.17}$$

Owing to compatibility of the pair $\mathcal{T}$ and $\mathcal{I}$ along with (3.17), one can have

$$\mathcal{T}\mathcal{I}x_m \to \mathcal{I}u \text{ when } m \to \infty. \tag{3.18}$$

On setting $x = u$ and $y = \mathcal{I}x_m$ in (3.15), one gets

$$d(\mathcal{T}u, \mathcal{T}\mathcal{I}x_m) \leq d(\mathcal{I}u, \mathcal{I}\mathcal{I}x_m) + \frac{1-k}{k} \max\{d(\mathcal{I}u, \mathcal{T}u), d(\mathcal{I}\mathcal{I}x_m, \mathcal{T}\mathcal{I}x_m)\}.$$

Since $\mathcal{I}$ is a nonexpansive, on making $m \to \infty$, one gets

$$d(\mathcal{T}u, \mathcal{I}u) \leq \frac{1-k}{k} d(\mathcal{I}u, \mathcal{T}u)$$

wherein $\frac{1}{2} < k < 1$. Thus

$$\mathcal{T}u = \mathcal{I}u. \tag{3.19}$$

In view of (3.18) and (3.19), we conclude that $\mathcal{T}\mathcal{I}x_m \to \mathcal{T}u$, which together with (3.17) gives rise that $\mathcal{T}$ and $\mathcal{I}$ are reciprocally continuous and hence also subsequentially continuous.

In what follows, we denote $seg[x, q] = \{\mathcal{W}(x, q, k) : 0 \leq k \leq 1\}$ where $\mathcal{W}$ is a convex structure on $(\mathcal{X}, d)$.

Theorem 3.4. Let $\mathcal{K}$ be a nonempty closed convex subset of a convex metric space $(\mathcal{X}, d)$ satisfying the property (I). If $\mathcal{T}$ and $\mathcal{I}$ are compatible self maps defined on

$\mathcal{K}$ such that $\mathcal{I}(\mathcal{K}) = \mathcal{K}$, $q \in Fix(\mathcal{I})$, $\mathcal{I}$ is $\mathcal{W}$-affine and nonexpansive which also satisfy

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{I}x, \mathcal{I}y) + \frac{1-k}{k} \max\{d(\mathcal{I}x, seg[\mathcal{T}x, q]), d(\mathcal{I}y, seg[\mathcal{T}y, q])\} \tag{3.20}$$

for all $x, y \in \mathcal{K}$ wherein $\frac{1}{2} < k < 1$, then $\mathcal{T}$ and $\mathcal{I}$ have a common fixed point provided one of the following conditions holds:
(**i**) $cl\mathcal{T}(\mathcal{K})$ is compact and $\mathcal{T}$ is continuous,
(**ii**) $\mathcal{K}$ is compact and $\mathcal{T}$ is continuous,
(**iii**) $Fix(\mathcal{I})$ is bounded and $\mathcal{T}$ is compact.

Proof: Choose a sequence $\{k_n\} \subset (\frac{1}{2}, 1)$ with $k_n \to 1$ as $n \to \infty$. For each $n \in \mathbb{N}$, define $\mathcal{T}_n : \mathcal{K} \to \mathcal{K}$ as

$$\mathcal{T}_n x = \mathcal{W}(\mathcal{T}x, q, k_n) \tag{3.21}$$

for some $q \in \mathcal{K}$. Obviously, for each n,$\mathcal{T}_n$ maps $\mathcal{K}$ into itself as $\mathcal{K}$ is convex. Now, we show that the maps $\mathcal{T}_n$ and $\mathcal{I}$ are compatible. To accomplish this, consider an arbitrary sequence $\{x_m\} \subset \mathcal{K}$ such that $\mathcal{T}_n x_m \to u$ and $\mathcal{I}x_m \to u$ when $m \to \infty$.

Using definition of $\mathcal{T}_n$, one can have

$$\begin{aligned} d(\mathcal{T}x_m,\ \mathcal{T}_n x_m) = d(\mathcal{T}x_m, \mathcal{W}(\mathcal{T}x_m, q, k_n)) &\leq k_n\, d(\mathcal{T}x_m, \mathcal{T}x_m) + (1-k_n) d(\mathcal{T}x_m, q) \\ &= (1-k_n) d(\mathcal{T}x_m, q) \end{aligned}$$

which on making $m \to \infty$, gives rise

$$d(\lim_m \mathcal{T}x_m, u) \leq (1-k_n) d(\lim_m \mathcal{T}x_m, q).$$

Again, on making $n \to \infty$, one gets

$$d(\lim_m \mathcal{T}x_m, u) \leq 0,$$

implying thereby $\mathcal{T}x_m \to u$ as $m \to \infty$.

Owing to $\mathcal{T}x_m \to u$, $\mathcal{I}x_m \to u$, and compatibility of the maps $\mathcal{T}$ and $\mathcal{I}$ we have

$$d(\mathcal{T}\mathcal{I}x_m, \mathcal{I}\mathcal{T}x_m) \to 0 \text{ when } m \to \infty.$$

By the property (I) and compatibility of the maps $\mathcal{T}$ and $\mathcal{I}$, we have

$$\begin{aligned} 0 &\leq \lim_m d(\mathcal{T}_n \mathcal{I}x_m, \mathcal{I}\mathcal{T}_n x_m) \\ &= \lim_m d(\mathcal{W}(\mathcal{T}\mathcal{I}x_m, q, k_n), \mathcal{I}\mathcal{W}(\mathcal{T}x_m, q, k_n)) \\ &= \lim_m d(\mathcal{W}(\mathcal{T}\mathcal{I}x_m, q, k_n), \mathcal{W}(\mathcal{I}\mathcal{T}x_m, \mathcal{I}q, k_n)) \\ &= \lim_m d(\mathcal{W}(\mathcal{T}\mathcal{I}x_m, q, k_n), \mathcal{W}(\mathcal{I}\mathcal{T}x_m, q, k_n)) \\ &\leq k_n \lim_m d(\mathcal{T}\mathcal{I}x_m, \mathcal{I}\mathcal{T}x_m) = 0 \end{aligned} \tag{3.22}$$

which shows that $\{\mathcal{T}_n\}$ and $\mathcal{I}$ are compatible for each n wherein the sequence $\{x_m\} \subset \mathcal{K}$ is arbitrarily chosen.
Next, we show that $\{\mathcal{T}_n\}$ (for each n) and $\mathcal{I}$ are subsequentially continuous on $\mathcal{K}$. In case $\mathcal{T}$ and $\mathcal{I}$ are compatible, then owing to Lemma 3.2, the maps $\mathcal{T}$ and $\mathcal{I}$ are reciprocally continuous. Further, we suppose that $\mathcal{T}_n x_m \to u$ and $\mathcal{I}x_m \to u$, for $\{x_m\} \subset \mathcal{K}$, therefore in conclusion, we have $\mathcal{T}x_m \to u$ and $\mathcal{I}x_m \to u$ as $m \to \infty$. Also, as the maps $\mathcal{T}$ and $\mathcal{I}$ are reciprocally continuous, therefore $\mathcal{T}\mathcal{I}x_m \to \mathcal{T}u$ and $\mathcal{I}\mathcal{T}x_m \to \mathcal{I}u$. Moreover, owing to nonexpansiveness of $\mathcal{I}$, one can have

$$\mathcal{I}\mathcal{T}_n x_m \to \mathcal{I}u. \tag{3.23}$$

Now, making use of definition of $\mathcal{T}_n$, the property(I) and reciprocal continuity of the maps $\mathcal{T}$ and $\mathcal{I}$, one can write

$$d(\mathcal{T}_n u, \mathcal{T}_n \mathcal{I}x_m) = d(\mathcal{W}(\mathcal{T}u, q, k_n), \mathcal{W}(\mathcal{T}\mathcal{I}x_m, q, k_n)) \leq k_n\, d(\mathcal{T}u, \mathcal{T}\mathcal{I}x_m)$$

which on making $m \to \infty$ gives rise

$$\mathcal{T}_n \mathcal{I}x_m \to \mathcal{T}_n u \tag{3.24}$$

Now, in view of (3.23) and (3.24), one infers that the maps $\mathcal{T}_n$ and $\mathcal{I}$ are reciprocally continuous and hence also subsequentially continuous.

Also, for all $x, y \in \mathcal{K}$, one can write

$$d(\mathcal{T}_n x, \mathcal{T}_n y) \leq k_n d(\mathcal{T}x, \mathcal{T}y) \leq k_n [d(\mathcal{I}x, \mathcal{I}y) + \frac{1-k_n}{k_n} \max\{d(\mathcal{I}x, seg[\mathcal{T}x, q]), d(\mathcal{I}y, seg[\mathcal{T}y, q])\}]$$

i.e.

$$d(\mathcal{T}_n x, \mathcal{T}_n y) \leq k_n\, d(\mathcal{I}x, \mathcal{I}y) + \ (1-k_n) \max\{d(\mathcal{I}x, seg[\mathcal{T}x, q]), d(\mathcal{I}y, seg[\mathcal{T}y, q])\}$$

for all $x, y \in \mathcal{K}$ and $\frac{1}{2} < k_n < 1$.

Since $\mathcal{K}$ is closed, therefore using Corollary 3.2, $\mathcal{T}_n$ (for every $n \in \mathbb{N}$) and $\mathcal{I}$ have common fixed point x_n in $\mathcal{K}$, i.e.

$$x_n = \mathcal{T}_n x_n = \mathcal{I}x_n.$$

Since $\mathcal{W}$ is continuous and $cl\mathcal{T}(\mathcal{K})$ is compact, then $cl\mathcal{T}_n(\mathcal{K})$ is also compact. The compactness of $cl\mathcal{T}(\mathcal{K})$ implies that there exists a subsequence $\mathcal{T}x_m$ of $\mathcal{T}x_n$ such that $\mathcal{T}x_m \to y$ as $m \to \infty$. Then by definition $\mathcal{T}_m x_m$, we have

$$d(\mathcal{T}_m x_m, y) = d(y, \mathcal{W}(\mathcal{T}x_m, q, k_m)) \leq k_m\, d(y, \mathcal{T}x_m) + (1-k_m)\, d(y, q)$$

which shows that $x_m \to y$ as $m \to \infty$. Since $\mathcal{T}$is continuous, therefore $\mathcal{T}x_m \to \mathcal{T}y$. By using uniqueness of limit, one concludes

$$\mathcal{T}y = y. \tag{3.25}$$

Using compatibility of the maps $\mathcal{T}$ and $\mathcal{I}$ and Lemma 3.2, one can write $\mathcal{I}y = \mathcal{T}y$, which in turn (due to (3.25)) yields that $\mathcal{I}y = \mathcal{T}y = y$. Thus $Fix(\mathcal{T}) \cap Fix(\mathcal{I}) \neq \emptyset$.

Since $\mathcal{K}$ is compact and $\mathcal{T}$ is continuous, therefore $\mathcal{T}(\mathcal{K})$ is compact and henceforth result follows from (i).

As in (i), there exists a unique $x_n \in \mathcal{K}$ such that $x_n = \mathcal{T}_n x_n = \mathcal{I}x_n$. As $\mathcal{T}$ is compact and $\{x_n\}$ being in $Fix(\mathcal{I})$ is bounded, so in view of compactness of the map $\mathcal{T}$, $\{\mathcal{T}x_n\}$ admits a convergent subsequence $\{\mathcal{T}x_m\}$, i.e. $\mathcal{T}x_m \to y$ as $m \to \infty$. Now, on the lines of the proof of part (i), one can have

$$Fix(\mathcal{T}) \cap Fix(\mathcal{I}) \neq \emptyset.$$

This completes the proof of the theorem.

Analogously to Lemma 3.2, one can also have the following:

Lemma 3.3. Let $\mathcal{K}$ be a nonempty subset of metric space $(\mathcal{X}, d)$. If the maps $\mathcal{T}$ and $\mathcal{I}$ are self-compatible on $\mathcal{K}$ such that (for all $x, y \in \mathcal{K}$)

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{I}x, \mathcal{I}y) + \frac{1-k}{k} \max\{d(\mathcal{I}x, \mathcal{T}x), d(\mathcal{I}y, \mathcal{T}y)\}$$

wherein $\frac{1}{2} < k < 1$ and $\mathcal{I}$ is continuous, then $\mathcal{T}$ and $\mathcal{I}$ are reciprocally continuous.

Proof: The proof of this lemma is same as that of Lemma 3.2.

The following theorem extends Theorem 2 of [12].

Theorem 3.5. Let $\mathcal{K}$ be a nonempty closed convex subset of a convex metric space $(\mathcal{X}, d)$ satisfying the property (I). If $\mathcal{T}$ and $\mathcal{I}$ are compatible self-maps defined on $\mathcal{K}$ satisfying the condition

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{I}x, \mathcal{I}y) + \tfrac{1-k}{k} \max\{d(\mathcal{I}x, seg[\mathcal{T}x, q]), d(\mathcal{I}y, seg[\mathcal{T}y, q])\}$$

for all $x, y \in \mathcal{K}$, $\frac{1}{2} < k < 1$ with $\mathcal{I}(\mathcal{K}) = \mathcal{K}$, $q \in Fix(\mathcal{I})$ and $\mathcal{I}$ is $\mathcal{W}$-affine and continuous, then $\mathcal{T}$ and $\mathcal{I}$ have a common fixed point provided one of the following conditions holds:

(**i**) $cl\mathcal{T}(\mathcal{K})$ is compact and $\mathcal{T}$ is continuous,
(**ii**) $\mathcal{K}$ is compact and $\mathcal{T}$ is continuous,
(**iii**) $Fix(\mathcal{I})$ is bounded and $\mathcal{T}$ is compact.

Proof: Since $\mathcal{T}$ and $\mathcal{I}$ are compatible and $\mathcal{I}$ is continuous, therefore in view of Lemma 3.3, one finds that the maps $\mathcal{T}$ and $\mathcal{I}$ are reciprocally continuous. The rest of the proof can be completed on the lines of the proof of Theorem 3.4.

4 Applications To Invariant Approximation

As applications of Theorems 3.3 and 3.4, we derive two results in invariant approximation theory for compatible mappings in the frame work of convex metric spaces.

Theorem 4.1. Let $\mathcal{T}$ and $\mathcal{I}$ be self-maps of a convex metric space $(\mathcal{X}, d)$ and $\mathcal{K}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{I})$ with $x_0 \in \mathcal{X}$. Suppose that $\mathcal{D} = \mathcal{P}_{\mathcal{K}}(x_0)$ is nonempty convex subset with $\mathcal{I}(\mathcal{D}) = \mathcal{D}$, $q \in Fix(\mathcal{I})$, $\mathcal{I}$ is a $\mathcal{W}$-affine and nonexpansive. If the maps $\mathcal{T}$ and $\mathcal{I}$ are compatible on $\mathcal{D}$ and also satisfy (for all $x, y \in \mathcal{D}' = \mathcal{D} \cup \{x_0\}$)

$$d(\mathcal{T}x, \mathcal{T}y) \leq \begin{cases} d(\mathcal{I}x, \mathcal{I}x_0) & \text{if } y = x_0 \\ d(\mathcal{I}x, \mathcal{I}y) + \frac{1-k}{k} \max\{d(\mathcal{I}x, seg[\mathcal{T}x, q]), d(\mathcal{I}y, seg[\mathcal{T}y, q])\} & \text{if } y \in \mathcal{D} \end{cases} \tag{4.1}$$

with $\frac{1}{2} < k < 1$, then $\mathcal{T}$ and $\mathcal{I}$ have a common fixed point in $\mathcal{D}$ provided one of the following conditions holds:

(**i**) $cl\mathcal{T}(\mathcal{K})$ is compact and $\mathcal{T}$ is continuous.
(**ii**) $\mathcal{K}$ is compact and $\mathcal{T}$ is continuous.
(**iii**) $Fix(\mathcal{I})$ is bounded and $\mathcal{T}$ is compact.

Proof: Firstly, we show that $\mathcal{T}$ is a self-map on $\mathcal{D}$ i.e. $\mathcal{T} : \mathcal{D} \to \mathcal{D}$. To do this, let $y \in \mathcal{D}$ so that $\mathcal{I}y \in \mathcal{D}$ as $\mathcal{I}(\mathcal{D}) = \mathcal{D}$. In case $y \in \partial\mathcal{K}$, then $\mathcal{T}y \in \mathcal{K}$ as $\mathcal{T}(\partial\mathcal{K}) \subseteq \mathcal{K}$. Owing to the fact that $\mathcal{T}x_0 = x_0 = \mathcal{I}x_0$, one may have (from (4.1))

$$d(\mathcal{T}y, x_0) = d(\mathcal{T}y, \mathcal{T}x_0) \leq d(\mathcal{I}y, \mathcal{I}x_0) = d(\mathcal{I}y, x_0) = d(x_0, \mathcal{K})$$

which shows that $\mathcal{T}y \in \mathcal{D}$, and in all $\mathcal{T}$ and $\mathcal{I}$ are self-maps on $\mathcal{D}$. Thus all the conditions of Theorem 3.4 are satisfied and hence there exists a $u \in \mathcal{D}$ such that $\mathcal{T}u = u = \mathcal{I}u$.

Similarly, the following theorem extends Theorem 3 from [12].

Theorem 4.2. Let $\mathcal{T}$ and $\mathcal{I}$ be self-maps of a convex metric space $(\mathcal{X}, d)$ and $\mathcal{K}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{I})$ with $x_0 \in \mathcal{X}$. Suppose that $\mathcal{D} = \mathcal{P}_{\mathcal{K}}(x_0)$ is nonempty convex subset with $\mathcal{I}(\mathcal{D}) = \mathcal{D}$, $q \in Fix(\mathcal{I})$, $\mathcal{I}$ is a $\mathcal{W}$-affine and continuous. If the maps $\mathcal{T}$ and $\mathcal{I}$ are compatible on $\mathcal{D}$ and also satisfy (for all $x, y \in \mathcal{D}' = \mathcal{D} \cup \{x_0\}$)

$$d(\mathcal{T}x, \mathcal{T}y) \leq \begin{cases} d(\mathcal{I}x, \mathcal{I}x_0) & \text{if } y = x_0 \\ d(\mathcal{I}x, \mathcal{I}y) + \frac{1-k}{k} \max\{d(\mathcal{I}x, seg[\mathcal{T}x, q]), d(\mathcal{I}y, seg[\mathcal{T}y, q])\} & \text{if } y \in \mathcal{D} \end{cases}$$

with $\frac{1}{2} < k < 1$, then $\mathcal{T}$ and $\mathcal{I}$ have a common fixed point in $\mathcal{D}$ provided one of the following conditions holds:

(**i**) $cl\mathcal{T}(\mathcal{K})$ is compact and $\mathcal{T}$ is continuous.
(**ii**) $\mathcal{K}$ is compact and $\mathcal{T}$ is continuous.
(**iii**) $Fix(\mathcal{I})$ is bounded and $\mathcal{T}$ is compact.

Proof: The proof of this theorem is similar to that of Theorem 4.1, hence it is omitted.

Remark 4.1. Example 12 of [12] can be utilized to demonstrate Theorems 4.1 and 4.2.

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